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HARMONIOUS COLORING ON DOUBLE STAR GRAPH FAMILIES

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Abstract. In this paper we investigate the harmonious chromatic number for the central graph, middle graph, total graph and line graph of double star graph $K_{1,n,n}$ denoted by $C(K_{1,n,n})$, $M(K_{1,n,n})$, $T(K_{1,n,n})$ and $L(K_{1,n,n})$ respectively. We prove that for the line graph of double star graph, the harmonious chromatic number and the achromatic number are equal.

1. Preliminaries

The central graph [4] $C(G)$ of a graph G is formed by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The total graph [3] of G has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G.

The line graph [1, 3] of G denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $2n+1$ vertices and $2n$ edges. Let $V(K_{1,n,n}) = \{v\} \cup \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$ and $E(K_{1,n,n}) = \{e_1, e_2, \dots, e_n\} \cup \{s_1, s_2, s_3, \dots, s_n\}$.

A harmonious coloring of a simple graph G is proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number $\chi_H(G)$ is the least number of colors in such a coloring.

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An achromatic coloring of a graph G is a proper vertex coloring of G in which every pair of colors appears on at least one pair of adjacent vertices. The achromatic number of G denoted $\chi_c(G)$, is the greatest number of colors in an achromatic coloring of G.

A collection of articles in harmonious coloring and achromatic coloring can be found in the bibliography [2].

2. Harmonious Coloring on central graph of double star graph

Proposition 2.1. *For any double star graph $K_{1,n,n}$, the harmonious chromatic number,*

$$\chi_H(C(K_{1,n,n})) = 3n + 1.$$

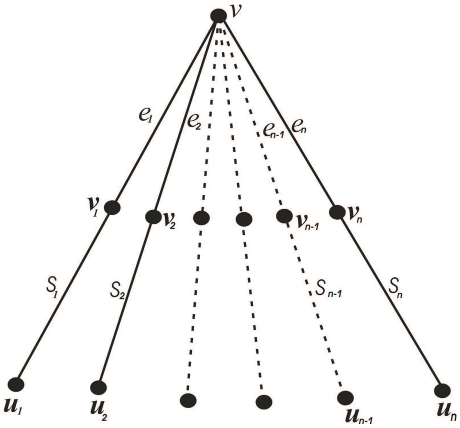

Figure 1: Double star graph $K_{1,n,n}$

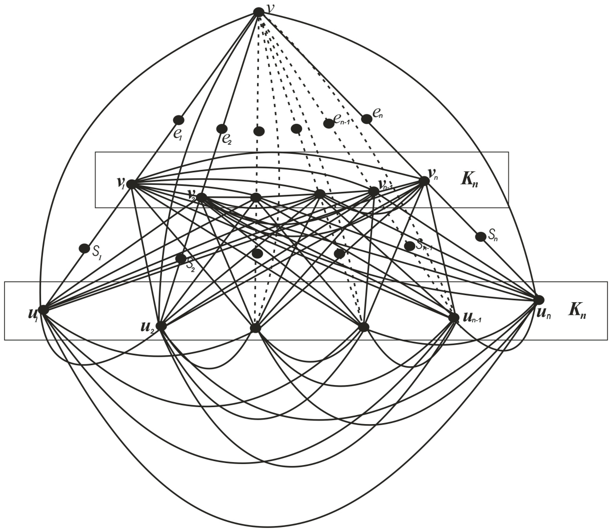

Figure 2: Central graph of double star graph $C(K_{1,n,n})$.

Proof. By definition of central graph, let the edge vv_i and v_iu_i $(1 \le i \le n)$ be subdivided by the vertices $e_i(1 \le i \le n)$ and $s_i(1 \le i \le n)$ in $C(K_{1,n,n})$. Clearly $V(C(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$. The vertices $v_i(1 \le i \le n)$ induce a clique of order n (say K_n) and the vertices $v, u_i(1 \le i \le n)$ induce a clique of order $n+1$ (say K_{n+1}) in $C(K_{1,n,n})$ respectively. The number of edges in $C(K_{1,n,n})$ is $3n + 2n^2$. Thus we have $\chi_H(C(K_{1,n,n})) \ge 3n + 1$.

All the vertices v, e_i, v_i, u_i are mutually at distance at most 2, and so must all have distinct colors and the vertices s_i could have the n colors used on e_i but in different order.

Now consider the vertex set $V(C(K_{1,n,n}))$ and the color class $C = \{c_i : 1 \le i \le 3n+1\}$, assign a proper harmonious coloring to $C(K_{1,n,n})$ as follows: For $(1 \le i \le n)$, assign the color c_i for u_i and assign the color c_{n+1} to v. For $(1 \le i \le n)$, assign the color c_{n+1+i} for e_i. For $(1 \le i \le n)$, assign the color c_{2n+1+i} for v_i. For $(1 \le i \le n-1)$, assign the color c_{2n+1} for s_i and assign the color c_{n+2} to s_n. Therefore, $\chi_H(C(K_{1,n,n})) \le 3n+1$. Hence $\chi_H(C(K_{1,n,n})) = 3n+1$. □

3. Harmonious coloring on middle graph and total graph of double star graph

Proposition 3.1. *For any double star graph $K_{1,n,n}$, the harmonious chromatic number,*

$$\chi_H(M(K_{1,n,n})) = 2n+1, \forall n \ge 3.$$

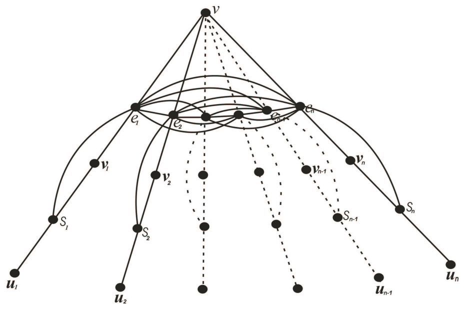

Figure 3: Middle graph of double star graph $M(K_{1,n,n})$.

Proof. By definition of middle graph, each edge vv_i and $v_iu_i(1 \le i \le n)$ in $K_{1,n,n}$ are subdivided by the vertices u_i and s_i in $M(K_{1,n,n})$ and the vertices $v, e_1, e_2, \ldots, e_n$ induce a clique of order $n+1$(say K_{n+1}) in $M(K_{1,n,n})$. i.e.,$V(M(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$. The number of edges in $M(K_{1,n,n})$ is $\dfrac{n^2+9n}{2}$. Thus we have $\chi_H(M(K_{1,n,n})) \ge 2n+1$.

The lower bound can be established from the fact that $2n+1$ vertices v, e_i, v_i are all mutually at distance at most 2. Hence the $2n+1$ vertices v, v_i, e_i require distinct colors. The vertices u_i can be colored with the same colors as e_i, but in a different order.

The following $(2n+1)$ - coloring for $M(K_{1,n,n})$ is harmonious: For $(1 \le i \le n)$, assign the color c_i for e_i and assign c_{n+1} to v. For $(1 \le i \le n)$, assign the color c_{n+1+i} for v_i. For $(1 \le i \le n-1)$, assign the color c_{n+2+i} for s_i and assign the color c_{n+2} to s_n. For $(1 \le i \le n-2)$, assign the color c_{i+2} for u_i and assign the colors c_1 to u_{n-1} and c_2 to u_n. Therefore, $\chi_H(M(K_{1,n,n})) \le 2n+1$. Hence $\chi_H(M(K_{1,n,n})) = 2n+1$. Note that $\chi_H(M(K_{1,1,1})) = 4$ and $\chi_H(M(K_{1,2,2})) = 6$. □

Proposition 3.2. *For any double star graph $K_{1,n,n}$, the harmonious chromatic number,*

$$\chi_H(T(K_{1,n,n})) = 2n+1.$$

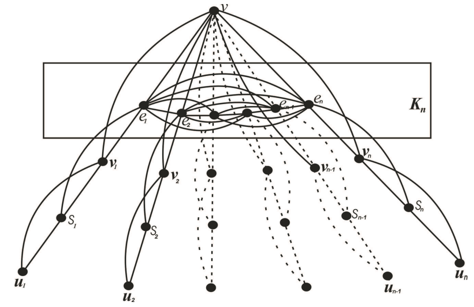

Figure 4: Total graph of double star graph $T(K_{1,n,n})$.

Proof. By the definition of total graph, we have $V(T(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$, in which the vertices v, e_1, e_2,..., e_n induce a clique of order $n+1$ (say K_{n+1}). The number of edges in $T(K_{1,n,n})$ is $\dfrac{n^2+13n}{2}$. Thus we have $\chi_H(T(K_{1,n,n})) \ge 2n+1$.

The lower bound can be established from the fact that $2n+1$ vertices v, e_i, v_i are all mutually at distance at most 2. The following $(2n+1)$ - coloring for $M(K_{1,n,n})$ is harmonious: For $(1 \le i \le n)$, assign the color c_i for e_i and assign c_{n+1} to v. For $(1 \le i \le n)$, assign the color c_{n+1+i} for v_i. For $(1 \le i \le n-1)$, assign the color c_{n+2+i} for s_i and assign the color c_{n+2} to s_n. For $(1 \le i \le n-2)$, assign the color c_{i+2} for u_i and assign the colors c_1 to u_{n-1} and c_2 to u_n. Therefore, $\chi_H(T(K_{1,n,n})) \le 2n+1$. Hence $\chi_H(T(K_{1,n,n})) = 2n+1$. □

4. Harmonious and achromatic coloring on line graph of double star graph

Proposition 4.1. *For any double star graph $K_{1,n,n}$, the harmonious chromatic number,*

$$\chi_H(L(K_{1,n,n})) = n+1.$$

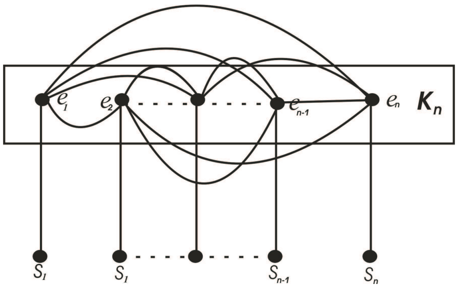

Figure 5: Line graph of double star graph $L(K_{1,n,n})$.

Proof. By the definition of line graph, each edge of $K_{1,n,n}$ taken to be as vertex in $L(K_{1,n,n})$. The vertices e_1, e_2,..., e_n induce a clique of order n in $L(K_{1,n,n})$.i.e., $V(L(K_{1,n,n})) = \{e_i : 1 \le i \le n\} \cup$

$\{s_i : 1 \le i \le n\}$. The number of edges in $L(K_{1,n,n})$ is $\dfrac{n^2+n}{2}$. Thus we have $\chi_H(L(K_{1,n,n})) \ge n+1$.

Now consider the vertex set $V(L(K_{1,n,n}))$ and color class $C = \{c_1, c_2, \dots, c_n, c_{n+1}\}$, assign a proper harmonious coloring to $L(K_{1,n,n})$ as follows: For $(1 \le i \le n)$, assign the color c_i for e_i. For $(1 \le i \le n)$, assign the color c_{n+1} to s_i. Therefore, $\chi_H(L(K_{1,n,n})) \le n+1$. Hence $\chi_H(L(K_{1,n,n})) = n+1$. □

Theorem 4.2 ([5])**.** *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c(L(K_{1,n,n})) = n+1.$$

Proof. As the number of edges in $L(K_{1,n,n})$ is $\dfrac{n^2+n}{2} < \dbinom{n+2}{2}$. Thus we have $\chi_c(L(K_{1,n,n})) \ge n+1$. Now consider the vertex set $V(L(K_{1,n,n}))$ and the color class $C = \{c_1, c_2, c_3, \dots c_n, c_{n+1}\}$, assign the achromatic coloring to $L(K_{1,n,n})$ as follows: For $(1 \le i \le n)$, assign the color c_i for e_i. For $(1 \le i \le n)$, assign the color c_{n+1} to s_i. Therefore, $\chi_c(L(K_{1,n,n})) \le n+1$. Hence $\chi_c(L(K_{1,n,n})) = n+1$. □

5. Main theorem

Now we characterize the graph for which the harmonious chromatic number and achromatic number are same. The proof of the Main Theorem follows from Proposition 4.1 and Theorem 4.2.

Theorem 5.1. *For any double star graph $K_{1,n,n}$,*

$$\chi_c(L(K_{1,n,n})) = \chi_H(L(K_{1,n,n})) = n+1.$$

6. Conclusion

In this present paper, we have proved for the line graph of double star graph, the harmonious chromatic number and the achromatic number are equal. As a motivation this work can be extended by classifying the different families of graphs for which these two numbers are equal.

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