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THE GENERAL $\Gamma-$ COMPATIBLE ROOK LENGTH POLYNOMIALS

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Abstract. Rook placements and rook polynomials have been studied by mathematicians since the early 1970's. Since then many relationships between rook placements and other subjects have been discovered (cf. [1], [6-15]). In [2] and [3], K. Ding introduced the rook length polynomials and the $\gamma-$compatible rook length polynomials. In [3] and [4], he used these polynomials to establish a connection between rook placements and algebraic geometry for the first time.

In this paper, we give explicit formulas for the $\gamma-$compatible rook length polynomials in more general cases than considered in [3]. In particular, we generalize the formula for the rook length polynomial in the parabolic case in [2] to the $\gamma-$compatible rook length polynomial.

1. Introduction

Let $\mathbf{M}^r_{m,n}(\mathbf{C})$ be the set of all $m \times n$ matrices of rank r over the complex field $\mathbf{C}$, where m and n are positive integers and r is a nonnegative integer. Also let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of some positive integer with $\lambda_1 \geq \cdots \geq \lambda_m > 0$. A *Ferrers board* F_λ is a subarray of an $m \times n$ matrix, where $n = \lambda_1$ and the *ith* row has length λ_i for $1 \leq i \leq m$.

Let $\mathbf{M}^r_\lambda = \left\{ a \in \mathbf{M}^r_{m,n}(\mathbf{C}) \mid a_{ij} = 0 \text{ for } (i,j) \notin F_\lambda \right\}$. An element of $\mathbf{M}^r_\lambda$ is called a *rook placement* of rank r on F_λ if it is a $(0,1)$ matrix with exactly r 1's (or rooks) and at most one 1 (or rook) in each row and column.

Let R^r_λ be the set of all the rook placements of rank r on F_λ. For any $\sigma \in R^r_\lambda$, the *length function* $l(\sigma)$ is the minimum number of adjacent row and/or column transpositions required to get the 1's in the upper right hand corner such that all intermediate rook placements are in the Ferrers board F_λ.

In the sequel, whenever we display a rook placement σ on F_λ, we omit the elements σ_{ij} for $(i,j) \notin F_\lambda$ in order to make the shape of the Ferrers board F_λ more evident. We will also write $\sigma(i) = j$ if and only if $\sigma_{ij} = 1$ for $(i,j) \in F_\lambda$.

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For example, let $\lambda = (3,2,2)$ and let σ be defined by $\sigma(1) = 1$, $\sigma(2) = 3$, and $\sigma(3) = 2$. Then

$$\sigma = \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}$$

is a rook placement of rank $r = 3$ on F_λ with $l(\sigma) = 1$.

The *rook length polynomial* is then defined by

$$RL_r(\lambda, q) = \sum_{\sigma \in R_\lambda^r} q^{l(\sigma)}.$$

In [2], K. Ding showed the relationship between the rook length polynomials and Garsia-Remmel polynomials. In [4], he showed that the Poincare polynomial for homology (and cohomology) of the partition varieties $\mathbf{B}/\mathbf{M}_\lambda^m$ with real coefficients is $RL_m(\lambda, q^2)$, where $\mathbf{B}$ is the Borel subgroup of upper triangle matrices of $\mathbf{GL}_m(\mathbf{C})$.

Let $\gamma = (\gamma_1, \ldots, \gamma_t)$ be a partition of m. We say that λ is a $\gamma-$*compatible* partition if $\lambda = (k_1^{\gamma_1}, \ldots, k_t^{\gamma_t})$, where k_i, $1 \le i \le t$, are positive integers with $k_1 \ge \cdots \ge k_t$.

A rook placement σ of rank r on a Ferrers board F_λ, where λ is a $\gamma-$compatible partition, is said to be $\gamma-$*compatible* if σ is monotone increasing on each interval $\left(\sum_{j=1}^{j'} \gamma_j, \sum_{j=1}^{j'+1} \gamma_j\right]$, i.e., if $\sigma(i) < \sigma(i+1)$ whenever $\sum_{j=1}^{j'} \gamma_j < i < i+1 \le \sum_{j=1}^{j'+1} \gamma_j$, for fixed values of j', $0 \le j' < t$. (If $j' = 0$, then $\sum_{j=1}^{0} \gamma_j = 0$.)

For example, if $\gamma = (3,4)$ and $\lambda = (6,6,6,5,5,5,5) = (6^3, 5^4)$, then

$$\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is a γ-compatible rook placement on F_λ.

Let $R_\lambda^r(\gamma)$ be the set of all $\gamma-$compatible rook placements on the Ferrers board F_λ with rank r. The $\gamma-$*compatible rook length polynomial* is defined by

$$RL_r(\lambda, \gamma, q) = \sum_{\sigma \in R_\lambda^r(\gamma)} q^{l(\sigma)}.$$

In [3], K. Ding gave an explicit formula for $RL_r(\lambda, \gamma, q)$ when $r = m$ and proved that the Poincare polynomial for cohomology of the partition variety $\mathbf{P}_\gamma/\mathbf{M}_\lambda^m$ with real coefficients is the $\gamma-$compatible rook length polynomial with $r = m$, where $\mathbf{P}_\gamma$ denotes the parabolic subgroup of $\mathbf{G}_m$ of the form

$$P_\gamma = \begin{bmatrix} G_{\gamma_1} & * & \cdots & * \\ 0 & \ddots & * & \vdots \\ \vdots & 0 & \ddots & * \\ 0 & \cdots & 0 & G_{\gamma_t} \end{bmatrix}.$$

Here $G_{\gamma_i} = \mathbf{GL}_{\gamma_i}(\mathbf{C})$ and the $*$'s are arbitrary matrices of the appropriate sizes.

In this paper we give explicit and recurrence formulas for $RL_r(\lambda, \gamma, q)$, where $1 \leq r \leq \min(m, n)$. We also give an explicit formula for the case where λ is parabolic and $r = m$.

2. A recurrence formula for the general $\gamma-$ compatible rook length polynomial

For integers a and b such that $a \geq b \geq 0$, the *Gaussian binomial coefficient* is defined by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]!_q}{[b]!_q\,[a-b]!_q},$$

where $[s]!_q = (1)_q (2)_q \cdots (s)_q$ with $(k)_q = \left(1 + q + \cdots + q^{k-1}\right)$ for $1 \leq k \leq s$ and $[0]!_q = 1$.

We will make use of the following result results from [3] in our proof of Theorem 3. The second result is a local formula for computing the length of a rook placement:

Proposition 1. *For integers s and t such that $s \geq 0$ and $t \geq 1$, we have*

$$\sum_{0 \leq a_1 < \cdots < a_t \leq s} q^{a_1 + \cdots + a_t} = \begin{bmatrix} s+t \\ s \end{bmatrix}_q.$$

Proposition 2.(Local Formula) *Let $\sigma \in R_\lambda^r$, then*

$$l(\sigma) = \sum_{i=1}^{r} (u_i + v_i + w_i),$$

where u_i is the number of zero columns to the right of the ith rook, v_i is the number of rooks above and to the right (or 'northeast') of the ith rook, and w_i is the number of zero rows above the ith rook.

Theorem 3. *Let* $\gamma = (\gamma_1, \ldots, \gamma_t)$ *be a partition of* m *and* $\lambda = (k_1^{\gamma_1}, \ldots, k_t^{\gamma_t})$ *be a* $\gamma-$*compatible partition, where* k_i, $1 \le i \le t$, *are positive integers with* $k_1 \ge \cdots \ge k_t$. *Let* k_{t+1} *and* γ_{t+1} *be positive integers where* $k_{t+1} \le k_t$. *Then* $\lambda \cup k_{t+1}^{\gamma_{t+1}} = \left(k_1^{\gamma_1}, \ldots, k_{t+1}^{\gamma_{t+1}}\right)$ *is* $\gamma \cup \gamma_{t+1} = (\gamma_1, \ldots, \gamma_{t+1})-$*compatible and*

$$RL_r\left(\lambda \cup k_{t+1}^{\gamma_{t+1}}, \gamma \cup \gamma_{t+1}, q\right) = \sum_{r_{t+1}=0}^{\min(r,\gamma_{t+1},k_{t+1})} q^{r_{t+1}(\Gamma - r + r_{t+1})} \begin{bmatrix} \gamma_{t+1} \\ r_{t+1} \end{bmatrix}_q \begin{bmatrix} k_{t+1} \\ r_{t+1} \end{bmatrix}_q RL_{r-r_{t+1}}\left(\lambda - r_{t+1}^{\Gamma}, \gamma, q\right),$$

where $r = \sum_{i=1}^{t+1} r_i$, $\Gamma = \sum_{k=1}^{t} \gamma_k$ *and* $\lambda - r_{t+1}^{\Gamma} = \left((k_1 - r_{t+1})^{\gamma_1}, \ldots, (k_t - r_{t+1})^{\gamma_t}\right)$.

Proof. Consider a $\gamma \cup \gamma_{t+1}-$compatible rook placement σ on $F_{\lambda \cup k_{t+1}^{\gamma_{t+1}}}$ of rank r. There are γ_{t+1} rows and r_{t+1} rooks in the last block of σ where $0 \le r_{t+1} \le \min(r, \gamma_{t+1}, k_{t+1})$. Suppose that the jth rook of the last block lies in the $x_{r_{t+1}-j+1}$ column from the right and the $y_j th$ row from the top of the last block. Then $1 \le y_1 < \cdots < y_{r_{t+1}} \le \gamma_{t+1}$ and $1 \le x_1 < \cdots < x_{r_{t+1}} \le k_{t+1}$.

Let α_j be the number of zero rows in $\lambda \cup k_{t+1}^{\gamma_{t+1}}$ above the jth rook, β_j be the number of rooks northeast of the jth rook and δ_j the number of zero columns to the right of the jth rook.

Since $\sum_{j=1}^{t}(\gamma_i - r_i)$ is the total number of zero rows in the first t blocks and $(y_j - j)$ is the number of zero rows above the jth rook in the last block, then $\alpha_j = \sum_{j=1}^{t}(\gamma_i - r_i) + (y_j - j) = \Gamma - r + r_{t+1} + y_j - j$. Since σ is $\gamma \cup \gamma_{t+1}-$compatible, then if there is a rook to the northeast of the jth rook, the rook must be in one of the first t blocks. So the column in which the rook is located is a zero column in the last block. Since $x_{r_{t+1}-j+1} - (r_{t+1} - j + 1)$ is the number of zero columns to the right of the jth rook in the last block, then $\beta_j + \delta_j = x_{r_{t+1}-j+1} - (r_{t+1} - j + 1)$.

By removing the last block and all the columns which contain the rooks in the last block from σ, we obtain a Ferrers board of shape

$$\lambda - r_{t+1}^{\Gamma} = \left((k_1 - r_{t+1})^{\gamma_1}, \ldots, (k_t - r_{t+1})^{\gamma_t}\right).$$

Thus the original rook placement σ induces a $\gamma-$compatible rook placement σ' of rank $r - r_{t+1}$ on this new Ferrers board $F_{\lambda - r_{t+1}^{\Gamma}}$. Moreover, in removing the last block the values in the local formula for the remaining rooks remain unchanged. Consequently, these rooks contribute $l(\sigma')$ to the value of $l(\sigma)$ and so

$$\begin{aligned} l(\sigma) &= l(\sigma') + \sum_{j=1}^{r_{t+1}} (\alpha_j + \beta_j + \delta_j) \\ &= l(\sigma') + \sum_{j=1}^{r_{t+1}} \left(\Gamma - r - 1 + y_j + x_{r_{t+1}-j+1}\right) \\ &= l(\sigma') + r_{t+1}(\Gamma - r - 1) + \sum_{j=1}^{r_{t+1}} (y_j + x_j). \end{aligned}$$

Conversely, suppose we are given $\sigma' \in R^{r-r_{t+1}}_{\lambda-r^{\Gamma}_{t+1}}(\gamma)$, where $0 \leq r_{t+1} \leq \min(r, \gamma_{t+1}, k_{t+1})$, and integers x_i and y_i, $1 \leq i \leq r_{t+1}$, where $1 \leq x_1 < \cdots < x_{r_{t+1}} \leq k_{t+1}$ and $1 \leq y_1 < \cdots < y_{r_{t+1}} \leq \gamma_{t+1}$. Then we can reverse the above process to obtain a unique rook placement $\sigma \in R^{r}_{\lambda\cup k^{\gamma_{t+1}}_{t+1}}(\gamma \cup \gamma_{t+1})$. More specifically, we first insert r_{t+1} zero columns in σ' at the positions $x_1, \ldots, x_{r_{t+1}}$ from the right. Then we attach a block with γ_{t+1} rows and k_{t+1} columns to σ' at the bottom. This last block will contain r_{t+1} rooks in the positions determined by the x_i's and the y_i's.

Thus,

$$
\begin{aligned}
&RL_r\left(\lambda \cup k^{\gamma_{t+1}}_{t+1}, \gamma \cup \gamma_{t+1}, q\right)\\
&= \sum_{\sigma\in RL^{r}_{\lambda\cup k^{\gamma_{t+1}}_{t+1}}(\gamma\cup\gamma_{t+1})} q^{l(\sigma)}\\
&= \sum_{\substack{0\leq r_t\leq \min(r,k_{t+1},\gamma_{t+1}),\\ \sigma'\in R^{r-r_t}_{\lambda-r^{\Gamma}_{t+1}}(\gamma),\\ 1\leq x_1<\cdots<x_{r_{t+1}}\leq k_{t+1},\\ 1\leq y_1<\cdots<y_{r_{t+1}}\leq \gamma_{t+1}}} q^{l\left(\sigma'\right)+r_{t+1}(\Gamma-r-1)+\sum_{j=1}^{r_{t+1}}(x_j+y_j)}\\
&= \sum_{r_{t+1}=0}^{\min(r,\gamma_{t+1},k_{t+1})} \left(q^{r_{t+1}(\Gamma-r-1)} \sum_{\substack{1\leq x_1<\cdots<x_{r_{t+1}}\leq k_{t+1},\\ 1\leq y_1<\cdots<y_{r_{t+1}}\leq \gamma_{t+1}}} q^{\sum_{j=1}^{r_{t+1}}(x_j+y_j)} \sum_{\sigma'\in R^{r-r_t}_{\lambda-r^{\Gamma}_{t+1}}(\gamma)} q^{l\left(\sigma'\right)} \right)\\
&= \sum_{r_{t+1}=0}^{\min(r,\gamma_{t+1},k_{t+1})} \left(q^{r_{t+1}(\Gamma-r-1)} \sum_{\substack{1\leq x_1<\cdots<x_{r_{t+1}}\leq k_{t+1},\\ 1\leq y_1<\cdots<y_{r_{t+1}}\leq \gamma_{t+1}}} q^{\sum_{j=1}^{r_{t+1}} x_j + \sum_{j=1}^{r_{t+1}} y_j} \right.\\
&\qquad\qquad \left. * RL_{r-r_{t+1}}\left(\lambda - r^{\Gamma}_{t+1}, \gamma, q\right)\right)\\
&= \sum_{r_{t+1}=0}^{\min(r,\gamma_{t+1},k_{t+1})} \left(q^{r_{t+1}(\Gamma-r-1)} q^{r_{t+1}(r_{t+1}+1)} \begin{bmatrix} k_{t+1} \\ r_{t+1} \end{bmatrix}_q \begin{bmatrix} \gamma_{t+1} \\ r_{t+1} \end{bmatrix}_q \right.\\
&\qquad\qquad \left. * RL_{r-r_{t+1}}\left(\lambda - r^{\Gamma}_{t+1}, \gamma, q\right)\right)\\
&= \sum_{r_{t+1}=0}^{\min(r,\gamma_{t+1},k_{t+1})} \left(q^{r_{t+1}(\Gamma-r+r_{t+1})} \begin{bmatrix} k_{t+1} \\ r_{t+1} \end{bmatrix}_q \begin{bmatrix} \gamma_{t+1} \\ r_{t+1} \end{bmatrix}_q RL_{r-r_{t+1}}\left(\lambda - r^{\Gamma}_{t+1}, \gamma, q\right)\right).
\end{aligned}
$$

Corollary 4. *Let $r = m$. Then $r_i = \gamma_i$, $1 \leq i \leq t+1$, and*

$$
RL_m\left(\lambda \cup k^{\gamma_{t+1}}_{t+1}, \gamma \cup \gamma_{t+1}, q\right) = \begin{bmatrix} k_{t+1} \\ \gamma_{t+1} \end{bmatrix}_q RL_{m-\gamma_{t+1}}\left(\lambda - \gamma^{\Gamma}_{t+1}, \gamma, q\right),
$$

where $\lambda - \gamma_{t+1}^{\Gamma} = \left((k_1 - \gamma_{t+1})^{\gamma_1}, \ldots, (k_t - \gamma_{t+1})^{\gamma_t}\right).$

Let $\gamma = (1, \ldots, 1) = (1^m)$. Then the $\gamma-$compatible rook length polynomial is the rook length polynomial, i.e.,

$$RL_r(\lambda, \gamma, q) = RL_r(\lambda, q).$$

In this case, $k_i = \lambda_i$, $1 \leq i \leq m$, and $\Gamma = \sum_{k=1}^{m} \gamma_k = m$.

The following corollary of Theorem 3 appears as Corollary 5.35 of [2].

Corollary 5. *Let* $\gamma = (1, \ldots, 1) = (1^m)$ *and* λ_{m+1} *be an integer such that* $1 \leq \lambda_{m+1} \leq \lambda_m$. *Then*

$$RL_r(\lambda \cup \lambda_{m+1}, q) = RL_r(\lambda, q) + q^{m-r+1} (\lambda_{m+1})_q \, RL_{r-1}(\lambda - 1^m, q),$$

where $\lambda - 1^m = (\lambda_1 - 1, \ldots, \lambda_m - 1).$

3. The general formula for the $\gamma-$ compatible rook length polynomial

Theorem 6. *Let* $\gamma = (\gamma_1, \ldots, \gamma_t)$ *be a partition of* m *and* $\lambda = (k_1^{\gamma_1}, \ldots, k_t^{\gamma_t})$ *be a* $\gamma-$*compatible partition, where* k_i, $1 \leq i \leq t$, *are positive integers with* $k_1 \geq \cdots \geq k_t$. *Then the* $\gamma-$*compatible rook length polynomial for* $0 \leq r \leq \min(m, k_1)$ *is given by*

$$RL_r(\lambda, \gamma, q) = \sum_{\mathbf{\Delta}} \prod_{i=1}^{t} \begin{bmatrix} k_i - r_{i+1} - r_{i+2} - \cdots - r_t \\ r_i \end{bmatrix}_q \begin{bmatrix} \gamma_i \\ r_i \end{bmatrix}_q q^{\sum_{i=2}^{t} \sum_{j=1}^{i-1} r_i(\gamma_j - r_j)},$$

where $\mathbf{\Delta} = \{(r_1, r_2, \ldots, r_t) \mid 0 \leq r_i \leq min(\gamma_i, k_i),\ r_1 + \ldots + r_t = r\}$ *and* r_i *equals the number of rooks in block* i, $1 \leq i \leq t$.

The proof will be by induction on the number of blocks t. The base case, when $t = 1$, is stated and proved in the following lemma.

Lemma 7. *Let* $\lambda = (n^m)$ *and* $\gamma = ((m))$. *In particular,* λ *is* $\gamma-$*compatible and* F_λ *is an* $m \times n$ *rectangular Ferrers board. Then for* $0 \leq r \leq \min(m, n)$

$$RL_r(\lambda, \gamma, q) = \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} m \\ r \end{bmatrix}_q.$$

Proof. If $r = 0$, $RL_r(\lambda, \gamma, q) = 1$. So assume that $r \geq 1$. For any $\sigma \in R^r_\lambda(\gamma)$, let the jth rook of σ be in the $x_{r-j+1}th$ column from the right and the $y_j th$ row from the top where $1 \leq j \leq r$, $1 \leq x_1 < \cdots < x_r \leq n$, and $1 \leq y_1 < \cdots < y_r \leq m$. Since $x_j - j$ counts the number of zero columns to the right of the $(r-j+1)th$ rook, then $u = \sum_{j=1}^{r}(x_j - j)$ counts the total number of zero columns to the right of all r rooks in σ. Since $y_j - j$ counts the number of zero rows above the jth rook, then $w = \sum_{j=1}^{r}(y_j - j)$ counts the total number of zero rows above all r rooks in σ. By the definition of $\gamma-$compatible, $\sigma(i) < \sigma(i+1)$ and so there are no rooks above and to the right of any of the rooks. So $l(\sigma) = u + w$. Thus,

$$\begin{aligned} RL_r((n^m),(m),q) &= \sum_{\sigma \in R^r_\lambda(\gamma)} q^{l(\sigma)} \\ &= \sum_{\substack{1 \leq x_1 < \cdots < x_r \leq n, \\ 1 \leq y_1 < \cdots < y_r \leq m}} q^{u+w} \\ &= \sum_{1 \leq x_1 < \cdots < x_r \leq n} q^u \sum_{1 \leq y_1 < \cdots < y_r \leq m} q^w \\ &= \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} m \\ r \end{bmatrix}_q . \end{aligned}$$

Proof of Theorem 6. We have already proven the base case as Lemma 7. For the inductive step, we assume that formula is true for $t-1$ blocks, where $t > 1$.

By the recurrence formula (Theorem 3), we have

$$RL_r(\lambda, \gamma, q) = \sum_{r_t=0}^{\min(r, \gamma_{t,k_t})} q^{r_t(\Gamma - r + r_t)} \begin{bmatrix} \gamma_t \\ r_t \end{bmatrix}_q \begin{bmatrix} k_t \\ r_t \end{bmatrix}_q RL_{r-r_t}\left(\lambda - r_t^\Gamma, \gamma', q\right),$$

where $\Gamma = \sum_{i=1}^{t-1} \gamma_i = m - \gamma_t, \lambda - \gamma_t^\Gamma = \left((k_1 - \gamma_t)^{\gamma_1}, \ldots, (k_t - \gamma_t)^{\gamma_{t-1}}\right)$ and $\gamma' = (\gamma_1, \ldots, \gamma_{t-1})$.

By the induction hypothesis,

$$\begin{aligned} &RL_{r-r_t}\left(\lambda - r_t^\Gamma, \gamma', q\right) \\ &= \sum_{\mathbf{\Delta}'} \prod_{i=1}^{t-1} \begin{bmatrix} (k_i - r_t) - r_{i+1} - r_{i+2} - \cdots - r_{t-1} \\ r_i \end{bmatrix}_q \begin{bmatrix} \gamma_i \\ r_i \end{bmatrix}_q q^{\sum_{i=2}^{t-1} \sum_{j=1}^{i-1} r_i(\gamma_j - r_j)}, \end{aligned}$$

where $\mathbf{\Delta}' = \{(r_1, r_2, \ldots, r_{t-1}) \mid 0 \leq r_i \leq \min(\gamma_i, k_i),\ r_1 + \ldots + r_{t-1} = r - r_t\}$.

So

$$
\begin{aligned}
RL_r(\lambda,\gamma,q) &= \sum_{0\le r_t\le \min(k_t,\gamma_t,r)} \Bigg(q^{r_t(m-\gamma_t-r+r_t)} \begin{bmatrix} k_t \\ r_t \end{bmatrix}_q \begin{bmatrix} \gamma_t \\ r_t \end{bmatrix}_q \\
&\quad * \sum_{\Delta'} \prod_{i=1}^{t-1} \begin{bmatrix} (k_i-r_t)-r_{i+1}-r_{i+2}-\cdots-r_{t-1} \\ r_i \end{bmatrix}_q \begin{bmatrix} \gamma_i \\ r_i \end{bmatrix}_q q^{\sum_{i=2}^{t-1}\sum_{j=1}^{i-1} r_i(\gamma_j-r_j)} \Bigg) \\
&= \sum_{0\le r_t\le \min(k_t,\gamma_t,r)} q^{r_t(m-\gamma_t-r+r_t)} \\
&\quad * \sum_{\Delta'} \prod_{i=1}^{t} \begin{bmatrix} k_i-r_{i+1}-r_{i+2}-\cdots-r_t \\ r_i \end{bmatrix}_q \begin{bmatrix} \gamma_i \\ r_i \end{bmatrix}_q q^{\sum_{i=2}^{t-1}\sum_{j=1}^{i-1} r_i(\gamma_j-r_j)} \\
&= \sum_{\mathbf{\Delta}} \prod_{i=1}^{t} \begin{bmatrix} k_i-r_{i+1}-r_{i+2}-\cdots-r_t \\ r_i \end{bmatrix}_q \begin{bmatrix} \gamma_i \\ r_i \end{bmatrix}_q q^{\sum_{i=2}^{t}\sum_{j=1}^{i-1} r_i(\gamma_j-r_j)}.
\end{aligned}
$$

From Theorem 6, we obtain Theorem 33 of [3] as a corollary.

Corollary 8. *If $r = m$, then $r_i = \gamma_i$, $1 \le i \le t$ and $\mathbf{\Delta}$ contains only one condition, namely $r_i = \gamma_i$, $1 \le i \le t$. Thus when $r = m$,*

$$RL_m(\lambda,\gamma,q) = \prod_{i=1}^{t} \begin{bmatrix} k_i-\gamma_{i+1}-\gamma_{i+2}-\cdots-\gamma_t \\ \gamma_i \end{bmatrix}_q .$$

Example. Let $\gamma = (1,3)$ and $\lambda = (4,3,3,3)$. Then λ is $\gamma-$compatible since $\lambda = \left(4^1, 3^3\right)$. Let $r = 3$. Since $0 \le r_i \le \min(\gamma_i, k_i)$, $i = 1,2$, $\Delta = \{(r_1,r_2) = (0,3), (r_1,r_2) = (1,2)\}$. By Theorem 6, we get

$$
\begin{aligned}
RL_3((4,3,3,3),(1,3),q) &= q^3 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \\
&= q^3 + \frac{(3)!_q\,(3)!_q}{(2)!_q} \\
&= q^3 + (1+q)\left(1+q+q^2\right)^2 \\
&= 1 + 3q + 5q^2 + 6q^3 + 3q^4 + q^5.
\end{aligned}
$$

We now calculate the $\gamma-$compatible rook length polynomial directly from its definition.

Let $\sigma_i, 1 \le i \le 19$, be defined as follows:

$$\sigma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{bmatrix}, \quad \sigma_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{bmatrix}, \quad \sigma_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \end{bmatrix}, \quad \sigma_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 1 \end{bmatrix}, \quad \sigma_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_{10} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{bmatrix}, \quad \sigma_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 1 \end{bmatrix}, \quad \sigma_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_{13} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{bmatrix}, \quad \sigma_{14} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 1 \end{bmatrix}, \quad \sigma_{15} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_{16} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 0 \end{bmatrix}, \quad \sigma_{17} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 0 \end{bmatrix}, \quad \sigma_{18} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 0 \end{bmatrix},$$

$$\sigma_{19} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{bmatrix}.$$

Then by the definition of the length function, we have

$$\begin{aligned} &l(\sigma_1) = 3, \quad l(\sigma_2) = 4, \quad l(\sigma_3) = 3, \quad l(\sigma_4) = 2, \quad l(\sigma_5) = 1, \quad l(\sigma_6) = 2, \\ &l(\sigma_7) = 5, \quad l(\sigma_8) = 4, \quad l(\sigma_9) = 3, \quad l(\sigma_{10}) = 0, \quad l(\sigma_{11}) = 1, \quad l(\sigma_{12}) = 2, \\ &l(\sigma_{13}) = 1, \quad l(\sigma_{14}) = 2, \quad l(\sigma_{15}) = 3, \quad l(\sigma_{16}) = 2, \quad l(\sigma_{17}) = 3, \quad l(\sigma_{18}) = 4, \end{aligned}$$

and $l(\sigma_{19}) = 3$.
So

$$RL_3\left((4,3,3),(1,3),q\right) = \sum_{i=i}^{18} q^{l(\sigma_i)} = 1 + 3q + 5q^2 + 6q^3 + 3q^4 + q^5.$$

4. A formula for the parabolic $\gamma-$ compatible rook length polynomial

We say that λ is *parabolic* of type $\mu = (\mu_1, \ldots, \mu_k)$ (or $\mu-$*parabolic*) if $m = n$ and there exist positive integers $\mu_1, \ldots, \mu_k$ such that

$$M_\lambda^r = \left\{ \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & A_{kk} \end{bmatrix} \right\},$$

where A_{ii} is a $\mu_i \times \mu_i$ submatrix for $1 \le i \le k$. If λ is $\mu-$parabolic and $r = m$, then the invertible elements in M_λ^m form a parabolic subgroup of $GL_m(\mathbf{C})$.

A rook placement of rank r on a Ferrers board F_λ, where λ is $\mu-$parabolic, is said to be $\mu-$*parabolic*. Let $RL_\lambda^r(\gamma, \mu)$ be the set of all $\gamma-$compatible $\mu-$parabolic rook placements of rank r on the Ferrers board F_λ. The $\gamma-$*compatible* $\mu-$*parabolic rook length polynomial* is given by

$$RL_r(\lambda, \gamma, \mu, q) = \sum_{\sigma \in RL_\lambda^r(\gamma,\mu)} q^{l(\sigma)}.$$

We now give a formula for the $\gamma-$compatible $\mu-$parabolic rook length polynomial for $r = m$.

If $n_1 + n_2 + \cdots + n_t = n$, where $n, n_1, \ldots, n_t$ are positive integers, then the *Gausian multinomial coefficient* is given by

$$\begin{bmatrix} n \\ n_1, n_2, .., n_t \end{bmatrix}_q = \frac{[n]!_q}{[n_1]!_q\,[n_2]!_q \cdots [n_t]!_q}.$$

Theorem 9. *Suppose that* $r = m = n$ *and that the partition* $\lambda = (k_1^{\gamma_1}, \ldots, k_t^{\gamma_t})$, $k_1 \ge k_2 \ge \cdots \ge k_t > 0$, *is both* $\gamma = (\gamma_1, \ldots, \gamma_t)-$*compatible and parabolic of type* $\mu = (\mu_1, \ldots, \mu_l)$. *Clearly, there are integers* $0 = s_0 < s_1 < \cdots < s_l \le t$ *such that* $\gamma_{(s_i+1)} + \cdots + \gamma_{s_{(i+1)}} = \mu_{i+1}$, *for* $0 \le i \le l-1$, $k_{(s_i+1)} = \cdots = k_{s_{(i+1)}}$, *for* $0 \le i \le l-1$, *and* $\lambda = (k_1^{\mu_1}, \ldots, k_l^{\mu_l})$. *Then*

$$RL_m(\lambda, \gamma, \mu, q) = \prod_{i=0}^{l-1} \begin{bmatrix} \mu_{(i+1)} \\ \gamma_{(s_i+1)}, \gamma_{(s_i+2)}, \ldots, \gamma_{s_{(i+1)}} \end{bmatrix}_q.$$

Proof. We apply Theorem 6 with $r_i = \gamma_i$ for $1 \le i \le t$.

$$RL_r(\lambda, \gamma, \mu, q) = \sum\nolimits_{\sigma \in RL_\lambda^r(\gamma,\mu)} q^{l(\sigma)}$$

$$= \prod_{i=1}^{t} \begin{bmatrix} k_i - \gamma_{i+1} - \gamma_{i+2} - \cdots - \gamma_t \\ \gamma_i \end{bmatrix}_q$$

$$= \prod_{i=0}^{l-1} \prod_{j=(s_i+1)}^{s_{(i+1)}} \begin{bmatrix} k_j - \gamma_{j+1} - \gamma_{j+2} - \cdots - \gamma_t \\ \gamma_j \end{bmatrix}_q$$

$$= \prod_{i=0}^{l-1} \prod_{j=(s_i+1)}^{s_{(i+1)}} \begin{bmatrix} k_j - \gamma_{j+1} - \cdots - \gamma_{s_{(i+1)}} - (\mu_{i+2} + \cdots + \mu_l) \\ \gamma_j \end{bmatrix}_q$$

$$= \prod_{i=0}^{l-1} \prod_{j=(s_i+1)}^{s_{(i+1)}} \begin{bmatrix} k_j - (\mu_{i+2} + \cdots + \mu_l) - \gamma_{j+1} - \cdots - \gamma_{s_{(i+1)}} \\ \gamma_j \end{bmatrix}_q$$

$$= \prod_{i=0}^{l-1} \prod_{j=(s_i+1)}^{s_{(i+1)}} \begin{bmatrix} \mu_{i+1} - \gamma_{j+1} - \cdots - \gamma_{s_{(i+1)}} \\ \gamma_j \end{bmatrix}_q$$

$$= \prod_{i=0}^{l-1} \prod_{j=(s_i+1)}^{s_{(i+1)}} \begin{bmatrix} \gamma_{(s_i+1)} + \gamma_{(s_i+2)} + \cdots + \gamma_{s_{(i+1)}} - \gamma_{j+1} - \cdots - \gamma_{s_{(i+1)}} \\ \gamma_j \end{bmatrix}_q$$

$$= \prod_{i=0}^{l-1} \begin{bmatrix} \gamma_{(s_i+1)} \\ \gamma_{(s_i+1)} \end{bmatrix}_q \begin{bmatrix} \gamma_{(s_i+1)} + \gamma_{(s_i+2)} \\ \gamma_{(s_i+2)} \end{bmatrix}_q \cdots \begin{bmatrix} \gamma_{(s_i+1)} + \cdots + \gamma_{s_{(i+1)}} \\ \gamma_{s_{(i+1)}} \end{bmatrix}_q$$

$$= \prod_{i=0}^{l-1} \frac{\left[\gamma_{(s_i+1)} + \cdots + \gamma_{s_{(i+1)}}\right]_q!}{\left[\gamma_{(s_i+1)}\right]_q! \left[\gamma_{(s_i+2)}\right]_q! \cdots \left[\gamma_{s_{(i+1)}}\right]_q!}$$

$$= \prod_{i=0}^{l-1} \begin{bmatrix} \mu_{i+1} \\ \gamma_{(s_i+1)}, \gamma_{(s_i+2)}, \ldots, \gamma_{s_{(i+1)}} \end{bmatrix}_q .$$

Corollary 10. *Let* $\gamma = (1^m)$. *Then*

$$RL_m(\lambda, \mu, q) = RL_m(\lambda, (1^m), \mu, q) = \prod_{i=0}^{l-1} \begin{bmatrix} \mu_{i+1} \\ 1 \end{bmatrix}_q = \prod_{i=0}^{l-1} (\mu_{i+1})_q .$$

This result appears as Corollary 1.19 in [2].

Example. Let $\lambda = (6, 6, 4, 4, 4, 1)$, $\gamma = (1^6)$ and $\mu = (2, 3, 1)$. Then λ is $\gamma-$compatible and $\mu-$parabolic. Then we have

$$RL_6(\lambda, \mu, q) = (2)_q (3)_q (1)_q = q^3 + 2q^2 + 2q + 1.$$

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