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GENERALIZED VECTOR VALUED DOUBLE SEQUENCE SPACE USING MODULUS FUNCTION

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Abstract. In this paper, we introduce a generalized vector valued paranormed double sequence space $F^2(E, p, f, s)$, using modulus function f, where $p = (p_{nk})$ is a sequence of non-negative real numbers, $s \geq 0$ and the elements are chosen from a seminormed space (E, q_E). Results regarding completeness, normality, K_2-space, co-ordinatewise convergence etc. are derived. Further, a study of multiplier sets, ideals, notion of statistical convergence and (p_{nk})-Cesáro summability in the space $F^2(E, p, f, s)$ is also made.

1. Introduction & Motivation

Ratha and Srivastava [12] and Ghosh and Srivastava [6] introduced and studied generalized classes of composite vector valued single sequence spaces $\mathrm{F}(E_k, \Lambda)$ and $\mathrm{F}(E_k, f)$ respectively, which are defined as

$$F(E_k, \Lambda) = \Big\{x = (x_k): \ x_k \in E_k \text{ for each } k \text{ and } (g_{E_k}(v_k x_k)) \in F\Big\}$$

and

$$F(E_k, f) = \Big\{x = (x_k): \ x_k \in E_k \text{ and the sequence } (f(g_{E_k}(x_k)) \in F\Big\},$$

where F is a normal sequence space with a monotone paranorm g_F, (E_k, g_{E_k}) is Banach space over the field of complex numbers $\mathbf{C}$, f is a modulus function and $\Lambda(z) = \sum_k \frac{z^k}{\nu_k}$, $\nu = (\nu_k)$ is a sequence of non-zero complex numbers satisfying

$$\nu = \liminf_{k\to\infty} |\nu_k|^{\frac{1}{k}}, \ 0 < \nu \leq \infty.$$

With suitable topologies, the authors have investigated various topological properties for these spaces. The study of these spaces includes many known spaces as particular cases. For example, by specifying F, E & f, one can obtain $w_0(f)$ & $w_\infty(f)$ of Maddox [7], $w_0(f, p)$ & $w_\infty(f, p)$ of Bilgin [3], $w(f)$ of Öztürk & Bilgin [11] and others.

To continue the study, we introduce a new space $F^2(E, p, f, s)$ of vector valued double sequences which unifies some of the earlier classes on double sequences as particular cases.

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Also, some important results have been derived on various aspects of double sequences which can be treated as generalization of the results derived by Gökhan et al. ([5] [4]) and Mursaleen et al. [9].

2. Sequence space $F^2(E, p, f, s)$

Let (E, q_E) be a seminormed space over the complex field $\mathbb{C}$. Let $S_2(E)$ denote the linear space of all double sequences $x = (x_{nk})$ with $x_{nk} \in E$ under the usual coordinatewise addition and scalar multiplication. Let F^2 be a normal scalar double sequence space with monotone paranorm g_{F^2} such that co-ordinatewise convergence implies convergence in paranorm g_{F^2}, i.e.,

$$a_{nk}^{j,l} \to 0(j, l \to \infty) \text{ for each n, k } \implies g_{F^2}(a^{j,l}) \to 0(j, l \to \infty) \tag{2.1}$$

where $(a_{nk}^{j,l}) = a^{j,l} \in F^2$ for each j and $l \in \mathbb{N}$.

Throughout this chapter, by a convergent double sequence we mean a convergent in Pringsheim's sense.[1]

Let f be a modulus function and p=(p_{nk}) be a sequence of strictly positive real numbers and $s \geq 0$. We introduce a new class $F^2(E, p, f, s)$ of vector valued double sequences as follows:

$$F^2(E, p, f, s) = \Big\{ x = (x_{nk}) \in S_2(E) : \ x_{nk} \in E \text{ for each } n, k \in N \text{ and the sequence } \Big((nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\Big) \in F^2 \Big\}. \tag{2.2}$$

Further, we define a topology on $F^2(E, p, f, s)$ by a paranorm g which is given by

$$g(x) = g_{F^2}\Big[(nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}/M}\Big], \text{ for } x = (x_{nk}) \in F^2(\text{E,p,f,s}) \tag{2.3}$$

where M=max$(1, H)$, $H = \sup_{n,k} p_{nk} < \infty$ and $\inf p_{nk} > 0$.
It is seen that $F^2(E, p, f, s)$ turns out to be a complete paranormed space of vector valued double sequences.

It can also be seen that for suitable choice of the sequence space F^2, E and the modulus function f, the space $F^2(E, p, f, s)$ includes many of the known scalar as well as vector valued sequence spaces as particular cases.

Application:

1. If we take $E = \mathbb{C}, \ f(x) = x, \ s = 0$ and $F^2 = \ell_2^\infty$, the space $F^2(E, p, f, s)$ gives rise the space $\ell_2^\infty(p)$ of Gökhan et al. [5].
2. If we take $E = \mathbb{C}, \ f(x) = x, \ s = 0$ and $F^2 = c_2$, the space $F^2(E, p, f, s)$ gives rise the space $c_2^P(p)$ of Gökhan et al. [4].

[1] see section 1.6 of Chapter 1.

3. If we take $E = \mathbb{C},\ f(x) = x,\ s = 0$ and $F^2 = c_2 \cap \ell_2^\infty$, the space $F^2(E, p, f, s)$ gives rise the space $c_2^{PB}(p)$ of Gökhan et al.[4].

4. If we take $E = \mathbb{C},\ f(x) = x,\ s = 0, p_{nk} \equiv 1$ and $F^2 = w^2(p),\ \ell_2^\infty,\ c_2,\ c_0^2,$, the space $F^2(E, p, f, s)$ gives rise the spaces of Tripathy [13].

3. Main Results

Theorem 3.1. *$F^2(E, p, f, s)$ is a linear space, where $p = (p_{nk})$ is a bounded sequence of strictly positive real numbers & $\inf p_{nk} > 0$.*

Proof. Let x=(x_{nk}), y=$(y_{nk}) \in F^2(E, p, f, s)$ and $\lambda, \mu \in \mathbb{C}$. Then

$$\begin{aligned}
&(nk)^{-s}\Big\{f\Big(q_E(\lambda x_{nk} + \mu y_{nk})\Big)\Big\}^{p_{nk}} \\
&\quad\le (nk)^{-s}\Big\{f\Big(q_E(\lambda x_{nk}) + q_E(\mu y_{nk})\Big)\Big\}^{p_{nk}} \\
&\quad\le (nk)^{-s}\Big\{f\Big(q_E(\lambda x_{nk})\Big) + f\Big(q_E(\mu y_{nk})\Big)\Big\}^{p_{nk}} \\
&\quad\le D(nk)^{-s}\Big[\Big\{f\Big(|\lambda| q_E(x_{nk})\Big)\Big\}^{p_{nk}} + \Big\{f\Big(|\mu| q_E(y_{nk})\Big)\Big\}^{p_{nk}}\Big] \\
&\quad\le D(1 + [|\lambda|])^H (nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}} + D(1 + [|\mu|])^H (nk)^{-s}\Big\{f\Big(q_E(y_{nk})\Big)\Big\}^{p_{nk}}
\end{aligned}$$

where D=max$(1, 2^{H-1})$. Since F^2 is normal, $\lambda x + \mu y \in F^2(E, p, f, s)$.

Theorem 3.2. *F^2(E, p, f, s) is a paranormed space under the paranorm g given by (2.3), where $p = (p_{nk})$ is a bounded sequence of strictly positive real numbers & $\inf p_{nk} > 0$.*

Proof. It is clear from the definition of g that $g(\bar{\theta}) = 0$ and $g(x) = g(-x)$, where $\bar{\theta}$ is the null element. Again taking $\lambda = 1,\ \mu = 1$ in the Theorem 3.1 and the fact that g_{F^2} is a monotone paranorm on F^2, we get g(x+y)$\le$ g(x)+g(y). It is left to prove the continuity of scalar multiplication under g.

Suppose $\{\lambda_m\}$ is a sequence of scalars such that $\lambda_m \to \lambda$ as $m \to \infty$ and let $x^{j,l} \xrightarrow{g} x$ as $j,\ l \to \infty$. To show $g(\lambda_m x^{j,l} - \lambda x) \to 0$ as $j,\ l \to \infty$ where $x^{j,l} = (x_{nk}^{j,l}) \in F^2(E, p, f, s)$.

Let

$$a_{nk}^{jm} = (nk)^{-s}\Big\{f\Big(|\lambda_m - \lambda| q_E(x_{nk})\Big)\Big\}^{p_{nk}/M}. \tag{3.1}$$

As $\lambda_m \to \lambda$ as $m \to \infty$, for sufficiently large m, we can assume that $|\lambda_m - \lambda| < 1$.

Consider

$$\begin{aligned}
g(\lambda_m x^{j,l} - \lambda x) &= g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(q_E\Big[\lambda_m x_{nk}^{j,l} - \lambda x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&\le g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(q_E\Big[(\lambda_m - \lambda)(x_{nk}^{j,l} - x_{nk}) \\
&+ \lambda(x_{nk}^{j,l} - x_{nk}) + (\lambda_m - \lambda)x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&\le g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(q_E\Big[(\lambda_m - \lambda)(x_{nk}^{j,l} - x_{nk})\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&+ g_{F_2}\Big[(nk)^{-s}\Big\{f\Big(q_E\Big[\lambda(x_{nk}^{j,l} - x_{nk})\Big]\Big)\Big\}^{p_{nk}/M}\Big] \\
&+ g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(q_E\Big[(\lambda_m - \lambda)x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&= g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(|\lambda_m - \lambda| q_E\Big[x_{nk}^{j,l} - x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&+ g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(|\lambda| q_E\Big[x_{nk}^{j,l} - x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&+ g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(|\lambda_m - \lambda| q_E\Big[x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&\equiv \text{I+II+III}. \qquad (3.2)
\end{aligned}$$

Since $\lambda_m \to \lambda$ as $m \to \infty$, so

$$\begin{aligned}
\text{I} &\equiv g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(|\lambda_m - \lambda| q_E\Big[x_{nk}^{j,l} - x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&\le g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(q_E\Big[x_{nk}^{j,l} - x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] = g(x^{j,l} - x).
\end{aligned}$$

Again since $f(\lambda) < (1 + [|\lambda|])f(1)$, so as $m \to \infty$

$$\begin{aligned}
\text{II} &\equiv g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(|\lambda| q_E\Big[x_{nk}^{j,l} - x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&\le g_{F^2}\Big[\Big((nk)^{-s}\Big\{(1 + [|\lambda|])f\Big(q_E\Big[x_{nk}^{j,l} - x_{nk}\Big]\Big)\Big\}^{p_{nk}/M}\Big)\Big] \\
&= (1 + [|\lambda|])g(x^{j,l} - x).
\end{aligned}$$

Hence from (3.2) using (3.1) we get

$$\begin{aligned}
&g(\lambda_m x^{j,l} - \lambda x) \\
&= g(x^{j,l} - x) + (1 + [|\lambda|])g(x^{j,l} - x) + \Big[g_{F^2}(a_{nk}^m)\Big]
\end{aligned}$$

Also, since

$$f\Big(|\lambda_m - \lambda| q_E(x_{nk})\Big) < f\Big(q_E(x_{nk})\Big)$$

holds because $|\lambda_m - \lambda| < 1$ for sufficiently large m and F^2 is normal so $a^m = (a^m_{nk}) \in F^2$ for sufficiently large m. Obviously for each n, k, $a^m_{nk} \to 0$ as $m \to \infty$. So by the condition (2.1), we get $g_{F^2}(a^m_{nk}) \to 0$ for sufficiently large m. Again II and III tend to zero as $j,\ l \to \infty$ & $m \to \infty$, because $\lambda_m \to \lambda$ & $x^{j,l} \xrightarrow{g} x$. Hence we get

$$g(\lambda_m x^{j,l} - \lambda x) \to 0 \text{ as } m \to \infty \text{ and } j,\ l \to \infty.$$

Hence the proof.

Theorem 3.3. F^2(E,p,f,s) is a K_2-space if F^2 is a K_2-space.

Proof. Define $P_{nk} : F^2(E,p,f,s) \to E$ as $P_{nk}(x) = x_{nk},\ n,k = 1,2,3,\ldots$, where $x = (x_{nk}) \in F^2(E,p,f,s)$. To show P_{nk} is continuous.

Let $(x^{j,l}) = ((x^{j,l}_{nk}))$ be a sequence in F^2(E,p,f,s) such that

$$g(x^{j,l}) \to 0 \text{ as } j,l \to \infty.$$

Since F^2 is a K_2-space, $g(x^{j,l}) \to 0$ as $j,\ l \to \infty$ implies that

$$(nk)^{-s}\Big\{f\Big(q_E(x^{j,l}_{nk})\Big)\Big\}^{p_{nk}/M} \to 0 \text{ as } j,\ l \to \infty, \text{ for each n,k.}$$

We claim that $q_E(x^{j,l}_{nk}) \to 0$ as $j,\ l \to \infty$, because f is continuous and increasing. This implies

$$q_E\Big(P_{nk}(x^{j,l})\Big) = q_E\Big(x^{j,l}_{nk}\Big) \to 0 \text{ for } j,\ l \to \infty.$$

Hence the proof.

Theorem 3.4. *F^2(E,p,f,s) is a normal space.*

Proof. The proof is straightforward, so we omit it.

Theorem 3.5. *F^2(E,p,f,s) is complete with respect to the paranorm g if (E, q_E) is complete, and F^2 is normal K_2-space, where (p_{nk}) is bounded sequence of strictly positive real numbers such that* $\inf p_{nk} > 0$.

Proof. Let $x^{j,l} = (x^{j,l}_{nk})$ be a Cauchy sequence in F^2(E, p, f, s). So

$$g(x^{j,l} - x^{r,t}) \to 0 \text{ as } j,\ l,\ r,\ t \to \infty$$

i.e.,

$$g_{F_2}\Big[\Big((nk)^{-s}\Big\{f\Big(q_E(x^{j,l}_{nk} - x^{r,t}_{nk})\Big)\Big\}^{p_{nk}/M}\Big)\Big] \to 0 \text{ as } j,\ l,\ r,\ t \to \infty.$$

Since F_2(E,p,f,s) is a K_2 space and f is continuous, so for each n,k

$$\Big(nk\Big)^{-s/M} q_E\Big(x^{j,l}_{nk} - x^{r,t}_{nk}\Big) \to 0 \text{ as } j,\ l,\ r,\ t \to \infty.$$

Hence for fixed n, k

$$q_E\Big(x^{j,l}_{nk} - x^{r,t}_{nk}\Big) \to 0 \text{ as } j,\ l,\ r,\ t \to \infty.$$

This implies that for fixed n, k, $(x^{j,l}_{nk})$ behaves as a Cauchy sequence in E. But (E, q_E) is complete, so there exist $x = (x_{nk}) \in$ E such that

$$q_E\Big(x^{j,l}_{nk} - x_{nk}\Big) \to 0 \text{ as } j,\ l \to \infty.$$

So

$$a^{j,l}_{nk} \to 0(\text{ as } j,\ l \to \infty) \text{ for each } n, k \quad (\text{since } f \text{ is continuous})$$

where

$$a^{j,l}_{nk} = (nk)^{-s}\Big\{f\Big(q_E(x^{j,l}_{nk} - x_{nk})\Big)\Big\}^{p_{nk}/M}, n,\ k = 1, 2, \dots \tag{3.3}$$

Since for each n, k, $a^{j,l}_{nk} \to 0$ as $j,\ l \to \infty$, so choose $\delta^{j,l}_{nk}$ such that

$$a^{j,l}_{nk} \le \delta^{j,l}_{nk}(nk)^{-s}\Big\{f\Big(q_E(x^{j,l}_{nk})\Big)\Big\}^{p_{nk}/M} \text{ where } 0 < \delta^{j,l}_{nk} < 1. \tag{3.4}$$

Clearly $a^{j,l} \in F^2$, for each m, because F^2 is normal.
Hence,

$$g_{F_2}\Big[\Big((nk)^{-s}\Big\{f\Big(q_E(x^{j,l}_{nk} - x_{nk})\Big)\Big\}^{p_{nk}/M}\Big)\Big] \to 0 \text{ as } j,\ l \to \infty$$

i.e., $g(x^{j,l} - x) \to 0$ as $j, l \to \infty$.
Now using (3.3) and (3.4) we get

$$(nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}/M} \le (\delta^{j,l}_{nk} + 1)(nk)^{-s}\Big\{f\Big(q_E(x^{j,l}_{nk})\Big)\Big\}^{p_{nk}/M}.$$

Since F^2 is normal space and $x^{j,l} \in F^2$(E,p,f,s), $x = (x_{nk}) \in F^2$(E,p,f,s). Hence the proof.

Theorem 3.6. *Let f be a modulus function such that $f(uv) = f(u)f(v)$ where u, v are positive scalars. Let E be a commutative normal sequence algebra under $\circ$, where $x_{nk} \circ y_{nk} = x_{nk}y_{nk}$ and F^2 is a normal sequence algebra under the multiplication $\circ'$, defined as $(a_{nk}) \circ' (b_{nk}) = (a_{nk}b_{nk})$, where $(a_{nk}),\ (b_{nk}) \in F^2$. Then $F^2(E, p, f, s)$ is a commutative sequence algebra.*

Proof. Let $x = (x_{nk})$ and $y = (y_{nk}) \in F^2(E, p, f, s)$. Consider

$$\begin{aligned}
&(nk)^{-2s}\Big\{f\Big(q_E(x_{nk} \circ y_{nk})\Big)\Big\}^{p_{nk}} \\
&= (nk)^{-2s}\Big\{f\Big(q_E(x_{nk}y_{nk})\Big)\Big\}^{p_{nk}} \\
&\le (nk)^{-2s}\Big\{f\Big(q_E(x_{nk})q_E(y_{nk})\Big)\Big\}^{p_{nk}} \text{ (since E is a normed algebra)} \\
&= (nk)^{-2s}\Big\{f\Big(q_E(x_{nk})\Big)f\Big(q_E(y_{nk})\Big)\Big\}^{p_{nk}} \text{ (by given condition)} \\
&= (nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}(nk)^{-s}\Big\{f\Big(q_E(y_{nk})\Big)\Big\}^{p_{nk}} \in F^2
\end{aligned}$$

as $x, y \in F^2(E, p, f, s)$, which implies $\Big(x_{nk}y_{nk}\Big) \in F^2(E, p, f, s)$. So $F^2(E, p, f, s)$ is a sequence algebra. Further, it can be seen easily that $F^2(E, p, f, s)$ is a commutative sequence algebra as E is commutative.

Remark 3.1. The condition $\inf p_{nk} > 0$ is not required when $F^2 \equiv c_0^2,\ c^2,\ \ell_p^2,\ p \geq 1$. But for $F^2 \equiv \ell_\infty^2$ the condition $\inf p_{nk} > 0$ is required. For the sake of completeness we have chosen this condition in general.

Now, we prove the following lemma which will be used in sequel:

Lemma 3.1. *Let f_1, f_2 be modulus functions and $0 < \delta < 1$. Let $f_1(t) > \delta$ for $t \geq 0$, then*

$$(f_2 \circ f_1)(t) \leq \frac{2f_2(1)}{\delta} f_1(t).$$

Proof. Since for $f_1(t) > \delta$,

$$f_1(t) < \frac{f_1(t)}{\delta} < 1 + \Big[\frac{f_1(t)}{\delta}\Big]$$

we have

$$(f_2 \circ f_1)(t) \leq \Big(1 + [\frac{f_1(t)}{\delta}]\Big) f_2(1) \leq 2\frac{f_1(t)}{\delta} f_2(1).$$

Some inclusion relations which are known for single sequence spaces are extended analogously to double sequence spaces as follows:

Theorem 3.7. *Let F^2 be a normal sequence space. Then the following inequalities hold:*

1. *If $\sup_{t>0} \frac{f_1(t)}{f_2(t)} < \infty$, then $F^2(E, p, f_2, s) \subseteq F^2(E, p, f_1, s)$.*
2. *$F^2(E, p, f_1, s) \bigcap F^2(E, p, f_2, s) \subseteq F^2(E, p, f_1 + f_2, s)$.*
3. *$F^2(E, p, f_1, s) \subseteq F^2(E, p, f_2 \circ f_1, s)$ if $(nk)^{-s} \in F^2$, where $(f_2 \circ f_1)(t) = f_2(f_1(t))$ and $\inf p_{nk} > 0$ & $\sup p_{nk} < \infty$.*
4. *If $s_1 \leq s_2$, then $F^2(E, p, f_1, s_1) \subseteq F^2(E, p, f_1, s_2)$.*

Proof (i). Let $x = (x_{nk}) \in F^2(E, p, f_2, s)$.
Since $\sup_{t>0} \frac{f_1(t)}{f_2(t)} < \infty$ is given, $\exists\ L > 0$ such that $f_1(t) \leq Lf_2(t)$ for all t > 0 and hence

$$(nk)^{-s}\Big\{f_1\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}} \leq \max\Big(1,\ L^H\Big)(nk)^{-s}\Big\{f_2\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}} \text{ for each n and k.}$$

Since F^2 is normal, so the result follows.

Proof (ii). Let $x = (x_{nk}) \in F^2(E, p, f_1, s) \bigcap F^2(E, p, f_2, s)$. Consider

$$\begin{aligned}(nk)^{-s}\Big\{(f_1+f_2)\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}} &= (nk)^{-s}\Big[\Big\{f_1\Big(q_E(x_{nk})\Big)\Big\} + \Big\{f_2\Big(q_E(x_{nk})\Big)\Big\}\Big]^{p_{nk}} \\ &\le D(nk)^{-s}\Big[\Big\{f_1\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}} + \Big\{f_2\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\Big]\end{aligned}$$

for each n and k and D=max$(1, 2^{H-1})$. Result follows as F^2 is normal sequence space.

Proof (iii). Let us choose δ such that $0 < \delta < 1$. Let

$$N_1 = \Big\{(n,k) \in N \times N : f_1\Big(q_E(x_{nk})\Big) \le \delta\Big\}$$
$$N_2 = \Big\{(n,k) \in N \times N : f_1\Big(q_E(x_{nk})\Big) > \delta\Big\}.$$

If (n,k)$\in N_1$, then

$$(f_2 \circ f_1)\Big(q_E(x_{nk})\Big) \le f_2(\delta).$$

Hence

$$(nk)^{-s}\Big((f_2 \circ f_1)\Big(q_E(x_{nk})\Big)\Big)^{p_{nk}} \le \eta_1 (nk)^{-s}, \tag{3.5}$$

where

$$\eta_1 = \max\Big[\Big\{f_2(\delta)\Big\}^{\inf p_{nk}}, \Big\{f_2(\delta)\Big\}^{\sup p_{nk}}\Big].$$

Again for (n,k)$\in N_2$,

$$\begin{aligned}(nk)^{-s}\Big((f_2 \circ f_1)\Big(q_E(x_{nk})\Big)\Big)^{p_{nk}} &\le (nk)^{-s}\Big[\frac{2f_2(1)}{\delta} f_1\Big(q_E(x_{nk})\Big)\Big]^{p_{nk}} \text{ (by Lemma 3.1)} \\ &\le \eta_2 (nk)^{-s}\Big[f_1\Big(q_E(x_{nk})\Big)\Big]^{p_{nk}} \qquad (3.6)\end{aligned}$$

where

$$\eta_2 = \max\Big\{\Big\{\frac{2f_2(1)}{\delta}\Big\}^{\inf p_{nk}}, \Big\{\frac{2f_2(1)}{\delta}\Big\}^{\sup p_{nk}}\Big\}.$$

Let $\eta = \max(\eta_1,\ \eta_2)$.

From (3.5) and (3.6) we get for $(n,k) \in N_1 \cup N_2$,

$$(nk)^{-s}\Big((f_2 \circ f_1)\Big(q_E(x_{nk})\Big)\Big)^{p_{nk}} \le \eta\Big[(nk)^{-s} + (nk)^{-s}\Big[f_1\Big(q_E(x_{nk})\Big)\Big]^{p_{nk}}\Big].$$

Since $((nk)^{-s}) \in F^2$ and F^2(E, f, p, s) is normal, so the result follows.

Proof (iv). For $s_1 \le s_2$

$$(nk)^{-s_2}\Big[f\Big(q_E(x_{nk})\Big)\Big]^{p_{nk}} \le (nk)^{-s_1}\Big[f\Big(q_E(x_{nk})\Big)\Big]^{p_{nk}} \text{ for every n, k.}$$

By using the normality of F^2, the result is obtained.

4. Multiplier set of $F^2(E,p,f,s)$

This section deals with some inclusion relations between the set $F^2(E,p,f,s)$ and its multiplier set.

We define multiplier set of $F^2(E,p,f,s)$ as

$$M^2[F^2(E,p,f,s)] \\ = \Big\{a=(a_{nk})\in E : \Big(a_{nk}x_{nk}\Big)\in F^2(E,p,f,s) \text{ for all } x=(x_{nk})\in F^2(E,p,f,s)\Big\}$$

where E is taken as normed algebra. Now, we prove the following theorems:

Theorem 4.1. *Let E be normed algebra and F^2 be a normal sequence space. Then*

$$\ell_2^\infty(E)\subseteq M^2[F^2(E,p,f,s)],$$

where

$$\ell_2^\infty(E)=\Big\{a=(a_{nk}) : a_{nk}\in E \text{ and } \sup_{\mathrm{n,k}} \mathrm{q_E}\Big(\mathrm{a_{nk}}\Big)<\infty\Big\}.$$

Proof. Let $a=(a_{nk})\in\ell_2^\infty(E)$ and $x=(x_{nk})\in F^2(E,p,f,s)$.

Let B= $\sup_{n,k} q_E(a_{nk})<\infty$. Now,

$$\begin{aligned}&(nk)^{-s}\Big\{f\Big(q_E(a_{nk}x_{nk})\Big)\Big\}^{p_{nk}}\\ &\quad\le (nk)^{-s}\Big\{f\Big(q_E(a_{nk})q_E(x_{nk})\Big)\Big\}^{p_{nk}} \text{ (since E is normed algebra)}\\ &\quad< (1+[B])^H(nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\end{aligned}$$

where $[B^H]$ denotes the integral part of B^H.
Since F^2 is normal, this implies $(a_{nk}x_{nk})\in F^2(E,p,f,s)$ and consequently $(a_{nk})\in M^2[F^2(E,p,f,s)]$. Hence the proof.

Theorem 4.2. *For any modulus function satisfying $f(\alpha\beta)\le f(\alpha)+f(\beta), \alpha,\beta\in [0,\ \infty)$,*

$$F^2(E,p,f,s)\subseteq M^2[F^2(E,p,f,s)],$$

where E is a normed algebra.

Proof. Let $x=(x_{nk})\in F^2(E,p,f,s)$. We want to show that $x=(x_{nk})\in M^2[F^2(E,p,f,s)]$, i.e., to show $(x_{nk}y_{nk})\in F^2(E,p,f,s)$ for all $y=(y_{nk})\in F^2(E,p,f,s)$.

Consider

$$q_E(x_{nk}y_{nk})\le q_E(x_{nk})q_E(y_{nk})$$

since E is a normed algebra.

Then

$$f\Big(q_E(x_{nk}y_{nk})\Big) \le f\Big(q_E(x_{nk})q_E(y_{nk})\Big) \le f\Big(q_E(x_{nk})\Big) + f\Big(q_E(y_{nk})\Big).$$

Thus

$$\begin{aligned}(nk)^{-s}\Big\{f\Big(q_E(x_{nk}y_{nk})\Big)\Big\}^{p_{nk}} &\le (nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big) + f\Big(q_E(y_{nk})\Big)\Big\}^{p_{nk}} \\ &\le D(nk)^{-s}\Big[\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}} + \Big\{f\Big(q_E(y_{nk})\Big)\Big\}^{p_{nk}}\Big]\end{aligned}$$

where D=max$(1, 2^{H-1})$.

This implies $xy \in F^2(E,p,f,s)$ and hence $x \in M^2[F^2(E,p,f,s)]$.

5. Ideals of $F^2(E,p,f,s)$

Let I^2 be a normal subspace of F^2, where F^2 is a sequence algebra. Let E be commutative normed algebra and S$_2(E)$ is the linear space of all sequences $x = (x_{nk})$ with $x_{nk} \in E$ under the usual coordinatewise addition and scalar multiplication.

$$\begin{aligned}&I^2(E,p,f,s)\\ &\quad = \Big\{x = (x_{nk}) : x_{nk} \in S(E) \text{ for each n,k and } \Big((nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\Big) \in I^2\Big\}\end{aligned}$$

It is easy to check that $I^2(E,p,f,s)$ is a subspace of $F^2(E,p,f,s)$.

Theorem 5.1. *If I^2 is closed subspace of F^2 and F^2 is a normal K_2-space then for $0 < p_{nk} \le 1$, $I^2(E,p,f,s)$ is a closed subspace of F^2(E,p,f,s).*

Proof. It is easy to show that $I^2(E,p,f,s)$ is a subspace of $F^2(E,p,f,s)$. Next, to show it is closed, we take $x = (x_{nk}) \in \overline{I^2(E,p,f,s)}$, the closure of $I^2(E,p,f,s)$. This implies the existence of a sequence $x^{j,l} = \Big((x^{j,l}_{nk})\Big) \in I^2(E,p,f,s)$ such that

$$g(x^{j,l} - x) \to 0 \text{ as } j,\ l \to \infty$$

for some $x = (x_{nk}) \in F^2(E,p,f,s)$.

Consequently,

$$g_{F^2}\Big[\Big((nk)^{-s}\Big\{f\Big(q_E(x^{j,l}_{nk} - x_{nk})\Big)\Big\}^{p_{nk}}\Big)\Big] \to 0 \text{ as } m \to \infty. \tag{5.1}$$

(since M=max$(1, \sup p_{nk})$=1)

Since F^2 is K_2-space and f is continuous at 0, so,

$$q_E(x^{j,l}_{nk} - x_{nk}) \to 0 \text{ as } j,\ l \to \infty \text{ for each n, k.}$$

Consider

$$\begin{aligned}\Big\{f\Big(q_E(x_{nk}^{j,l})\Big)\Big\}^{p_{nk}} &\leq \Big\{f\Big(q_E(x_{nk}^{j,l}-x_{nk}+x_{nk})\Big)\Big\}^{p_{nk}}\\ &\leq \Big\{f\Big(q_E(x_{nk}^{j,l}-x_{nk})\Big)+f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\\ &\leq \Big\{f\Big(q_E(x_{nk}^{j,l}-x_{nk})\Big)\Big\}^{p_{nk}}+\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\ (\text{as } 0<p_{nk}\leq 1)\end{aligned}$$

Therefore

$$\Big\{f\Big(q_E(x_{nk}^{j,l})\Big)\Big\}^{p_{nk}}-\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}} \leq \Big\{f\Big(q_E(x_{nk}^{j,l}-x_{nk})\Big)\Big\}^{p_{nk}}. \tag{5.2}$$

Since F^2 is normal and g_{F^2} is a monotone paranorm, so (5.2) implies that

$$\bigg((nk)^{-s}\bigg(\Big\{f\Big(q_E(x_{nk}^{j,l})\Big)\Big\}^{p_{nk}}-\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\bigg)\bigg)\in F^2. \tag{5.3}$$

So we get from (5.2) and (5.3)

$$\begin{aligned}&g_{F^2}\bigg[(nk)^{-s}\bigg(\Big\{f\Big(q_E(x_{nk}^{j,l})\Big)\Big\}^{p_{nk}}-\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\bigg)\bigg]\\ &\leq g_{F^2}\bigg[(nk)^{-s}\Big\{f\Big(q_E(x_{nk}^{j,l}-x_{nk})\Big)\Big\}^{p_{nk}}\bigg]\\ &= g(x^{j,l}-x)\end{aligned}$$

Using (5.1) we get,

$$g_{F^2}\bigg[(nk)^{-s}\bigg(\Big\{f\Big(q_E(x_{nk}^{j,l})\Big)\Big\}^{p_{nk}}\bigg)-(nk)^{-s}\bigg(\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\bigg)\bigg]\to 0 \text{ as } m\to\infty. \tag{5.4}$$

Since I^2 is closed in F^2, it is clear from (5.4) that,

$$\bigg((nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\bigg)\in I^2.$$

Hence $x=(x_{nk})\in I^2(E,p,f,s)$.

Theorem 5.2. *Let I^2 be an ideal of F^2. Further, let the modulus function f satisfies $f(uv)=f(u)f(v)$ where u, v are scalars. Then $I^2(E,p,f,s)$ is an ideal of $F^2(E,p,f,s)$.*

Proof. For $x=(x_{nk})\in I^2(E,p,f,s)$ and $r=(r_{nk})\in F^2(E,f,p,s)$,

$$\begin{aligned}(nk)^{-2s}\Big\{f\Big(q_E(r_{nk}x_{nk})\Big)\Big\}^{p_{nk}} &\leq (nk)^{-2s}\Big\{f\Big(q_E(r_{nk})q_E(x_{nk})\Big)\Big\}^{p_{nk}}\\ &= (nk)^{-s}\Big\{f\Big(q_E(r_{nk})\Big)\Big\}^{p_{nk}}(nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\in I^2\end{aligned}$$

As because I^2 is an ideal of F^2,

$$\bigg((nk)^{-s}\Big\{f\Big(q_E(r_{nk})\Big)\Big\}^{p_{nk}}(nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\bigg)\in I^2.$$

Further normality of I^2 implies

$$\Big((nk)^{-2s}\Big\{f\Big(q_E(r_{nk}x_{nk})\Big)\Big\}^{p_{nk}}\Big) \in I^2$$

and hence $rx \in I^2(E, p, f, s)$.

Similarly it can be shown that $xr \in I^2(E, f, s)$ which completes the proof.

Theorem 5.3. *If I^2 is a subspace of ℓ_2^∞, for any unbounded function f, $I^2(E, p, f, s)$ is an ideal of $\ell_2^\infty(E, p, f, s)$.*

Proof. Let $x = (x_{nk}) \in I^2(E, p, f, s)$ and $\ell = (\ell_{nk}) \in \ell_2^\infty(E, p, f, s)$. So

$$\sup_{n,k}(nk)^{-s}\Big\{f\Big(q_E(\ell_{nk})\Big)\Big\}^{p_{nk}} < \infty \tag{5.5}$$

But f is unbounded and in order to hold (5.5), it follows that $\ell = (\ell_{nk}) \in \ell_2^\infty(E)$.

Let

$$T = \sup_{n,k} q_E(\ell_{nk}).$$

Then

$$\begin{aligned}(nk)^{-s}\Big\{f\Big(q_E(\ell_{nk}x_{nk})\Big)\Big\}^{p_{nk}} &\le (nk)^{-s}\Big\{f\Big(q_E(\ell_{nk})q_E(x_{nk})\Big)\Big\}^{p_{nk}}\\ &\le (nk)^{-s}\Big\{f\Big(Tq_E(x_{nk})\Big)\Big\}^{p_{nk}}\\ &\le (1+[T])^H(nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}.\end{aligned}$$

Hence by the normality of I^2, it follows that $\ell x \in I^2(E, p, f, s)$.

Similarly we can show that $x\ell \in I^2(E, p, f, s)$.

6. Statistical convergence and strongly (p_{nk})-Cesáro summability

The concepts of Cesáro summability and strongly p-Cesáro summability for double sequences are introduced by Moricz [8] while the notion of statistical convergence for double sequences has been discussed by Mursaleen et al. [9].

Mursaleen et al. [9] first introduced and extended the concept of statistical convergence for double sequences of real or complex numbers after defining the analogue concept of natural density for double sequences as follows:

Let K $\subseteq \mathbb{N}\times\mathbb{N}$ be a two dimensional set of positive integers and let

$$K(n,m) = C\Big(\Big\{(i,j) : i \le n \text{ and } j \le m\Big\}\Big).$$

where C(A) denotes the cardinality of the set A.

If the sequence $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense [1], then we say that K has double natural density $\delta_2(K)$ and is written as

$$\delta_2(K) = P - \lim_{n,m} \frac{K(n,m)}{nm}$$

Mursaleen et al. [9] defined analogously the statistical convergence and statistical Cauchy convergence for double sequences $x = (x_{nk})$ as follows:

Definition 6.1. A real double sequence $x = (x_{ij})$ is said to be statistically convergent to the number ℓ, if for each $\epsilon > 0$, the set

$$\Big\{(i,j) : i \leq n \text{ and } \mathrm{j} \leq \mathrm{m} : |\mathrm{x_{ij}} - \ell| \geq \epsilon\Big\}$$

has double natural density zero in the Pringsheim's sense [1], i.e.,

$$P - \lim_{m,n} \frac{1}{mn} C\Big(\Big\{(i,j) : i \leq n \ \&\ j \leq m,\ |x_{ij} - \ell| \geq \epsilon\Big\}\Big) = 0$$

and this is denoted as $\mathrm{st}_2 - \lim_{i,j} x_{ij} = \ell$. We denote the set of all statistically convergent sequences (in Pringsheim's sense) by st_2.

Definition 6.2. A real double sequence $x = (x_{ij})$ is said to be statistically Cauchy, if for each $\epsilon > 0$ there exist $A = A(\epsilon)$ and $B = B(\epsilon)$ such that for all i, p $\geq$ A, j, q $\geq$ B, the set

$$\Big\{(i,j) : i \leq r \text{ and } \mathrm{j} \leq \mathrm{s} : |\mathrm{x_{ij}} - \mathrm{x_{pq}}| \geq \epsilon\Big\}$$

has double natural density zero in Pringsheim's sense.

In this section, we have extended the concepts of statistical convergence and Cesáro-summability to the generalized vector valued double sequence space $F^2(E,p,f,s)$ as follows:

Definition 6.3. A vector valued double sequence x=$(x_{ij}) \in F^2(E,p,f,s)$ is said to be statistically convergent to L if for each $\epsilon > 0$, the set

$$\Big\{(i,j),\ i \leq n,\ j \leq k :\ (ij)^{-s}\Big\{f\Big(q_E(x_{ij} - L)\Big)\Big\}^{p_{ij}} \geq \epsilon\Big\}$$

has double natural density zero.

In this case we write $x_{ij} \overset{s^2(E,p,f,s)}{\longrightarrow} L$. It is easy to check that L is unique.

Definition 6.4. Let $x = (x_{ij})$ be a vector valued double sequence in F^2(E, p, f, s) and $p = (p_{ij})$ be a sequence of strictly positive real numbers. Then $x = (x_{ij})$ is said to be strongly (p_{ij})-Cesáro-type summable to ℓ if

$$\lim_{n,k\to\infty} \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell\Big)\Big\}^{p_{ij}}\Big] = 0.$$

Note 6.1. Let $s^2_{E,p,f,s}$, $w^2_{E,p,f,s}$ denote the space of all statistically convergent vector valued double sequences and the space of all strongly (p_{ij})-Cesáro-type summable vector valued double sequences respectively.

Theorem 6.1. *A vector valued double sequence $x = (x_{ij}) \in F^2(E, p, f, s)$ is statistically convergent to ℓ if it is strongly (p_{ij})-Cesáro-type suummable to ℓ.*

Proof. Let

$$I_1(\epsilon) = \Big\{(i,j), i \leq n, j \leq k : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\}.$$

Let us assume that $x = (x_{ij})$ is strongly (p_{ij})-Cesáro suummable to ℓ. Then

$$\begin{aligned}
&\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] \\
&= \frac{1}{nk}\Big[\sum_{(i,j)\in I_1}\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] + \sum_{(i,j)\notin I_1}\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big]\\
&\geq \frac{1}{nk}\sum_{(i,j)\in I_1}\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big]\\
&\geq \epsilon\frac{1}{nk}C\Big(\Big\{(i,j) \in \mathbb{N}\times\mathbb{N}, i \leq n,\ j \leq k : \Big[(nk)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\}\Big)
\end{aligned}$$

implies that $x = (x_{ij})$ is statistically convergent to ℓ.

Theorem 6.2. $w^2_{E,p,f,s} \cap \ell^2_\infty(E, p, f, s) = s^2_{E,p,f,s} \cap \ell^2_\infty(E, p, f, s)$, *where*

$$\ell^2_\infty(E, p, f, s) = \Big\{x = (x_{nk}) \in S_2(E) : \ x_{nk} \in E \ \textit{and} \ \Big((nk)^{-s}\Big\{f\Big(q_E(x_{nk})\Big)\Big\}^{p_{nk}}\Big) \in \ell^2_\infty\Big\}.$$

Proof. Let $x = (x_{ij}) \in s^2_{E,p,f,s} \cap \ell^2_\infty(E, p, f, s)$ and let

$$I_2(\epsilon) = \Big\{(i,j),\ i \leq n,\ j \leq k : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big\}^{p_{ij}}\Big)\Big] \geq \frac{\epsilon}{2}\Big\}.$$

Let

$$T = \sup_{i,j}\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big].$$

Since $x = (x_{ij})$ is bounded statistically convergent, we can choose N such that for all $n, k \geq N$,

$$\frac{1}{nk}C\Big(\Big\{(i,j)\ i \leq n,\ j \leq k : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] \geq \frac{\epsilon}{2}\Big\}\Big) < \frac{\epsilon}{2T}.$$

Thus

$$\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\Big[(Ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big]$$
$$=\frac{1}{nk}\sum_{(i,j)\in I_2(\epsilon)}\Big[(Ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big]+\frac{1}{nk}\sum_{(i,j)\notin I_2(\epsilon)}\Big[(Ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big]$$
$$<\frac{1}{nk}nk\frac{\epsilon}{2T}T+\frac{1}{nk}nk\frac{\epsilon}{2}=\epsilon$$

Hence $x=(x_{ij})$ is strongly (p_{ij})-Cesáro-type summable to ℓ.

We have proved more generalized form of some well known results of Mursaleen et al. [2003] regarding statistical convergence as follows:

Theorem 6.3. *A vector valued double sequence $x=(x_{ij})$ in $F^2(E,p,f,s)$ is statistically convergent to a number ℓ if and only if there exists a subset $R=\{(i,j)\}\subseteq N\times N$, $i,j=1,2,\ldots$ such that $\delta_2(R)=1$ and*

$$\lim_{\substack{i,j\to\infty\\(i,j)\in R}} q_E(x_{ij}-\ell)=0.$$

Proof. Let $x=(x_{ij})$ be statistically convergent to ℓ.

Let

$$R_\eta=\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}:\ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big]\geq 1/\eta\Big\}$$

and

$$T_\eta=\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}:\ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big]< 1/\eta\Big\}$$

$(\eta=1,2,\ldots)$

Then $\delta_2(R_\eta)=0$. Again, (T_i) is a sequence of sets such that $T_i\supseteq T_{i+1}$ and $\delta_2(T_\eta)=1,\ \eta=1,2,\ldots$. Now, we have to show that for $(i,j)\in T_\eta$, (x_{ij}) is convergent to ℓ.

Now, if possible, let $x=(x_{ij})$ be not convergent to ℓ, for all $(i,j)\in T_\eta$.

Then there is a $\epsilon>0$ such that, for infinitely many i, j,

$$\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big]\geq\epsilon$$

Let

$$T_\epsilon=\Big\{(i,j):\ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big]<\epsilon\Big\}$$

where $\epsilon>1/\eta$. Then $\delta_2(T_\epsilon)=0$.

Since $T_\eta\subseteq T_\epsilon$, it follows that $\delta_2(T_\eta)=0$, a contradiction. Thus $x=(x_{ij})$ is convergent to ℓ.

Conversely, suppose that there exists a subset R=$\Big\{$(i,j) $\Big\}\subseteq N\times$N such that δ_2(R)=1 and

$$\lim_{i,j\to\infty} q_E(x_{ij}-\ell)=0.$$

So there exists a +ve integer N_0 such that for every $\epsilon > 0$,

$$\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big] < \epsilon$$

for all $i,\ j \geq N_0$.
Now,

$$\begin{aligned} R_\epsilon &= \Big\{(i,j)\in\mathbb{N}\times\mathbb{N}:\ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij}-\ell)\Big)\Big\}^{p_{ij}}\Big]\geq\epsilon\Big\} \\ &\subseteq \mathbb{N}\times\mathbb{N}-\Big\{(n_{N_0+1},k_{N_0+1}),(n_{N_0+2},k_{N_0+2}),\ldots,\Big\}. \end{aligned}$$

Then

$$\delta_2(R_\epsilon)\leq 1-\delta_2\Big(\Big\{(n_{N_0+1},k_{N_0+1}),(n_{N_0+2},k_{N_0+2}),\ldots,\Big\}\Big)=1-1=0$$

Consequently x is statistically convergent to ℓ.

Coroloary 6.1. *If $s^2_{E,f,p,s}-\lim x_{ij}=\ell$, then there exists a sequence $y=(y_{ij})$ such that $\lim_{i,j} y_{ij}\overset{q_E}{=}\ell$ and $\delta_2\Big(\Big\{(i,j): x_{ij}\neq y_{ij}\Big\}\Big)=1$ i.e., $x_{ij}=y_{ij}$ for all most all i,j.*

Theorem 6.4. *The set $s^2_{E,f,p,s}\cap\ell^2_\infty(E,p,f,s)$ is a closed linear subspace of the normed linear space $\ell^2_\infty(E,p,f,s)$.*

Proof. Let $x^{(mn)}=(x^{(mn)}_{ij})$ be any Cauchy sequence in the space $s^2_{E,f,p,s}\cap\ell^2_\infty(E,p,f,s)$. Let $x^{(mn)}\to x\in\ell^2_\infty(E,p,f,s)$. Since $x^{(mn)}\in s^2_{E,f,p,s}$, there exist $a_{mn}\in E$ such that $s^2_{E,f,p,s}-\lim_{i,j}x^{(mn)}_{ij}=a_{mn}$ for $m,n=1,2,\ldots$.

Since $x^{(mn)}\to x$, for every $\epsilon>0$, there exist a positive integer $n_0\in\mathbb{N}$ such that

$$g(x^{(mn)}-x^{(pq)})<\frac{\epsilon}{3} \tag{6.1}$$

for every $m,p\geq n_0,\ n,q\geq n_0$, where g denotes the norm.

By Theorem 6.3. there exist subsets K_1 and K_2 of $\mathbb{N}\times\mathbb{N}$ with $\delta_2\Big(K_1\Big)=1=\delta_2\Big(K_2\Big)$ and

$$\lim_{\substack{i,j\to\infty\\(i,j)\in K_1}} x^{(mn)}_{ij}\overset{g}{=}a_{mn} \text{ and } \lim_{\substack{i,j\to\infty\\(i,j)\in K_2}} x^{(pq)}_{ij}\overset{g}{=}a_{pq} \tag{6.2}$$

We choose $(k_1,k_2)\in K_1\cap K_2$ (where $\delta_2\Big(K_1\cap K_2\Big)=1$).
Then by (6.2) we have

$$g(x^{(mn)}_{k_1k_2}-a_{mn})<\frac{\epsilon}{3} \tag{6.3}$$

and

$$g(x^{(pq)}_{k_1k_2}-a_{pq})<\frac{\epsilon}{3} \tag{6.4}$$

Therefore for each $m, p \geq n_0$ and $n, q \geq n_0$, using (6.1), (6.3) and (6.4) we have

$$\begin{aligned} g(a_{pq} - a_{mn}) &\leq g(x_{k_1k_2}^{(mn)} - a_{mn}) + g(x_{k_1k_2}^{(pq)} - a_{pq}) + g(x_{k_1k_2}^{(mn)} - x_{k_1k_2}^{(pq)}) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Hence the sequence (a_{mn}) is a Cauchy sequence in E. Since E is a Banach space, it is complete. Let

$$\lim_{m,n} q_E(a_{mn}) = a \tag{6.5}$$

We will show that x is statistically convergent to a.

Since $x^{(mn)}$ is convergent to x, for every $\epsilon > 0$, there exist $N_1(\epsilon)$ such that for $i, j \geq N_1(\epsilon)$,

$$g(x_{ij}^{(mn)} - x_{ij}) < \frac{\epsilon}{3}$$

Also since (6.5) holds, we have for every $\epsilon > 0$, there exist $N_2(\epsilon)$ such that for $i, j \geq N_2(\epsilon)$,

$$g(a_{mn} - a) < \frac{\epsilon}{3}$$

Again since $s^2_{E,f,p,s} - \lim_{i,j} x_{ij}^{(mn)} = a_{mn}$, there exists a set $R = \Big\{(i,j)\Big\} \subseteq \mathbb{N} \times \mathbb{N}$, $i, j = 1, 2, \ldots$ such that $\delta_2(R) = 1$ and for every $\epsilon > 0$, there exist $N_3(\epsilon)$ such that for $i, j \geq N_3(\epsilon)$, $(i, j) \in R$,

$$g(x_{ij}^{(mn)} - a_{mn}) < \frac{\epsilon}{3}$$

Let

$$N(\epsilon) = \max\Big(N_1(\epsilon),\ N_2(\epsilon),\ N_3(\epsilon)\Big)$$

Then for every $\epsilon > 0$, there exist $N(\epsilon)$ such that for $i, j \geq N(\epsilon)$, $(i, j) \in R$,

$$g(x_{ij} - a) \leq g(x_{ij} - x_{ij}^{(mn)}) + g(x_{ij}^{(mn)} - a_{mn}) + g(a_{mn} - a) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Therefore x is statistically convergent to a, i.e., $x \in s^2_{E,f,p,s} \cap \ell^2_\infty(E, p, f, s)$. Hence the proof.

Theorem 6.5. *The set* $s^2_{E,p,f,s} \cap \ell^2_\infty(E, p, f, s)$ *is a nowhere dense in* $\ell^2_\infty(E, p, f, s)$.

Proof. It is shown by T. Neubrum et al. [10] that every closed subspace of an arbitrary linear normed space S different from S is a nowhere dense set in S and using Theorem 5.5.4, it is enough to show that $s^2_{E,p,f,s} \cap \ell^2_\infty(E, p, f, s) \neq \ell^2_\infty(E, p, f, s)$ in order to establish our claim.

Let us take $F^2 = \mathbb{R} \times \mathbb{R},\ E = \mathbb{R},\ p_{ij} = 1$. Let $x = (x_{ij})$ be such that

$$x_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are even} \\ 0 & \text{otherwise} \end{cases}$$

Let $f(x) = x$ and $s = 0$. Then $x = (x_{ij})$ is not statistically convergent, but it is bounded. Hence the result.

Definition 6.5. A sequence $x = (x_{ij})$ is said to be statistically Cauchy if for any given $\epsilon > 0$, there exist $N_1(\epsilon)$ and $N_2(\epsilon)$ such that for all $i, r \geq N_1$ and $j, t \geq N_2$,

$$\Big\{(i,j), i \leq n, j \leq k : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - x_{rt})\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\}$$

has double natural density zero.

Theorem 6.6. *A sequence $x = (x_{ij})$ in $F^2(E,p,f,s)$ is statistically convergent if and only if it is statistically Cauchy.*

Proof. Let us assume that $x = (x_{ij})$ in $F^2(E,p,f,s)$ is statistically convergent to ℓ. Then for any given $\epsilon > 0$, the set

$$\Big\{(i,j), n \leq u, k \leq v : \ \Big[\Big((ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big)\Big]\Big) \geq \epsilon\Big\}$$

has double natural density zero.

Let N_1 and N_2 be so chosen that

$$\Big[(N_1N_2)^{-s}\Big\{f\Big(q_E(x_{N_1N_2} - \ell)\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon.$$

Now,

$$\begin{aligned}
&\Big\{(i,j), n \leq u, k \leq v : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - x_{N_1N_2})\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\} \\
&\quad \subseteq \Big\{(i,j), n \leq u, k \leq v : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\} \\
&\quad \bigcup \Big\{(i,j), n \leq u, k \leq v : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{N_1N_2} - \ell)\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\}
\end{aligned}$$

Hence

$$\begin{aligned}
&\delta_2\Big(\Big\{(i,j), n \leq u, k \leq v : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - x_{N_1N_2})\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\}\Big) \\
&\quad \leq \delta_2\Big(\Big\{(i,j), n \leq u, k \leq v : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\}\Big) \\
&\qquad + \delta_2\Big(\Big\{(i,j), n \leq u, k \leq v : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{N_1N_2} - \ell)\Big)\Big\}^{p_{ij}}\Big] \geq \epsilon\Big\}\Big) \\
&\quad = 0
\end{aligned}$$

where $\delta_2(A)$ denotes the double natural density of the set A. Thus $x = (x_{ij})$ is statistically Cauchy sequence.

Conversely, let $x = (x_{ij})$ be statistically Cauchy sequence, but not statistically convergent. Then

$$\delta_2\Big(\Big\{(i,j), i \leq n, j \leq k : \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - x_{N_1N_2})\Big)\Big\}^{p_{ij}}\Big]\Big) \geq \epsilon\Big\}\Big) = 0$$

i.e.,

$$\delta_2\Big(\Big\{(i,j), i \leq n, j \leq k: \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - x_{N_1N_2})\Big)\Big\}^{p_{ij}}\Big]\Big) < \epsilon\Big\}\Big) = 1. \tag{6.6}$$

So, in particular,

$$\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - x_{N_1N_2})\Big)\Big\}^{p_{ij}}\Big] \leq 2\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] < \epsilon \tag{6.7}$$

holds if

$$\Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big] < \epsilon/2$$

If possible, let $x = (x_{ij})$ be not statistically convergent. Then

$$\delta_2\Big\{(i,j), i \leq n, j \leq k: \ \Big[\Big((ij)^{-s}\Big\{f\Big(q_E(x_{ij} - \ell)\Big)\Big\}^{p_{ij}}\Big]\Big) < \epsilon\Big\}\Big) = 0.$$

Therefore the set

$$\delta_2\Big(\Big\{(i,j), i \leq n, j \leq k: \ \Big[(ij)^{-s}\Big\{f\Big(q_E(x_{ij} - x_{N_1N_2})\Big)\Big\}^{p_{ij}}\Big]\Big) < \epsilon\Big\}\Big) = 0$$

which contradicts (6.6). Hence $x = (x_{ij})$ is statistically convergent to ℓ.

7. Summary and Conclusion

Considering $F^2 = \mathbb{R}^2$, $E = \mathbb{R}$, $p_{ij} = 1$, $f = I$, $s = 0$, all the results of Mursaleen et al. [9] regarding statistical convergence can be obtained from these results. Moreover, if we take $p = (p_{ij})$ to be a sequence of constant terms, say, $p_{ij} = p$, where $0 < p < 1$, then $M = 1$ and restricting $F^2 = E = \mathbb{R}$, $f = I$, $s = 0$, our (p_{ij})-Cesáro-type summability reduces to p-Cesáro summability defined by Mursaleen et al.[9].

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