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SOME INEQUALITIES OF OSTROWSKI AND GRÜSS TYPE FOR TRIPLE INTEGRALS ON TIME SCALES

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Abstract. In this paper, we establish some inequalities of Ostrowski and Grüss type for triple integrals on arbitrary time scales involving three functions and their partial derivatives. We also discuss the discrete Ostrowski and Grüss type inequalities for triple sum on time scale.

1. Introduction

In 1938, A. Ostrowski [16] has proved the following inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a, b) and Let, on (a, b), $\left|\acute{f}(x)\right| \leq M$. Then, for every $x \in [a, b]$,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|\acute{f}\right\|_\infty,$$

the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

One of the most important integral inequalities proved by Gerhard Grüss [8] in 1935 is the Grüss Integral Inequality, which gives estimation for the integral of a product in terms of product of integrals and is defined as:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}\left(J-j\right)(N-n), \tag{1.1}$$

provided that f and g are two integrable functions on $[a, b]$ and satisfy the condition $j \leq f(x) \leq J$, $n \leq g(x) \leq N$, for all $x \in [a, b]$, where j, J, n, N are given real constants.

B. G. Pachpatte [18] has given some inequalities of Grüss type involving functions of two independent variables. Later on, in [19], he established some new inequalities for triple integrals involving three functions and their partial derivatives. The main aim of this paper is to establish the time scale version of these inequalities.

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2. Time scale essentials

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the set of real numbers. Some important examples of time scales are, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_\circ$.

If $\mathbb{T}$ is a time scale, then for $t \in \mathbb{T}$, we define the forward and backward jump operators respectively as $\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

A point t is said to be right-scattered if $\sigma(t) > t$ and is left-scattered if $\rho(t) < t$. A point that is at the same time right-scattered as well as left-scattered is called an isolated point.

The point t is called right-dense if $\sigma(t) = t$. If $\rho(t) = t$, then t is said to be left-dense. If $\rho(t) = t = \sigma(t)$, then the point t is called dense.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous (denoted by C_{rd}) if it is continuous at each right-dense point or maximal point of $\mathbb{T}$, and its left-sided limit $\lim_{s \to t} f(s) = f(t^-)$ exists at left dense points of $\mathbb{T}$.

Let $t \in \mathbb{T}$, then two functions $\Phi, \Psi : \mathbb{T} \to [0, +\infty)$ satisfying $\Phi(t) = \sigma(t) - t$, $\Psi(t) = t - \rho(t)$ are called graininess functions.

If $\mathbb{T}$ has a left-scattered maximal point t, then $\mathbb{T}^K = \mathbb{T} - \sigma(t)$, otherwise $\mathbb{T}^K = \mathbb{T}$.A function $g^\sigma : \mathbb{T} \to \mathbb{R}$ is defined as $g^\sigma(t) = g(\sigma(t))$ for all $t \in \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a function on time scale. Then, for $s \in \mathbb{T}^K$, we define $f^\Delta(s)$ to be the number, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood N of s such that for all $u \in N$,

$$\left| f^\sigma(s) - f(u) - f^\Delta(s)(\sigma(s) - u) \right| \le \varepsilon |\sigma(s) - u|.$$

In this case, $f^\Delta(s)$ is called the delta derivative of f at s. f is said to be delta differentiable on $\mathbb{T}$, if f is differentiable at each $s \in \mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is said to be $\Delta-$derivative for all $f : \mathbb{T} \to \mathbb{R}$ if, $F^\Delta(t) = f(t)$ $t \in \mathbb{T}^K$, and in this case, we define the Δintegrable of f

$$\int_a^b f(t)\,\Delta t = F(b) - F(a),$$

for each $a, b \in \mathbb{T}$.

Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two time scales. Let σ_i, ρ_i and Δ_i be the forward jump operator, the backward jump operator and the delta differentiation, respectively on $\mathbb{T}_i$, for i =1,2. Let $a, b \in \mathbb{T}_1$, $c, d \in \mathbb{T}_2$, with $a < b$, $c < d$. $[a, b)$ and $[c, d)$ are the half-closed bounded intervals in $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively. Let us introduce a "rectangle" in $\mathbb{T}_1 \times \mathbb{T}_2$ by

$$R = [a, b) \times [c, d) = \{(t_1, t_2) : t_1 \in [a, b), t_2 \in [c, d)\}.$$

Let f be a real valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. This function f is said to be rd-continuous in t_2 if $a_1 \in \mathbb{T}_1$, then function $f(a_1, t_2)$ is rd-continuous on $\mathbb{T}_2$, and this function f is said to be rd-continuouse in t_1 if $a_2 \in \mathbb{T}_2$, then $f(t_1, a_2)$ is rd-continuous on $\mathbb{T}_1$.

CC_{rd} denotes the set of functions $f(t_1, t_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ having the properties

(i) f is rd-continuouse in t_1 and t_2.

(ii) if $(x_1, x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_1 right dense and x_2 right dense, then f is continuous at (x_1, x_2).

(iii) if x_1 and x_2 are left dense limits, then the limit of $f(t_1, t_2)$ exists as (t_1, t_2)approaches to (x_1, x_2) along any path in the region:

$$R_{LL}(x_1, x_2) = \{(t_1, t_2) : t_1 \in [a, x_1] \cap \mathbb{T}_1, [c, x_2] \cap \mathbb{T}_2\}.$$

Let CC_{rd}^1 denote the set of all functions in CC_{rd} for which both the Δ_1 partial derivative and Δ_2 partial derivative exist and are in CC_{rd}. In [3] Bohner has defined the norm as

$$\|f\| = \sup_{(x,y)\in[a,b]\times[c,d]} \left|f(x,y)\right| + \sup_{(x,y)\in R} \left|\frac{\partial f(x,\sigma_2(y))}{\Delta_1 x}\right| + \sup_{(x,y)\in R} \left|\frac{\partial f(\sigma_1(x), y)}{\Delta_2 x}\right|$$

where $f \in CC_{rd}^1([a,b] \times [c,d], \mathbb{R})$.

Let Φ, Ψ be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$, then

(i) $\displaystyle\int_a^b \left(\alpha\Phi(t) + \beta\Psi(t)\right)\Delta t = \alpha\int_a^b \Phi(t)\Delta t + \beta\int_a^b \Psi(t)\Delta t;$

(ii) $\displaystyle\int_a^b \Phi(t)\Delta t = -\int_b^a \Phi(t)\Delta t;$

(iii) $\displaystyle\int_a^b \Phi(t)\Delta t = \int_a^c \Phi(t)\Delta t + \int_c^b \Phi(t)\Delta t;$

(iv) $\displaystyle\int_a^b \Phi(\sigma(t))\Psi^{\Delta}(t)\Delta t = (\Phi\Psi)(b) - (\Phi\Psi)(a) - \int_a^b \Phi^{\Delta}(t)\Psi(t)\Delta t;$

(v) $\displaystyle\int_a^b \Phi(t)\Psi^{\Delta}(t)\Delta t = (\Phi\Psi)(b) - (\Phi\Psi)(a) - \int_a^b \Phi^{\Delta}(t)\Psi(\sigma(t))\Delta t;$

(vi) $\displaystyle\int_a^a \Phi(t)\Delta t = 0.$

The weight function $\eta\colon [a,b] \to [0,\infty)$ is a non-negative integrable function and

$$\int_a^b \eta(t)\,dt < \infty.$$

3. Main results

Here, we give some notations used to simplify the theorems:.

$$P[f] = (\beta_2 - \alpha_2)(\beta_3 - \alpha_3)\int_{\alpha_1}^{\beta_1} f(\sigma_1(r), y, z)\Delta_1 r$$

$$
\begin{aligned}
&+\left(\beta_1-\alpha_1\right)\left(\beta_3-\alpha_3\right)\int_{\alpha_2}^{\beta_2} f\left(x,\sigma_2\left(s\right),z\right)\Delta_2 s\\
&+\left(\beta_1-\alpha_1\right)\left(\beta_2-\alpha_2\right)\int_{\alpha_3}^{\beta_3} f\left(x,y,\sigma_3\left(t\right)\right)\Delta_3 t,\\
Q\left[f\right] =& \left(\beta_3-\alpha_3\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} f\left(\sigma_1\left(r\right),\sigma_2\left(s\right),z\right)\Delta_2 s\Delta_1 r\\
&+\left(\beta_2-\alpha_2\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_3}^{\beta_3} f\left(\sigma_1\left(r\right),y,\sigma_3\left(t\right)\right)\Delta_3 t\Delta_1 r\\
&+\left(\beta_1-\alpha_1\right)\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} f\left(x,\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s,\\
L\left[f\right] =& \int_{\sigma_1(r)}^{x}\int_{\sigma_2(s)}^{y}\int_{\sigma_3(t)}^{z}\frac{\partial^3 f\left(u,v,w\right)}{\Delta_3 w\Delta_2 v\Delta_1 u}\Delta_3 w\Delta_2 v\Delta_1 u,\\
K =& \left(\beta_1-\alpha_1\right)\left(\beta_2-\alpha_2\right)\left(\beta_3-\alpha_3\right),
\end{aligned}
$$

$$
\begin{aligned}
A\left(\Phi,\Psi,\eta;P,Q;K\right)\left(x,y,z\right) =& \Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\\
&-\frac{1}{3K}\left[\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\left\{P\left[\Phi\right]-Q\left[\Phi\right]\right.\right.\\
&\left.+\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\right\}\\
&\times\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\left\{P\left[\Psi\right]-Q\left[\Psi\right]\right.\\
&\left.\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\right\}\\
&\times\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\left\{P\left[\eta\right]-Q\left[\eta\right]\right.\\
&\left.\left.+\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
B\left(\Phi,\Psi,\eta;L\right)\left(x,y,z\right) =& \Psi\left(x,y,z\right)\eta\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L\left[\Phi\right]\Delta_3 t\Delta_2 s\Delta_1 r\\
&+\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L\left[\Psi\right]\Delta_3 t\Delta_2 s\Delta_1 r\\
&+\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L\left[\eta\right]\Delta_3 t\Delta_2 s\Delta_1 r,
\end{aligned}
$$

$$
\begin{aligned}
Z\left(\Phi,\Psi,\eta;P,Q;K\right) =& \frac{1}{K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\Delta_3 z\Delta_2 y\Delta_1 x\\
&-\frac{1}{3K^2}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\left\{\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\left(P\left[\Phi\right]-Q\left[\Phi\right]\right)\right.\\
&+\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\left(P\left[\Psi\right]-Q\left[\Psi\right]\right)\\
&\left.+\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\left(P\left[\eta\right]-Q\left[\eta\right]\right)\right\}\Delta_3 z\Delta_2 y\Delta_1 x\\
&-\frac{1}{3}\left[\left(\frac{1}{k}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\Delta_3 z\Delta_2 y\Delta_1 x\right)\right.
\end{aligned}
$$

$$\times\left(\frac{1}{k}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\Delta_3 t\Delta_2 s\Delta_1 r\right)$$
$$+\left(\frac{1}{k}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\eta(x,y,z)\Phi(x,y,z)\Delta_3 z\Delta_2 y\Delta_1 x\right)$$
$$\times\left(\frac{1}{k}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\Delta_3 t\Delta_2 s\Delta_1 r\right)$$
$$+\left(\frac{1}{k}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi(x,y,z)\Psi(x,y,z)\Delta_3 z\Delta_2 y\Delta_1 x\right)$$
$$\times\left(\frac{1}{k}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\eta(\sigma_1(r),\sigma_2(s),\sigma_3(t))\Delta_3 t\Delta_2 s\Delta_1 r\right).$$

Theorem 3.1. *Let $\mathbb{T}_1,\mathbb{T}_2$ and $\mathbb{T}_3$ be arbitrary three time scales and Δ_1,Δ_2 and Δ_3 be the delta differentiation operators respectively on $\mathbb{T}_1,\mathbb{T}_2$and $\mathbb{T}_3$.If $[\alpha_i,\beta_i]\subseteq\mathbb{T}_1$, for $i=1,2,3$ and $M=\prod_{i=1}^{3}[\alpha_i,\beta_i]$ then for $(x,y,z),(r,s,t)\in M$ and for some continuous functions $\Phi,\Psi,\eta\in CC_{rd}^{1}$ $[M,\mathbb{R}]$, then the following inequality holds:*

$$\left|A(\Phi,\Psi,\eta;P,Q;K)(x,y,z)\right|\leq\frac{1}{3K}B\left(|\Phi|,|\Psi|,|\eta|;|L|\right)(x,y,z),$$

for all $(x,y,z)\in M$.

Proof. Consider

$$\begin{aligned}
L[\Phi] &= \int_{\sigma_1(r)}^{x}\int_{\sigma_2(s)}^{y}\int_{\sigma_3(t)}^{z}\frac{\partial^3\Phi(u,v,w)}{\Delta_3 w\Delta_2 v\Delta_1 u}\Delta_3 w\Delta_2 v\Delta_1 u\\
&= \int_{\sigma_1(r)}^{x}\int_{\sigma_2(s)}^{y}\left[\frac{\partial^2\Phi(u,v,w)}{\Delta_2 v\Delta_1 u}|_{\sigma_3(t)}^{z}\right]\Delta_2 v\Delta_1 u\\
&= \int_{\sigma_1(r)}^{x}\int_{\sigma_2(s)}^{y}\left[\frac{\partial^2\Phi(u,v,z)}{\Delta_2 v\Delta_1 u}-\frac{\partial^2\Phi(u,v,\sigma_3(t))}{\Delta_2 v\Delta_1 u}\right]\Delta_2 v\Delta_1 u\\
&= \int_{\sigma_1(r)}^{x}\left[\frac{\partial\Phi(u,v,z)}{\Delta_1 u}|_{\sigma_2(s)}^{y}-\frac{\partial\Phi(u,v,\sigma_3(t))}{\Delta_1 u}|_{\sigma_2(s)}^{y}\right]\Delta_1 u\\
&= \int_{\sigma_1(r)}^{x}\left[\frac{\partial\Phi(u,y,z)}{\Delta_1 u}-\frac{\partial\Phi(u,\sigma_2(s),z)}{\Delta_1 u}\right]\Delta_1 u\\
&\quad -\int_{\sigma_1(r)}^{x}\left[\frac{\partial\Phi(u,y,\sigma_3(t))}{\Delta_1 u}-\frac{\partial\Phi(u,\sigma_2(s),\sigma_3(t))}{\Delta_1 u}\right]\Delta_1 u\\
&= \Phi(u,y,z)|_{\sigma_1(r)}^{x}-\Phi(u,\sigma_2(s),z)|_{\sigma_1(r)}^{x}\\
&\quad -\Phi(u,y,\sigma_3(t))|_{\sigma_1(r)}^{x}+\Phi(u,\sigma_2(s),\sigma_3(t))|_{\sigma_1(r)}^{x}\\
&= \Phi(x,y,z)-\Phi(\sigma_1(r),y,z)-\Phi(x,\sigma_2(s),z)\\
&\quad +\Phi(\sigma_1(r),\sigma_2(s),z)-\Phi(x,y,\sigma_3(t))+\Phi(\sigma_1(r),y,\sigma_3(t))\\
&\quad +\Phi(x,\sigma_2(s),\sigma_3(t))-\Phi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\\
&= \Phi(x,y,z)-\left[\Phi(\sigma_1(r),y,z)+\Phi(x,\sigma_2(s),z)+\Phi(x,y,\sigma_3(t))\right]
\end{aligned}$$

$$+\left[\Phi(\sigma_1(r),\sigma_2(s),z)+\Phi\left(\sigma_1(r),y,\sigma_3(t)\right)+\Phi(x,\sigma_2(s),\sigma_3(t))\right]$$
$$-\Phi(\sigma_1(r),\sigma_2(s),\sigma_3(t)). \tag{3.1}$$

Similarly,

$$\begin{aligned} L[\Psi] = {} & \Psi(x,y,z)-\left[\Psi\left(\sigma_1(r),y,z\right)+\Psi(x,\sigma_2(s),z)+\Psi\left(x,y,\sigma_3(t)\right)\right] \\ & +\left[\Psi(\sigma_1(r),\sigma_2(s),z)+\Psi\left(\sigma_1(r),y,\sigma_3(t)\right)+\Psi(x,\sigma_2(s),\sigma_3(t))\right] \\ & -\Psi(\sigma_1(r),\sigma_2(s),\sigma_3(t)) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} L[\eta] = {} & \eta(x,y,z)-\left[\eta\left(\sigma_1(r),y,z\right)+\eta(x,\sigma_2(s),z)+\eta\left(x,y,\sigma_3(t)\right)\right] \\ & +\left[\eta(\sigma_1(r),\sigma_2(s),z)+\eta\left(\sigma_1(r),y,\sigma_3(t)\right)+\eta(x,\sigma_2(s),\sigma_3(t))\right] \\ & -\eta(\sigma_1(r),\sigma_2(s),\sigma_3(t)). \end{aligned} \tag{3.3}$$

Multiplying both sides of (3.1), (3.2) and (3.3) with $\Psi(x,y,z)\eta(x,y,z)$, $\eta(x,y,z)\Phi(x,y,z)$ and $\Phi(x,y,z)\Psi(x,y,z)$ respectively and then adding, we have:

$$\begin{aligned} & 3\Phi(x,y,z)\Psi(x,y,z)\eta(x,y,z) \\ & -\Psi(x,y,z)\eta(x,y,z)\left\{\left[\Phi\left(\sigma_1(r),y,z\right)+\Phi(x,\sigma_2(s),z)+\Phi\left(x,y,\sigma_3(t)\right)\right]\right. \\ & -\left[\Phi(\sigma_1(r),\sigma_2(s),z)+\Phi\left(\sigma_1(r),y,\sigma_3(t)\right)+\Phi(x,\sigma_2(s),\sigma_3(t))\right] \\ & \left.-\Phi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\right\} \\ & -\eta(x,y,z)\Phi(x,y,z)\left\{\left[\Psi\left(\sigma_1(r),y,z\right)+\Psi(x,\sigma_2(s),z)+\Psi\left(x,y,\sigma_3(t)\right)\right]\right. \\ & -\left[\Psi(\sigma_1(r),\sigma_2(s),z)+\Psi\left(\sigma_1(r),y,\sigma_3(t)\right)+\Psi(x,\sigma_2(s),\sigma_3(t))\right] \\ & \left.-\Psi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\right\} \\ & -\Phi(x,y,z)\Psi(x,y,z)\left\{\left[\eta\left(\sigma_1(r),y,z\right)+\eta(x,\sigma_2(s),z)+\eta\left(x,y,\sigma_3(t)\right)\right]\right. \\ & -\left[\eta(\sigma_1(r),\sigma_2(s),z)+\eta\left(\sigma_1(r),y,\sigma_3(t)\right)+\eta(x,\sigma_2(s),\sigma_3(t))\right] \\ & \left.-\eta(\sigma_1(r),\sigma_2(s),\sigma_3(t))\right\} \\ = {} & \Psi(x,y,z)\eta(x,y,z)L[\Phi]+\eta(x,y,z)\Phi(x,y,z)L[\Psi]+\Phi(x,y,z)\Psi(x,y,z)L[\eta]. \end{aligned}$$

Integrating both sides of the above equation over R with respect to $\sigma_1(r), \sigma_2(s)$ and $\sigma_3(t)$, we have

$$\begin{aligned} & 3\left(\beta_1-\alpha_1\right)\left(\beta_2-\alpha_2\right)\left(\beta_3-\alpha_3\right)\Phi(x,y,z)\Psi(x,y,z)\eta(x,y,z) \\ & -\Psi(x,y,z)\eta(x,y,z)\left\{\left[\left(\beta_2-\alpha_2\right)\left(\beta_3-\alpha_3\right)\int_{\alpha_1}^{\beta_1}\Phi\left(\sigma_1(r),y,z\right)\Delta_1 r\right.\right. \\ & +\left(\beta_1-\alpha_1\right)\left(\beta_3-\alpha_3\right)\int_{\alpha_2}^{\beta_2}\Phi(x,\sigma_2(s),z)\Delta_2 s \end{aligned}$$

$$+\left(\beta_1-\alpha_1\right)\left(\beta_2-\alpha_2\right)\int_{\alpha_3}^{\beta_3}\Phi\left(x,y,\sigma_3\left(t\right)\right)\Delta_3 t\Big]$$

$$-\Big[\left(\beta_3-\alpha_3\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\Phi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),z\right)\Delta_2 s\Delta_1 r$$

$$+\left(\beta_2-\alpha_2\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_3}^{\beta_3}\Phi\left(\sigma_1\left(r\right),y,\sigma_3\left(t\right)\right)\Delta_3 t\Delta_1 r$$

$$+\left(\beta_1-\alpha_1\right)\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(x,\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Big]$$

$$-\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\Big\}$$

$$-\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\Big\{\Big[\left(\beta_2-\alpha_2\right)\left(\beta_3-\alpha_3\right)\int_{\alpha_1}^{\beta_1}\Psi\left(\sigma_1\left(r\right),y,z\right)\Delta_1 r$$

$$+\left(\beta_1-\alpha_1\right)\left(\beta_3-\alpha_3\right)\int_{\alpha_2}^{\beta_2}\Psi\left(x,\sigma_2\left(s\right),z\right)\Delta_2 s$$

$$+\left(\beta_1-\alpha_1\right)\left(\beta_2-\alpha_2\right)\int_{\alpha_3}^{\beta_3}\Psi\left(x,y,\sigma_3\left(t\right)\right)\Delta_3 t\Big]$$

$$-\Big[\left(\beta_3-\alpha_3\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\Psi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),z\right)\Delta_2 s\Delta_1 r$$

$$+\left(\beta_2-\alpha_2\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_3}^{\beta_3}\Psi\left(\sigma_1\left(r\right),y,\sigma_3\left(t\right)\right)\Delta_3 t\Delta_1 r$$

$$+\left(\beta_1-\alpha_1\right)\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi\left(x,\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Big]$$

$$-\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\Big\}$$

$$-\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\Big\{\Big[\left(\beta_2-\alpha_2\right)\left(\beta_3-\alpha_3\right)\int_{\alpha_1}^{\beta_1}\eta\left(\sigma_1\left(r\right),y,z\right)\Delta_1 r$$

$$+\left(\beta_1-\alpha_1\right)\left(\beta_3-\alpha_3\right)\int_{\alpha_2}^{\beta_2}\eta\left(x,\sigma_2\left(s\right),z\right)\Delta_2 s$$

$$+\left(\beta_1-\alpha_1\right)\left(\beta_2-\alpha_2\right)\int_{\alpha_3}^{\beta_3}\eta\left(x,y,\sigma_3\left(t\right)\right)\Delta_3 t\Big]$$

$$-\Big[\left(\beta_3-\alpha_3\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\eta\left(\sigma_1\left(r\right),\sigma_2\left(s\right),z\right)\Delta_2 s\Delta_1 r$$

$$+\left(\beta_2-\alpha_2\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_3}^{\beta_3}\eta\left(\sigma_1\left(r\right),y,\sigma_3\left(t\right)\right)\Delta_3 t\Delta_1 r$$

$$+\left(\beta_1-\alpha_1\right)\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\eta\left(x,\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Big]$$

$$-\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\eta\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\Big\}$$

$$= \Psi(x,y,z)\eta(x,y,z)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L[\Phi]\,\Delta_3 t\Delta_2 s\Delta_1 r$$
$$+\eta(x,y,z)\Phi(x,y,z)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L[\Psi]\,\Delta_3 t\Delta_2 s\Delta_1 r$$
$$+\Phi(x,y,z)\Psi(x,y,z)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L[\eta]\,\Delta_3 t\Delta_2 s\Delta_1 r \tag{3.4}$$

Dividing both sides of equation (3.2) by $3K = 3(\beta_1-\alpha_1)(\beta_2-\alpha_2)(\beta_3-\alpha_3)$, we have:

$$\Phi(x,y,z)\Psi(x,y,z)\eta(x,y,z) - \frac{1}{3K}\Big[\Psi(x,y,z)\eta(x,y,z)\Big\{P[\Phi]-Q[\Phi]$$
$$+\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\,\Delta_3 t\Delta_2 s\Delta_1 r\Big\}$$
$$+\eta(x,y,z)\Phi(x,y,z)\left\{\begin{array}{c} P[\Psi]-Q[\Psi] \\ +\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\,\Delta_3 t\Delta_2 s\Delta_1 r\end{array}\right\}$$
$$+\Phi(x,y,z)\Psi(x,y,z)\left\{\begin{array}{c} P[\eta]-Q[\eta] \\ +\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\eta(\sigma_1(r),\sigma_2(s),\sigma_3(t))\,\Delta_3 t\Delta_2 s\Delta_1 r\end{array}\right\}\Big]$$
$$= \frac{1}{3K}\Big[\Psi(x,y,z)\eta(x,y,z)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L[\Phi]\,\Delta_3 t\Delta_2 s\Delta_1 r$$
$$+\eta(x,y,z)\Phi(x,y,z)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L[\Psi]\,\Delta_3 t\Delta_2 s\Delta_1 r$$
$$+\Phi(x,y,z)\Psi(x,y,z)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} L[\eta]\,\Delta_3 t\Delta_2 s\Delta_1 r. \tag{3.5}$$

Taking absolute value of both sides of the above equation, we have the required inequality

$$\left|A(\Phi,\Psi,\eta;P,Q;K)(x,y,z)\right| \le \frac{1}{3K}B(|\Phi|,|\Psi|,|\eta|;|L|)(x,y,z), \quad \text{for all } (x,y,z)\in M.$$

Theorem 3.2. *Let $\mathbb{T}_1,\mathbb{T}_2$ and $\mathbb{T}_3$ be three arbitrary time scales and Δ_1,Δ_2 and Δ_3 be the delta differentiation operators respectively on $\mathbb{T}_1,\mathbb{T}_2$ and $\mathbb{T}_3$. If $[\alpha_i,\beta_i]\subseteq\mathbb{T}_i$, for $i=1,2,3$ and $M=\prod_{i=1}^{3}[\alpha_i,\beta_i]$, then for $(x,y,z),(r,s,t)\in M$ and for some continuous functions $\Phi,\Psi,\eta\in CC_{rd}^{1}$ $[M,\mathbb{R}]$, the following inequality holds:*

$$\left|Z(\Phi,\Psi,\eta;P,Q;K)\right| \le \frac{1}{3K^2}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} B(|\Phi|,|\Psi|,|\eta|,|L|)(x,y,z)\,\Delta_3 z\Delta_2 y\Delta_1 x,$$

for all $(x,y,z)\in M$.

Proof. On integrating both sides of equation (3.7) in Theorem 3.1 with respect to (x,y,z) over M, we have:

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi(x,y,z)\Psi(x,y,z)\eta(x,y,z)\;\Delta_3 z\Delta_2 y\Delta_1 x$$

$$
\begin{aligned}
&-\frac{1}{3K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\left[\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\left\{P\left[\Phi\right]-Q\left[\Phi\right]\right.\right.\\
&\left.+\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\right\}\\
&+\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\left\{\begin{array}{c}P\left[\Psi\right]-Q\left[\Psi\right]\\+\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\end{array}\right\}\\
&\left.+\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\left\{\begin{array}{c}P\left[\eta\right]-Q\left[\eta\right]\\+\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\eta\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\end{array}\right\}\right]\Delta_3 z\Delta_2 y\Delta_1 x\\
=&\frac{1}{3K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\left[\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}L\left[\Phi\right]\Delta_3\left(\sigma_3\left(t\right)\right)\Delta_2 s\Delta_1 r\right.\\
&+\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}L\left[\Psi\right]\Delta_3\left(\sigma_3\left(t\right)\right)\Delta_2 s\Delta_1 r\\
&\left.+\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}L\left[\eta\right]\Delta_3 t\Delta_2 s\Delta_1 r\right]\Delta_3 z\Delta_2 y\Delta_1 x.
\end{aligned}
$$

Dividing both sides by K, we have:

$$
\begin{aligned}
&\frac{1}{K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\ \Delta_3 z\Delta_2 y\Delta_1 x\\
&-\frac{1}{3K^2}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\left\{\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\left(P\left[\Phi\right]-Q\left[\Phi\right]\right)+\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\left(P\left[\Psi\right]-Q\left[\Psi\right]\right)\right.\\
&\left.+\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\left(P\left[\eta\right]-Q\left[\eta\right]\right)\right\}\Delta_3 z\Delta_2 y\Delta_1 x\\
&-\frac{1}{3}\left[\left(\frac{1}{K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\Delta_3 z\Delta_2 y\Delta_1 x\right)\times\right.\\
&\left(\frac{1}{K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\right)\\
&+\left(\frac{1}{K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\Delta_3 z\Delta_2 y\Delta_1 x\right)\times\\
&\left(\frac{1}{K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Psi\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\right)\\
&+\left(\frac{1}{K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\Delta_3 z\Delta_2 y\Delta_1 x\right)\times\\
&\left.\left(\frac{1}{K}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\eta\left(\sigma_1\left(r\right),\sigma_2\left(s\right),\sigma_3\left(t\right)\right)\Delta_3 t\Delta_2 s\Delta_1 r\right)\right]\\
=&\frac{1}{3K^2}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\left[\Psi\left(x,y,z\right)\eta\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}L\left[\Phi\right]\Delta_3 t\Delta_2 s\Delta_1 r\right.\\
&+\eta\left(x,y,z\right)\Phi\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}L\left[\Psi\right]\Delta_3 t\Delta_2 s\Delta_1 r\\
&\left.+\Phi\left(x,y,z\right)\Psi\left(x,y,z\right)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}L\left[\eta\right]\Delta_3 t\Delta_2 s\Delta_1 r\right]\Delta_3 z\Delta_2 y\Delta_1 x.
\end{aligned}
$$

Taking absolute value of both sides, we get the required inequality:

$$\left|Z\left(\Phi,\Psi,\eta;P,Q;K\right)\right|\leq\frac{1}{3K^2}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}B\left(|\Phi|,|\Psi|,|\eta|;|L|\right)(x,y,z)\,\Delta_3 z\Delta_2 y\Delta_1 x,\ \text{for all } (x,y,z)\in M.$$

Discrete case:

In theorem 3.1 and 3.2 if we put $\mathbb{T}=\mathbb{Z}$ and choose $\alpha_i=1$ for $i=1,2,3$ and $\beta_1=\alpha+1$, $\beta_2=\beta+1$, $\beta_3=\gamma+1$, we get the respective inequalities for the discrete functions in the form of the following corrolaries :

Corollary 3.1. *Let $\mathbb{T}_1,\mathbb{T}_2$ and $\mathbb{T}_3$ be three arbitrary time scales, such that $C=\{1,2,3,.....,\alpha+1\}\subseteq\mathbb{T}_1$, $D=\{1,2,3,.....,\beta+1\}\subseteq\mathbb{T}_2$ and $E=\{1,2,3,.....,\gamma+1\}\subseteq\mathbb{T}_3$, where α,β and γ are natural numbers. Let $G=C\times D\times E$ and Δ_1,Δ_2 and Δ_3 be the difference operators respectively on $\mathbb{T}_1,\mathbb{T}_2$ and $\mathbb{T}_3$, then for the functions $\Phi,\Psi,\eta\colon G\to\mathbb{R}$*

$$\left|I\left(\Phi,\Psi,\eta;\bar{P},\bar{Q};k\right)(l,m,n)\right|\leq\frac{1}{3k}Q\left(|\Phi|,|\Psi|,|\eta|,|P|\right)(l,m,n),$$

for all $(l,m,n)\in G$.

Corollary 3.2. *Let $\mathbb{T}_1,\mathbb{T}_2$ and $\mathbb{T}_3$ be three arbitrary time scales such that $C=\{1,2,3,.....,\alpha+1\}\subseteq\mathbb{T}_1$, $D=\{1,2,3,.....,\beta+1\}\subseteq\mathbb{T}_2$ and $E=\{1,2,3,.....,\gamma+1\}\subseteq\mathbb{T}_3$, where α,β and γ are natural numbers. Let $G=C\times D\times E$ and Δ_1,Δ_2 and Δ_3 be the difference operators respectively on $\mathbb{T}_1,\mathbb{T}_2$ and $\mathbb{T}_3$, then for the functions $\Phi,\Psi,\eta\colon G\to\mathbb{R}$*

$$\left|M\left(\Phi,\Psi,\eta;\bar{P},\bar{Q};k\right)\right|\leq\frac{1}{3k^2}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}Q\left(|\Phi|,|\Psi|,|\eta|;|\bar{L}|\right)(l,m,n),$$

where

$$\begin{aligned}
\bar{P}[f] &= \beta\gamma\sum_{1}^{\alpha}f(\sigma_1(r),m,n)+\alpha\gamma\sum_{1}^{\beta}f(l,\sigma_2(s),n)+\alpha\beta\sum_{1}^{\gamma}f(l,m,\sigma_3(t))\\
\bar{Q}[f] &= \gamma\sum_{1}^{\alpha}\sum_{1}^{\beta}f(\sigma_1(r),\sigma_2(s),n)+\beta\sum_{1}^{\alpha}\sum_{1}^{\gamma}f(\sigma_1(r),m,\sigma_3(t))\\
&\quad+\alpha\sum_{1}^{\beta}\sum_{1}^{\gamma}f(l,\sigma_2(s),\sigma_3(t)),\\
\bar{L}[f] &= \sum_{r}^{l-1}\sum_{s}^{m-1}\sum_{t}^{n-1}\Delta_3\Delta_2\Delta_1 f(u,v,w),\\
k &= \alpha\beta\gamma,
\end{aligned}$$

$$\begin{aligned}
&I\left(\Phi,\Psi,\eta;\bar{P},\bar{Q};k\right)(l,m,n)\\
&\quad=\Phi(l,m,n)\Psi(l,m,n)\eta(l,m,n)
\end{aligned}$$

$$-\frac{1}{3k}\Psi(l,m,n)\eta(l,m,n)\left\{\bar{P}[\Phi]-\bar{Q}[\Phi]+\sum_{1}^{\alpha}\sum_{1}^{\beta}\sum_{1}^{\gamma}\Phi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\right\}$$
$$+\eta(l,m,n)\Phi(l,m,n)\left\{\bar{P}[\Psi]-\bar{Q}[\Psi]+\sum_{1}^{\alpha}\sum_{1}^{\beta}\sum_{1}^{\gamma}\Psi(\sigma_1(r),\sigma_2(s),\sigma_3(t))\right\}$$
$$+\Phi(l,m,n)\Psi(l,m,n)\left\{\bar{P}[\eta]-\bar{Q}[\eta]+\sum_{1}^{\alpha}\sum_{1}^{\beta}\sum_{1}^{\gamma}\eta(\sigma_1(r),\sigma_2(s),\sigma_3(t))\right\},$$

$$Q\left(\Phi,\Psi,\eta;\bar{L}\right)=\Psi(l,m,n)\eta(l,m,n)\sum_{1}^{\alpha}\sum_{1}^{\beta}\sum_{1}^{\gamma}\bar{L}[\Phi]+\eta(l,m,n)\Phi(l,m,n)\sum_{1}^{\alpha}\sum_{1}^{\beta}\sum_{1}^{\gamma}\bar{L}[\Psi]$$
$$+\Phi(l,m,n)\Psi(l,m,n)\sum_{1}^{\alpha}\sum_{1}^{\beta}\sum_{1}^{\gamma}\bar{L}[\eta],$$

and

$$M\left(\Phi,\Psi,\eta;\bar{P},\bar{Q};k\right)=\frac{1}{k}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}\Phi(l,m,n)\Psi(l,m,n)\eta(l,m,n)$$
$$-\frac{1}{3k^2}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}\left\{\Psi(l,m,n)\eta(l,m,n)\left(\bar{P}[f]-\bar{Q}[f]\right)\right.$$
$$+\eta(l,m,n)\Phi(l,m,n)\left(\bar{P}[f]-\bar{Q}[f]\right)$$
$$\left.+\Phi(l,m,n)\Psi(l,m,n)\left(\bar{P}[f]-\bar{Q}[f]\right)\right\}$$
$$-\frac{1}{3}\left[\left(\frac{1}{k}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}\Psi(l,m,n)\eta(l,m,n)\right)\left(\frac{1}{k}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}\Phi(l,m,n)\right)\right.$$
$$+\left(\frac{1}{k}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}\eta(l,m,n)\Phi(l,m,n)\right)\left(\frac{1}{k}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}\Psi(l,m,n)\right)$$
$$\left.+\left(\frac{1}{k}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}\Phi(l,m,n)\Psi(l,m,n)\right)\left(\frac{1}{k}\sum_{l=1}^{\alpha}\sum_{m=1}^{\beta}\sum_{n=1}^{\gamma}\eta(l,m,n)\right)\right].$$

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References

[1] R. Agarwal, M. Bohner and A. Peterson, *Inequalities on time scales: a survey*, Inequal. Appl., **4**(2001), 535–557.

[2] Farooq Ahmad, Arif Rafiq and Nazir Ahmad Mir, *Weighted Ostrowski-Grüss type inequality for differentiable mappings*, Glob. J. Pure Appl. Math., **2**(2006), 147–154.

[3] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Boston, MA. Birkhäser Boston Inc., 2001.

[4] M. Bohner and T. Mathews, *The Gruss inequality on time scales*, Commun. Math. Anal., **3**(2007), 1–8.

[5] M. Bohner and G. Sh. Guseinov, *Double integral calculus of variations on time scales*, Comput. Math. Appl., **54**(2007), 45–57.

[6] S. S. Dragomir and Young-Ho Kim, *On certian new integral inequalities and their applications*, J. Ineq. Pure Appl. Math., **3**(2002), Article 65.

[7] S. S. Dragomir and A. Sofo, *An integral inequality for twice differentiable mappings and applications*, Tamkang Journal of Mathematics, **31**(2000), 257–266.

[8] G. Grüss. *Uber das Maximum des absoluten Betrages von*, Math. Z., **39** (1935), 215–226.

[9] S. Hilger, Ein Maβkettenkül mit Anwendung auf Zentrumsmannigfaltigkeiten, PhD thesis, Universität Würzburg, 1988.

[10] Wenjun Liu and Quôc-Anh Ngô, *An Ostrowski-Grüss type inequality on time scales*, Comput. Math. Appl., **58**(2009), 1207–1210.

[11] W. J. Liu, Q. A. Ngô and W. B. Chen, *Ostrowski type inequalities on time scales for double integrals*, Acta Appl. Math., **110**(2010), 477–497.

[12] N. A. Mir and A. Rafiq, *A note on Ostrowski like inequalities in $L_1(a,b)$ spaces*, Gen. Math., **14**(2006), 23–30.

[13] N. A. Mir, A. Rafiq and F. Ahmad, *Weighted Ostrowski type inequality for differentiable mappings whose derivative belong to $L_\infty(a,b)$*, Gen. Math., **14**(2006), 27–38.

[14] Nazir Ahmad Mir, Arif Rafiq and Farooq Ahmad, *Generalization of integral inequalities for functions whose modulus of nth derivatives are convex*, Gen. Math., **14**(2007), 67–92.

[15] Nazir Ahmad Mir, Arif Rafiq and Muhammad Rizwan, *Ostrowski-Grüss cebysew type inequalities for functions whose modulus for second derivatives are convex*, Gen. Math., **16**(2008), 111–134.

[16] A. Ostrowski, *Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Helv., **10**(1938), 226–227.

[17] U. M. Özkan and H. Yildirim, *Ostrowski type inequality for double integrals on time scales*, Acta Appl. Math., **110**(2010), 283–288.

[18] B. G. Pachpatte, *On Grüss type inequality for double integrals*, J. Math. Anal. Appl., **267** (2002), 454–459.

[19] B. G. Pachpatte, *New Inequalities of Ostrowski and Grüss type for triple integrals*, Tamkang J. Math., **40** (2009), 117-127.

[20] Umut Mutlu Özkan and Huseyin Yildrim, *Grüss type inequalities for doubleintegrals on time scales*, Comput. Math. Appl., **57** (2009), 436–444.

[21] A. Rafiq, N. A. Mir and Fiza Zafar, *A generalized Ostrowski-Grüss type inequality for twice differentiable mappings in Euclidean norm*, Gen. Math., **16**(2008), 51–72.

[22] Arif Rafiq, Nazir Ahmad Mir, and Fiza Zafar, *A generalized Ostrowski type inequality for a random variable whose probability density function belongs to $L_\infty[a,b]$*, Demonstratio Math., **41**(2008), 723-732.

[23] A. Sofo, *Double integral inequalities based on multi-branch Peano kernels*, Math. Ineq. & Appl., **5**(2002), 491–504.

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