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MULTI-DIMENSIONAL HADAMARD'S INEQUALITIES

YIN CHEN

Abstract. In this paper, Hadamard's inequalities are extended to a convex function on a convex set in $\mathbf{R}^2$ or $\mathbf{R}^3$. In particular, it is proved that the average of convex function on a disc of radius r is between the average of the function on the circle of radius r and that on the circle of $\frac{2r}{3}$.

1. Introduction

Let f be a convex function defined on $[a,b]$. Then we have the following inequalities, which are called Hermite-Hadamard's inequalities, or simply Hadamard's inequalities,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

We want to extend Hadamard's inequalities to a convex function of several variables. Recall that a function defined on a convex domain D of a vector space is convex if for any non-negative constant $\alpha \in [0,1]$ and any two points $x_1, x_2 \in D$, the following inequality holds

$$f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2).$$

Let us consider a function $f(x,y)$ on a convex subset D of $\mathbf{R}^2$. Note that if $f(x,y)$ is a convex function on D, then it is convex on any line segment in D and in particular, it is convex of both x and y.

Hadamard's inequalities deal with a convex function on $[a,b]$. It states that the average of a convex function f on $[a,b]$ is between the values of f at the midpoint $x = \frac{a+b}{2}$ and the average of the values of f at the endpoints a and b. Let us consider a convex function f on a disk. A nature question is to ask if the similar inequalities hold for the function f on an annulus (contained in the disk): $a \le r \le b$ where $r = \sqrt{x^2+y^2}$. We show that we do have such similar inequalities. A particular case of these inequalities improves an inequality of Hadamard's type on a disk obtained by Chen [2] or Dragomir [3]. We also get some similar inequalities of Hadamard's type for a convex function on a regular polygon. We extend our

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results then to a convex function on a ball of $\mathbf{R}^3$. The reader is referred to [5] and [6] for the original papers on Hadamard's inequality, and [1], [4], [7], [8], [9], [10] or [13] for some new developments on this topic.

2. Hadamard's inequality on an annulus

In this section, we will fix the following notations

$$D(r) = \{(x, y) : x^2 + y^2 \le r^2\}$$

and

$$L(r) = \{(x, y) : x^2 + y^2 = r^2\}.$$

We will always consider a convex function on a disk centred at $(0,0)$. It is easy to see that our inequalities will be true for a convex function on a disk centred at any point in $\mathbf{R}^2$. Let f be a convex function on $D(R)$. For $0 \le r \le R$, we define

$$F(r,\theta) = f(r\cos\theta, r\sin\theta), \tag{2}$$

$$M_f(r) = \sup\{f(x, y) : x^2 + y^2 = r^2\}, \tag{3}$$

$$C_f(r) = \frac{1}{2\pi r}\int_{L(r)} f(x, y)\, ds, \tag{4}$$

and

$$B_f(r) = \frac{1}{\pi r^2}\iint_{D(r)} f(x, y)\, dx dy. \tag{5}$$

Clearly

$$C_f(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)\, d\theta, \tag{6}$$

and

$$B_f(r) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} F(rt,\theta)\, t\, d\theta\, dt \tag{7}$$

$$= 2\int_0^1 C_f(rt)\, t\, dt. \tag{8}$$

The following theorem is an analogue of Theorem 2.6.8 of [11] for the convex functions.

Theorem 2.1. *Let f be a convex function on $D(R)$. Then the functions $M_f(r)$, $C_f(r)$ and $B_f(r)$ are all increasing convex functions of r on $[0,R]$.*

Proof. We first observe that $F(r,\theta)$ is a convex function of r on $[0,R]$ for any $\theta \in [0,2\pi]$. Thus for any nonnegative constants α_1 and α_2 with $\alpha_1 + \alpha_2 = 1$ and for any $r_1, r_2 \in [0,R]$, we have

$$F(\alpha_1 r_1 + \alpha_2 r_2,\theta) \le \alpha_1 F(r_1,\theta) + \alpha_2 F(r_2,\theta). \tag{9}$$

Taking the supremum for θ over $[0,2\pi]$ on both sides of the above inequality gives the convexity of $M_f(r)$. If we integrate both sides of (9) for θ over $[0,2\pi]$, and then divide both sides by 2π, we know that $C_f(r)$ is convex. If we replace r_1 and r_2 by $r_1 t$ and $r_2 t$ in (9), multiply $\frac{t}{\pi}$ on both sides, and then integrate both sides for t over $[0,1]$ and for θ over $[0,2\pi]$, we will have that $B_f(r)$ is also convex.

Let r_1, r_2 be two constants such that $0 \le r_1 \le r_2 \le R$. Note that

$$r_1 = \left(\frac{1}{2} + \frac{r_1}{2r_2}\right) r_2 + \left(\frac{1}{2} - \frac{r_1}{2r_2}\right)(-r_2). \tag{10}$$

Since the three points $(r_2,\theta), (r_1,\theta)$ and $(r_2,\theta+\pi)$ are on the same line and f is convex on the line, by (10), we have

$$F(r_1,\theta) \le \left(\frac{1}{2} + \frac{r_1}{2r_2}\right) F(r_2,\theta) + \left(\frac{1}{2} - \frac{r_1}{2r_2}\right) F(r_2,\theta+\pi). \tag{11}$$

Taking supremum on both sides of (11) for $\theta \in [0,2\pi]$ gives that

$$M_f(r_1) \le \left(\frac{1}{2} + \frac{r_1}{2r_2}\right) M_f(r_2) + \left(\frac{1}{2} - \frac{r_1}{2r_2}\right) M_f(r_2) = M_f(r_2).$$

So $M_f(r)$ is increasing.

If we take integral on both sides of (11) for $\theta \in [0,2\pi]$ and multiply both sides by $\frac{1}{2\pi}$, we have

$$C_f(r_1) \le \left(\frac{1}{2} + \frac{r_1}{2r_2}\right) C_f(r_2) + \left(\frac{1}{2} - \frac{r_1}{2r_2}\right) C_f(r_2) = C_f(r_2).$$

This shows that $C_f(r)$ are increasing on $[0,R]$.

If we replace r_1 and r_2 by tr_1 and tr_2 in (11), multiply both sides by $\frac{t}{\pi}$, and then do the integrals for t over $[0,1]$ and for θ over $[0,2\pi]$, we have

$$B_f(r_1) \le \left(\frac{1}{2} + \frac{r_1}{2r_2}\right) B_f(r_2) + \left(\frac{1}{2} - \frac{r_1}{2r_2}\right) B_f(r_2) = B_f(r_2).$$

This shows that $B_f(r)$ is also increasing. □

Now let us consider Hadamard's inequalities over $a \le r \le b$.

Theorem 2.2. *Let $f(x,y)$ be a convex function on the disk $D(R)$. If $0 \le a < b \le R$, then*

$$\frac{1}{\pi(b^2-a^2)} \iint_{a\le r\le b} f(x,y)\,dx\,dy \ge C_f\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$
$$\frac{1}{\pi(b^2-a^2)} \iint_{a\le r\le b} f(x,y)\,dx\,dy \le \frac{2a+b}{3(a+b)} C_f(a) + \frac{a+2b}{3(a+b)} C_f(b).$$

Proof. For any $\theta \in [0,2\pi]$, $F(r,\theta)$ is a convex function of r on $[0,R]$. So there is a constant $A(\theta)$ not depending on r such that

$$F(r,\theta) \ge F(r_0,\theta) + A(\theta)(r - r_0),$$

where $r_0 = \frac{2(a^2+ab+b^2)}{3(a+b)}$. Therefore

$$\begin{aligned}\iint_{a\le r\le b} f(x,y)\,dx\,dy &= \int_0^{2\pi}\int_a^b F(r,\theta) r\,dr\,d\theta \\ &\ge \int_0^{2\pi}\int_a^b F(r_0,\theta) r\,dr\,d\theta + \int_0^{2\pi}\int_a^b A(\theta)(r^2 - r_0 r)\,dr\,d\theta \\ &= \frac{b^2-a^2}{2}\int_0^{2\pi} F(r_0,\theta)\,d\theta + \int_0^{2\pi} 0\,d\theta \\ &= \pi(b^2-a^2)C_f(r_0).\end{aligned}$$

This proves the first inequality.

To show the second inequality, we first note that

$$F(r,\theta) \le \frac{b-r}{b-a}F(a,\theta) + \frac{r-a}{b-a}F(b,\theta),$$

for any $r : a \le r \le b$ and any $\theta \in [0,2\pi]$, since $F(r,\theta)$ is convex function of r. It follows that

$$\begin{aligned}&\frac{1}{\pi(b^2-a^2)}\iint_{a\le r\le b} f(x,y)\,dx\,dy \\ &\le \frac{1}{\pi(b^2-a^2)}\int_0^{2\pi}\int_a^b \left[\frac{b-r}{b-a} rF(a,\theta) + \frac{r-a}{b-a} rF(b,\theta)\right] dr\,d\theta \\ &\le \frac{2a+b}{3(a+b)}C_f(a) + \frac{a+2b}{3(a+b)}C_f(b).\end{aligned}$$

This completes the proof. □

Letting $a \to 0$, we have Hadamard's inequalities on a disk.

Corollary 2.3. *Let f be a convex function on $D(R)$ and $0 \le b < R$. Then*

$$C_f(\frac{2}{3}b) \le B_f(b) \le \frac{1}{3}f(0,0) + \frac{2}{3}C_f(b). \tag{12}$$

By Theorem (2.1), we know that $f(0,0) \le C_f(b)$, thus Corollary (2.3) implies and improves the following Hadamard's inequality on a disk, obtained by Dragomir [3] and also by Chen [2].

Corollary 2.4. *Let f be a convex function on $[0,R]$. Then*

$$f(0,0) \le C_f(\frac{2}{3}b) \le B_f(b) \le C_f(b) \le M_f(b). \tag{13}$$

Let $f(x,y)=\sqrt{x^2+y^2}$. Then f is a convex function on $\mathbf{R}^2$. It is easy to find $M_f(r)=r$, $C_f(r)=r$ and $B_f(r)=\frac{2}{3}r$. Therefore the constant $\frac{2}{3}$ on both sides of (12) is the best possible.

3. Hadamrad's inequalities and centroid

Let $(\bar{x},\bar{y})$ be the centroid of a convex set D in $\mathbf{R}^2$. Let us compare the average value of f on D with the value of f at the centriod.

Theorem 3.1. *Let f be a convex function on a convex region $D\subset\mathbf{R}^2$. Then*

$$f(\bar{x},\bar{y})\le\frac{1}{A(D)}\iint_D f(x,y)\,dx\,dy \tag{14}$$

where $(\bar{x},\bar{y})$ is the centroid of D and $A(D)$ is the area of D.

Proof. Since $(\bar{x},\bar{y})$ is the centroid of D, we have

$$\bar{x}=\frac{\iint_D x\,dx\,dy}{\iint_D dx\,dy},\quad \bar{y}=\frac{\iint_D y\,dx\,dy}{\iint_D dx\,dy}.$$

It follows that

$$\iint_D (x-\bar{x})dx\,dy=0,\quad \iint_D (y-\bar{y})dx\,dy=0. \tag{15}$$

f being a convex function on D implies that f has support at $(\bar{x},\bar{y})$ ([12], page 108), that is

$$f(x,y)\ge f(\bar{x},\bar{y})+A(x-\bar{x})+B(y-\bar{y})$$

for some constants A and B and for any $(x,y)\in D$. Therefore

$$\begin{aligned}\iint_D f(x,y)\,dx\,dy &\ge \iint_D (f(\bar{x},\bar{y})+A(x-\bar{x})+B(y-\bar{y}))\,dx\,dy\\ &=\iint_D f(\bar{x},\bar{y})\,dx\,dy \quad \text{(by (15))}\\ &= f(\bar{x},\bar{y})A(D),\end{aligned}$$

which completes the proof. □

4. Hadamrad's inequality on a regular polygon

We now give an application to Theorem (3.1).

We see in section 2 that the average of a convex function on a disk is between the average of the function on the boundary and the average of the function on a shrunk curve to $\frac{2}{3}$ size of the boundary. We will show that this is true too for a convex function on a regular polygon.

First we need a lemma for the convex function on a triangle. Let Δ be a triangle and A,B,C be three vertices. Take points E and F on line segments AB and AC respectively such that $|AE|=\frac{2}{3}|AB|$ and $|AF|=\frac{2}{3}|AC|$. Then

Lemma 4.1.

$$\frac{1}{A(\Delta)}\iint_{\Delta} f(x,y)\,dx\,dy \ge \frac{1}{|EF|}\int_{EF} f(x,y)\,ds \tag{16}$$

$$\frac{1}{A(\Delta)}\iint_{\Delta} f(x,y)\,dx\,dy \le \frac{1}{3}f(x_A,y_A) + \frac{2}{3|BC|}\int_{BC} f(x,y)\,ds \tag{17}$$

where (x_A, y_A) is the coordinates of point A, and $A(\Delta)$ is the area of Δ.

Proof. Without loss of generality, let us consider a triangle ABC such that B is at $(0,0)$, C is at $(p,0)$ with $p>0$ and A is at (x_A, y_A). Let n be a positive integer larger than 2. Let D_0 be the point B, D_n be the point C and D_i be the points $(\frac{i}{n}p, 0)$ for $i = 1,2,\dots,n-1$. Clearly for $i = 1,2,\dots,n$, the small triangle $AD_{i-1}D_i$ has the area $\frac{A(\Delta)}{n}$ and has the centroid $(\frac{2i-1}{3n}p + \frac{x_A}{3}, \frac{y_A}{3})$. Applying Theorem (3.1) on each triangle $AD_{i-1}D_i$, we have

$$\frac{n}{A(\Delta)}\iint_{\Delta(AD_{i-1}D_i)} f(x,y)\,dx\,dy \ge f\left(\frac{2i-1}{3n}p + \frac{x_A}{3}, \frac{y_A}{3}\right).$$

Adding both sides for $i = 1,2,\dots,n$, we have

$$\frac{n}{A(\Delta)}\iint_{\Delta} f(x,y)\,dx\,dy \ge \sum_{i=1}^{n} f\left(\frac{2i-1}{3n}p + \frac{x_A}{3}, \frac{y_A}{3}\right),$$

or

$$\begin{aligned}\frac{1}{A(\Delta)}\iint_{\Delta} f(x,y)\,dx\,dy &\ge \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{2i-1}{3n}p + \frac{x_A}{3}, \frac{y_A}{3}\right)\\ &= \frac{3}{2p}\sum_{i=1}^{n} \frac{2p}{3n} f\left(\frac{2i-1}{3n}p + \frac{x_A}{3}, \frac{y_A}{3}\right)\\ &= \frac{1}{|EF|}\sum_{i=1}^{n} \frac{2p}{3n} f\left(\frac{2i-1}{3n}p + \frac{x_A}{3}, \frac{y_A}{3}\right).\end{aligned}$$

As $n \to \infty$, clearly last sum goes to the integral of f on EF

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{|EF|}\sum_{i=1}^{n} \frac{2p}{3n} f\left(\frac{2i-1}{3n}p + \frac{x_A}{3}, \frac{y_A}{3}\right) &= \frac{1}{|EF|}\int_{\frac{x_A}{3}}^{\frac{x_A+2p}{3}} f(x, \frac{y_A}{3}))\,dx\\ &= \frac{1}{|EF|}\int_{EF} f(x,y)\,ds.\end{aligned}$$

This proves (16).

To prove (17), we will use a theorem in [1] that says the average of a convex function on a triangle is less than or equal to the average of the values of the function at the three vertices. Applying this theorem on each small triangle $AD_{i-1}D_i$ for $i = 1,2,\dots,n$, we have

$$\frac{n}{A(\Delta)}\iint_{\Delta(AD_{i-1}D_i)} f(x,y)\,dx\,dy \le \frac{f(x_A,y_A) + f(\frac{i-1}{n}p, 0) + f(\frac{i}{n}p,0)}{3}.$$

Adding all these inequalities yields

$$\frac{n}{A(\Delta)}\iint_{\Delta} f(x,y)\,dx\,dy \le \sum_{i=1}^{n}\frac{f(x_A,y_A)+f(\frac{i-1}{n}p,0)+f(\frac{i}{n}p,0)}{3},$$

or

$$\begin{aligned}\frac{1}{A(\Delta)}\iint_{\Delta} f(x,y)\,dx\,dy &\le \sum_{i=1}^{n}\frac{f(x_A,y_A)+f(\frac{i-1}{n}p,0)+f(\frac{i}{n}p,0)}{3n}\\ &= \frac{f(x_A,y_A)}{3}+\sum_{i=1}^{n}\frac{f(\frac{i-1}{n}p,0)+f(\frac{i}{n}p,0)}{3n}\\ &= \frac{f(x_A,y_A)}{3}+\frac{f(0,0)-f(p,0)}{3n}+\frac{2}{3p}\sum_{i=1}^{n}\frac{p}{n}f(\frac{i}{n}p,0).\end{aligned}$$

Clearly, as $n\to\infty$, the above sum has a limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}f(\frac{i}{n}p,0)=\int_0^p f(x,0)\,dx=\int_{BC} f(x,y)\,ds.$$

This proves (17). □

Now we will extend these inequalities to a convex function on a regular polygon.

Theorem 4.2. *Let f be convex on a regular n-gon P_n, $C(a,b)$ be the center of P_n, D_n be the boundary of P_n and E_n be the boundary of the n-gon whose vertices are the points on the line segments connecting C and the vertices of P_n and $\frac{2}{3}$ of the length of the segments from C. Then*

$$\frac{1}{A(P_n)}\iint_{P_n} f(x,y)\,dx\,dy \ge \frac{1}{|E_n|}\int_{E_n} f(x,y)\,ds \tag{18}$$

$$\frac{1}{A(P_n)}\iint_{P_n} f(x,y)\,dx\,dy \le \frac{1}{3}f(a,b)+\frac{2}{3|D_n|}\int_{D_n} f(x,y)\,ds \tag{19}$$

Proof. Divide the n-gon to n identical triangles and apply Lemma (4.1) to each triangle. □

If $n\to\infty$ in (18) and (19), then we can get another proof for Corollary (2.3).

5. Three dimensional Hadamard's inequalities

Let $B(r)$ and $S(r)$ be the ball and the surface of the ball of radius r in the space, that is,

$$B(r)=\{(x,y,z): x^2+y^2+z^2\le r^2\}$$

and

$$S(r)=\{(x,y,z): x^2+y^2+z^2=r^2\}.$$

Let f be a convex function on $D(R)$. For $0\le r\le R$, we define

$$F(\rho,\phi,\theta)=f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi),$$

$$M_f(r) = \sup\{f(x,y,z) : x^2 + y^2 + z^2 = r^2\},$$
$$Q_f(r) = \frac{1}{4\pi r^2} \iint_{S(r)} f(x,y,z)\, dS,$$

and

$$P_f(r) = \frac{3}{4\pi r^3} \iiint_{B(r)} f(x,y,z)\, dx dy dz.$$

Clearly

$$Q_f(r) = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} F(r,\phi,\theta) \sin\phi \, d\theta d\phi,$$

and

$$\begin{aligned} P_f(r) &= \frac{3}{4\pi} \int_0^1 \int_0^{\pi} \int_0^{2\pi} F(rt,\theta)\, t \sin\phi \, dt\, d\theta d\phi dt \\ &= 3 \int_0^1 Q_f(rt)\, t^2\, dt. \end{aligned}$$

By the similar way as in the proof of Theorem (2.1), we can easily prove the following theorem.

Theorem 5.1. *Let f be a convex function on $B(R)$. Then the functions $M_f(r), P_f(r)$ and $Q_f(r)$ are all increasing convex functions of r on $[0,R]$.*

Theorem 5.2. *Let f be a convex function on $B(R)$ and $0 \le b < R$. Then*

$$Q_f(\frac{3}{4}b) \le P_f(b) \le \frac{1}{4} f(0,0,0) + \frac{3}{4} Q_f(b).$$

Proof. Let $\rho_0 = \frac{3}{4}b$. Since $F(\rho,\phi,\theta)$ is convex of ρ on $[0,R]$, it follows that there is $A(\phi,\theta)$ such that

$$F(\rho,\phi,\theta) \ge F(\rho_0,\phi,\theta) + A(\phi,\theta)(\rho - \rho_0). \tag{20}$$

Clearly

$$\int_0^b \int_0^{\pi} \int_0^{2\pi} A(\phi,\theta)(\rho - \rho_0)\, \rho^2 \sin\phi \, d\theta\, d\phi\, d\rho = 0. \tag{21}$$

By (20) and (21), we will have

$$\begin{aligned} P_f(b) &= \frac{3}{4\pi b^3} \int_0^{\pi} \int_0^{2\pi} \int_0^b F(\rho,\phi,\theta)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \\ &\ge \frac{3}{4\pi b^3} \int_0^{\pi} \int_0^{2\pi} \int_0^b F(\rho_0,\phi,\theta) \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \\ &= \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} F(\rho_0,\phi,\theta) \sin\phi \, d\theta\, d\phi \\ &= Q_f(\rho_0). \end{aligned}$$

On the other hand, since

$$\rho \le (1-\frac{\rho}{b})\,0+\frac{\rho}{b}\,b,$$

we have

$$F(\rho,\phi,\theta) \le (1-\frac{\rho}{b})\,f(0,0,0)+\frac{\rho}{b}F(b,\phi,\theta).$$

Thus

$$\begin{aligned}
P_f(b) &\le \frac{3}{4\pi b^3}\int_0^{\pi}\int_0^{2\pi}\int_0^{b} F(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&\ge \frac{3}{4\pi b^3}\int_0^{\pi}\int_0^{2\pi}\int_0^{b}(1-\frac{\rho}{b})f(0,0,0)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&\quad+\frac{3}{4\pi b^3}\int_0^{\pi}\int_0^{2\pi}\int_0^{b}\frac{\rho}{b}F(b,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \frac{1}{4}f(0,0,0)+\frac{3}{4}Q_f(b).
\end{aligned}$$

Using the same way in the proof of Theorem (3.1), we can easily see the following theorem.

Theorem 5.3. *Let f be a convex function on a convex domain $D\subset \mathbf{R}^3$. Then*

$$f(\bar{x},\bar{y},\bar{z}) \le \frac{1}{V(D)}\iiint_D f(x,y,z)\,dx\,dy\,dz$$

where $(\bar{x},\bar{y},\bar{z})$ is the centroid of D and $V(D)$ is the volume of D.

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Department of Mathematical Sciences, Lakehead University, Thunder Bay, Ontario, Canada, P7B 5E1.

E-mail: yin.chen@lakeheadu.ca