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DOMINATING SETS IN CAYLEY GRAPHS ON Z_n

T. TAMIZH CHELVAM AND I. RANI

Abstract. A Cayley graph is a graph constructed out of a group Γ and its generating set A. In this paper we attempt to find dominating sets in Cayley graphs constructed out of Z_n. Actually we find the value of domination number for $Cay(Z_n, A)$ and a minimal dominating set when $|A|$ is even and further we have proved that $Cay(Z_n, A)$ is excellent. We have also shown that $Cay(Z_n, A)$ is 2−excellent, when $n = t(|A| + 1) + 1$ for some integer $t, t > 0$.

1. Introduction

Let Γ be a finite group with e as the identity. A generating set of the group Γ is a subset A such that every element of Γ can be expressed as the product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. The Cayley graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, y)_a | x, y \in V(G)$, there exists $a \in A$ such that $y = xa\}$ and it is denoted by $Cay(\Gamma, A)$. The exclusion of e from A eliminates the possibility of loops in the graph. The inclusion of the inverse in A for every element of A means that an edge is in the graph regardless of which end vertex is considered. For $x, y \in V(G)$, there exists $g \in \Gamma$ such that $y = xg$. One can express g as product of $a_1, a_2, \ldots, a_n \in A$. Then y and x are connected by a path through $a_1, a_2, \ldots, a_n \in A$. Hence G is connected and $|A|$ is the degree of $Cay(\Gamma, A)$.

Suppose G is a graph, the open neighbourhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of vertices adjacent to v. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood $N(S)$ is defined to be $\cup_{v \in S} N(v)$, and the closed neighbourhood of S is $N[S] = N(S) \cup S$[1]. A set $S \subseteq V$, of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S[1]. A dominating set S is a minimal dominating set if no proper subset is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G[1] and the corresponding dominating set is called a γ-set. A graph G is said to be excellent if each vertex u of G is contained in some γ-set of G. The graph G is said to be k-excellent, if every subset S of $V(G)$ with $|S| = k$ is contained in some γ-set of G.

Throughout this paper, n is a fixed positive integer, $Z_n = \{0, 1, 2, \ldots, n-1\}$ and $G = Cay(Z_n, A)$, where A is a generating set. Unless otherwise specified A stands for

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the set $\{1, n-1, 2, n-2, \dots, k, n-k\}$ where $1 \le k \le (n-1)/2$. Hereafter $+$ stands for modulo n addition in Z_n. We use the following theorem to find the value of $\gamma(G)$.

Theorem 1.1.([1]) *For any graph G with n vertices and maximum degree Δ, we have $\lceil n/(\Delta(G)+1)\rceil \le \gamma(G) \le n - \Delta(G)$.*

2. Main Results

Theorem 2.1. *Let $G = Cay(Z_n, A)$ where $A = \{1, n-1, 2, n-2, \dots, k, n-k\}$ and n, k are positive integers with $1 \le k \le (n-1)/2$. Then $\gamma(G) = \lceil n/(|A|+1)\rceil$.*

Proof. Note that $|A| = 2k$ and let $\ell = \lceil n/(|A|+1)\rceil = \lceil n/(2k+1)\rceil$. Consider the set $D = \{0, (2k+1), 2(2k+1), 3(2k+1), \dots, (\ell-1)(2k+1)\}$.

Claim: D is a dominating set of G.

It is always true that $\bigcup_{i=0}^{\ell-1} N[(2k+1)i] \subset V(G)$ and so it is enough to prove that $V(G) \subseteq \bigcup_{i=0}^{\ell-1} N[(2k+1)i]$. Let $c \in V(G)$. By division algorithm, one can write $c = (2k+1)i + h$ for some i and h satisfying $0 \le i \le (\ell-1)$ and $0 \le h \le 2k$. Note that, for any $j, N[(2k+1)j] = \{(2k+1)j, (2k+1)j+1, (2k+1)j+(n-1), (2k+1)j+2, (2k+1)j+(n-2), \dots, (2k+1)j+k, (2k+1)j+(n-k)\}$.

We have the following cases:

Case(i): Suppose $0 \le h \le k$, then it is easy to see that $c \in N[(2k+1)i]$.

Case(ii): Suppose $(k+1) \le h \le 2k$ and $0 \le i \le \ell-2$. In this case $c = (2k+1)i+h = (2k+1)(i+1) + (n-(2k-h+1))$. By the assumption on h, $1 \le 2k-h+1 \le k$. This implies that $c \in N[(2k+1)(i+1)]$.

Case(iii): Assume that $k+1 \le h \le 2k$ and $i = \ell-1$.

Subcase(i): Suppose $(2k+1)$ divides n. In this case $c = (2k+1)(\ell-1)+h = n-(2k-h+1) \in N[0]$.

Subcase(ii): When $(2k+1)$ does not divide n, we have $c = (2k+1)(\ell-1)+h = (2k+1)\ell + (n-(2k-h+1)) > (2k+1)(n/(2k+1)) + (n-(2k-h+1))$. That is, $c > n-(2k-h+1)$. Since $c \le n-1, n-(2k-h+1) < c \le (n-1)$. Suppose $k+1 \le h < 2k$, we have $(2k-h+1) \le k$ and hence $c \in N[0]$. When $h = 2k$, we get a contradiction.

In all the above cases, we have proved that if $c \in V(G)$, then $c \in N[(2k+1)i]$, for some $i, 0 \le i \le (\ell-1)$. Hence D is a dominating set of G and so $\gamma(G) \le |D| = \ell = \lceil n/(|A|+1)\rceil$. Since $|A| = 2k$, by Theorem 1.1 we have $\ell \le \gamma(G) \le (n-2k)$ and we get that $\gamma(G) = \ell = \lceil n/(|A|+1)\rceil$.

Remark 2.2. When $|A| = n-1$, then $G = Cay(Z_n, A)$ is complete and hence $\gamma(G) = 1$.

Remark 2.3. $|A|$ is odd. For example, consider $Cay(Z_8, A)$ where $|A| = 3$. Note that the only generating sets of cardinality 3 are $\{1,4,7\}$ and $\{3,4,5\}$. When $A = \{1,4,7\}$, $\gamma(Cay(Z_8, A)) = 3 = \lceil 8/(3+1)\rceil + 1$ and as well as when $A = \{3,4,5\}$, $\gamma(Cay(Z_8, A)) = 3 = \lceil 8/(3+1)\rceil + 1$.

Remark 2.4. $\gamma(Cay(Z_n, A))$ depends upon A, the generating set of Z_n. For example, consider $Cay(Z_{10}, A))$ where $|A| = 4$. It is obvious that $\{1, 2, 8, 9\}$ and $\{1, 4, 6, 9\}$ are generating sets. When $A = \{1, 2, 8, 9\}, \gamma(Cay(Z_{10}, A)) = 2$ and when $A = \{1, 4, 6, 9\}$, $\gamma(Cay(Z_{10}, A)) = 3$.

Remark 2.5. When n is even and $|A| \geq n/2$ then $\gamma(G) = \lceil n/(|A|+1)\rceil \leq \lceil n/((n/2)+1)\rceil \leq \lceil 2n/(n+2)\rceil \leq 2$. When n is odd and $|A| \geq \lfloor n/2\rfloor, \gamma(G) = \lceil n/(|A|+1)\rceil \leq 2$.

Corollary 2.6. *For $n \geq 3, 1 \leq \gamma(G) \leq \lceil n/3\rceil$.*

Proof. When $n \geq 3$, we have $2 \leq |A| \leq (n-1)$. When $|A| = n-1, \gamma(G) = 1$. Further, when $|A| = 2$, by the Theorem 2.1, we have $\gamma(G) = \lceil n/3\rceil$.

In the following Lemma, we identify another γ-set for $Cay(Z_n, A)$ apart from the one identified in the proof of Theorem 2.1.

Lemma 2.7. *For any fixed k, if $n = (2k+1)t + h, 1 \leq h \leq 2k$ and $\ell = \lceil n/(2k+1)\rceil$ then $D = \{0, h, h+2k+1, h+2(2k+1), \ldots, h+(\ell-2)(2k+1)\}$ is a γ-set for G.*

Proof. Note that any element v of $V(G)$, could be written as $v - h = (2k+1)i + r$ for some i and r with $0 \leq i \leq \ell - 1$ and $0 \leq r \leq 2k$.

The following cases arise:

Case (i). Suppose $v = h + (2k+1)i + r$, $0 \leq i \leq (\ell-2)$ and $0 \leq r \leq k$ then $v \in N[h + (2k+1)i]$. Further when $i = \ell - 1$ and $0 \leq r \leq k, v \in N[0]$.

Case (ii). Suppose $v = h + (2k+1)i + r$ with $0 \leq i \leq (\ell-1)$ and $k+1 \leq r \leq 2k$.

Subcase (i). When $0 \leq i \leq (\ell-3)$, we can write v as $v = h + (2k+1)(i+1) + (n - ((2k+1) - r))$ and hence $v \in N[h + (2k+1)(i+1)]$ with $1 \leq (i+1) \leq (\ell-2)$.

Subcase (ii). When $i = (\ell-2)$, v could be written as $v = h + (2k+1)(\ell-1) + (n - ((2k+1) - r)) = n - ((2k+1) - r)$ hence $v \in N[0]$.

Subcase (iii). Suppose $i = (\ell-1)$, we have $v = h+(2k+1)(\ell-1)+r = h+(n+(r-h))$, with $-k+1 \leq r-h \leq 2k-1$. When $-k+1 \leq r-h \leq k$, $v \in N[h]$. When $k+1 \leq r-h \leq 2k-1$, we have $-k \leq r-h-(2k+1) \leq -2$ and v could be written as $v = h + (2k+1) + (n + (r-h) - (2k+1))$ hence $v \in N[h + (2k+1)]$.

In all the cases $v \in N[D]$ and hence D is a γ-set.

Lemma 2.8. *Assume that $A_1 = \{1, n-1\}$. Suppose D is a γ-set for $G_1 = Cay(Z_n, A_1)$, then D is a dominating set for $G_2 = Cay(Z_n, A_2)$ with a generating set A_2 of the form $\{1, n-1, 2, n-2, \ldots\}$ and $|A_2| > 2$.*

Proof. Since $A_1 \subset A_2, G_1 = Cay(Z_n, A_1)$ is a spanning subgraph of $G_2 = Cay(Z_n, A_2)$. Hence D is a dominating set of G_2.

Theorem 2.9. *The Cayley graph $G = Cay(Z_n, A)$ is excellent.*

Proof. By the Theorem 2.1, we have $\gamma(G) = \lceil n/(|A|+1)\rceil = \lceil n/(2k+1)\rceil$ and $D = \{0, 2k+1, 2(2k+1), \ldots, (\ell-1)(2k+1)\}$ where $\ell = \lceil n/(2k+1)\rceil$ is a γ-set . Suppose

$c \in V(G)$, then $c = (2k+1)i + h$ for some i and h with $0 \le i \le (\ell - 1)$ and $0 \le h \le 2k$. Consider $D + h = \{h, (2k+1) + h, 2(2k+1) + h, \ldots, (\ell - 1)(2k+1) + h\}$. Clearly $D + h$ is a γ-set containing c and hence G is excellent.

Theorem 2.10. *Let $G = Cay(Z_n, A)$ with $A = \{1, n-1, 2, n-2, \ldots, k, n-k\}$. Suppose $n = (2k+1)t + 1$ for some positive integer t, then G is 2-excellent.*

Proof. Let $p = 2k+1$. Note that $|A| = 2k$. By the Theorem 2.1, $\gamma(G) = \lceil n/(|A|+1)\rceil = \lceil ((2k+1)t+1)/(2k+1)\rceil = t+1$. Let $x, y \in V(G)$. Without loss of generality one may assume that $x < y$. Let us find a γ-set containing x and y.

Case(a). Suppose $y - x \le p$. Let $D = \{x, y, y+p, \ldots, y+(t-1)p\}$. Suppose we prove that D is a dominating set, then $|D| = t+1$ ensures that D is a γ-set containing x and y. One can easily see that the elements of $V(G)$ are of the form $V(G) = \{y, y+1, \ldots, y+2k, y+p, y+p+1, \ldots y+p+2k, \ldots, y+(t-1)p, y+(t-1)p+1, \ldots, y+(t-1)p+2k, y+tp\}$. Note that $N[y+ip] = \{y+ip, y+ip+1, y+ip+(n-1), \ldots, y+ip+k, y+ip+(n-k)\}$. Let $v \in V(G)$.

Subcase(i). Suppose $v = y + ip + c$ for some i and c satisfying $0 \le i \le t-1$ and $0 \le c \le k$, then $v \in N[y+ip] \subseteq N[D]$.

Subcase(ii). Suppose $v = y + ip + c$ for some i and c with $0 \le i \le (t-2)$ and $k+1 \le c \le 2k$, then $v = y+(i+1)p-(p-c) = y+(i+1)p+(n-p+c) = y+(i+1)p+n+c'$ where $c' = c-p$ and $-k \le c' \le -1$. Thus $v \in N[y+(i+1)p] \subseteq N[D]$.

Subcase(iii). When $v = y + (t-1)p + (k+1)$, then $v = y + n - 1 - k \le x + (2k+1) + n - (k+1) = x + k$. Therefore $v \le x + k$. Since $x < y$ and $y < v$, we have $v > x$. Hence $v \in N[x] \subseteq N[D]$.

Subcase(iv). Suppose $v = y + (t-1)p + c$ for some c, with $k+2 \le c \le 2k$, then $2 \le p - c + 1 \le k$. Now $v = y + (n-1) - p + c = y + n - (p-c+1) \in N[y] \subseteq N[D]$.

Subcase(v). Suppose $v = y + tp$ then $v = y + (n-1) \in N[y] \subseteq N[D]$.

Hence in this case $V = N[D]$.

Case (b). Assume that $y - x > p$ and $y - x$ is divisible by p. When this is the case $y = x + hp$ for some h. Consider $D = \{x, x+p, \ldots, x+hp, x+(h+1)p, x+(h+2)p, \ldots, x+tp\}$. As in Case(a), let us prove that D is a dominating set. Elements of $V(G)$ can be listed out as $\{x, x+1, \ldots, x+2k, x+p, x+p+1, \ldots, x+p+2k, \ldots, x+hp, x+hp+1, \ldots, x+hp+2k, x+(h+1)p, x+(h+1)p+1, \ldots, x+(h+1)p+2k, \ldots, x+(t-1)p, x+(t-1)p+1, \ldots, x+(t-1)p+2k, x+tp\}$. Let $v \in V(G)$.

Subcase(i). Suppose $v = x+ip+c$ for some i and c satisfying $0 \le i \le t$ and $0 \le c \le k$. In this case $v \in N[x+ip] \subseteq N[D]$.

Subcase(ii). Suppose $v = x + ip + c$ where $0 \le i \le (t-1)$ and $k+1 \le c \le 2k$ then $v = x + (i+1)p + (n+c-p) = x + (i+1)p + (n+c')$ where $c' = c - p$ and $-k \le c' \le -1$. Thus $v \in N[x+(i+1)p] \subseteq N[D]$ for $0 \le i \le (t-1)$.

Case(c). Suppose $y - x > p$ and $y - x$ is not divisible by p, then $y = x + hp + r, 1 \le r \le p-1$. Consider $D = \{x, x+p, x+2p, \ldots, x+hp, y, y+p, y+2p, \ldots, y+(t-h-1)p\}$.

As in the Cases (a) and (b) , let us prove that D is a dominating set. Elements of $V(G)$ can be listed out as $\{x, x+1, \ldots, x+2k, x+p, x+p+1, \ldots, x+p+2k, \ldots, x+hp, \ldots, x+hp+2k, x+(h+1)p, x+(h+1)p+1, \ldots, x+(h+1)p+2k, \ldots, x+(t-1)p, x+(t-1)p+1, \ldots x+(t-1)p+2k, x+tp\}$. Let $v \in V(G)$.

Subcase(i). Suppose $v = x + ip + c$ for some i and c satisfying $0 \leq i \leq h$ and $0 \leq c \leq k$. In this case $v \in N[x + ip] \subseteq N[D]$.

Subcase(ii). Suppose $v = x + ip + c$ where $0 \leq i \leq h - 1$ and $k + 1 \leq c \leq 2k$ then $v = x + ip + c = x + (i + 1)p + (n + c - p) \in N[x + (i + 1)p] \subseteq N[D]$ for $0 \leq i \leq h - 1$.

Subcase(iii). When $v = x + hp + c$ for some c satisfying $k + 1 \leq c \leq 2k$, then $-(k-1) \leq c-r \leq 2k-1$ and $v = y+(c-r)$. When $-(k-1) \leq c-r \leq k, v \in N[y]$ and when $k+1 \leq c-r \leq 2k-1$, then $v = y+p+(n+(c-r-p))$ and $-k \leq c-r-p \leq -2$ implies $v \in N[y + p] \subseteq N[D]$.

Subcase(iv). Suppose $v = x + ip + c$, $h + 1 \leq i \leq t - 1$, $0 \leq c \leq k$, then $-2k \leq c - r \leq k - 1$. When $-2k \leq c - r \leq -(k + 1)$, $v \in N[y + (i - h - 1)p] \subset N[D]$. When $-k \leq c - r \leq k - 1$, $v \in N[y + (i - h)p] \subseteq N[D]$.

Subcase(v). Suppose $v = x + ip + c$, $h + 1 \leq i \leq t - 1, k + 1 \leq c \leq 2k$, then $-(k - 1) \leq c - r \leq 2k - 1$. When $-(k - 1) \leq c - r \leq k, v \in N[y + (i - h)p] \subseteq N[D]$. When $k + 1 \leq c - r \leq 2k - 1$ then $v \in N[y + (i - h + 1)p] \subseteq N[D]$.

Subcase(vi). Suppose $v = x + tp = x + (n - 1) \in N[x] \subseteq N[D]$.

In all the cases D is a dominating set.

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Department of Mathematics, Manonmaniam Sundaranar University Tirunelveli 627 012 Tamil Nadu, India.

E-mail: tamche_59@yahoo.co.in

Department of Mathematics, Manonmaniam Sundaranar University Tirunelveli 627 012, Tamil Nadu, India.

E-mail: irvigna@rediffmail.com