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OSCILLATION OF THE SOLUTIONS OF SYSTEMS OF NONLINEAR PARABOLIC EQUATIONS WITH FUNCTIONAL ARGUMENTS

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Abstract. In the present paper the oscillatory properties of the solutions of systems of parabolic equations are investigated and oscillation criteria is derived for every solution of boundary value problems to be oscillatory or satisfies some limit condition. Our approach is to reduce the multi-dimensional problem to a one-dimensional problem for nonlinear functional differential inequalities.

1. Introduction

We are concerned with systems of parabolic equations with functional arguments

$$\text{(E)}\quad \frac{\partial}{\partial t}\left(u_r(x,t)+\sum_{i=1}^{l}h_i(t)u_r(x,\rho_i(t))\right)$$
$$-a_r(t)\Delta u_r(x,t)-\sum_{i=1}^{k}b_{ri}(t)\Delta u_r(x,\tau_i(t))$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{N}q_{rji}(x,t)\varphi_i(u_j(x,\sigma_i(t)))=0,$$
$$(x,t)\in G\times(0,\infty)\equiv\Omega,\ r=\{1,2,\dots,N\},$$

where Δ is the Laplacian in $\mathbb{R}^n$ and G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G.

We assume throughout this paper that :

(H1) $h_i(t)\in C^1([0,\infty);[0,\infty))$ $(i=1,2,\dots,l)$,
$a_r(t)\in C([0,\infty);[0,\infty))$,
$b_{ri}(t)\in C([0,\infty);[0,\infty))$ $(i=1,2,\dots,k)$;

(H2) $\rho_i(t)\in C^1([0,\infty);\mathbb{R})$, $\lim_{t\to\infty}\rho_i(t)=\infty$ $(i=1,2,\dots,l)$,
$\tau_i(t)\in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty}\tau_i(t)=\infty$ $(i=1,2,\dots,k)$,
$\sigma_i(t)\in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty}\sigma_i(t)=\infty$ $(i=1,2,\dots,m)$;

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(H3) $q_{rji}(x,t) \in C(\overline{\Omega};[0,\infty))$,

$$q_i(x,t) = \min_{1\le r\le N}\left\{q_{rri}(x,t) - \sum_{j=1,j\ne r}^{N} q_{jri}(x,t)\right\} \ge 0,$$

$q_i(t) = \min\{q_i(x,t); x \in \overline{G}\}$;

(H4) $\varphi_i(s) \in C(\mathbb{R};\mathbb{R})$, $\varphi_i(-s) = -\varphi_i(s)$, $\varphi_i(s) > 0$ for $s > 0$ and $\varphi_i(s)$ are convex in $(0,\infty)$ $(i = 1,2,\dots,m)$.

We consider two kinds of boundary conditions:

$$(\mathrm{B}_1) \qquad u_r(x,t) = 0 \quad \text{on} \quad \partial G \times [0,\infty),$$

$$(\mathrm{B}_2) \qquad \frac{\partial u_r}{\partial \nu}(x,t) + \mu_r(x,t)u_r(x,t) = 0 \quad \text{on} \quad \partial G \times [0,\infty),$$

where ν is the unit exterior normal vector to ∂G and $\mu_r(x,t) \in C(\partial G\times[0,\infty);[0,\infty))$ $(r = 1,2,\dots,N)$.

The first eigenvalue λ_0 of the eigenvalue problem

$$\Delta w + \lambda w = 0 \quad \text{in} \quad G,$$
$$w = 0 \quad \text{on} \quad \partial G$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G.

Definition 1. By vector *solution* of system (E) we mean a function $u(x,t) = \{u_1(x,t), u_2(x,t),\dots,u_N(x,t)\}^T \in C^2(\overline{G}\times[t_{-1},\infty);\mathbb{R}) \cap C^1(\overline{G}\times[\hat{t}_{-1},\infty);\mathbb{R}) \cap C(\overline{G}\times[\tilde{t}_{-1},\infty);\mathbb{R})$ which satisfies (E), where

$$t_{-1} = \min\left\{0, \min_{1\le i\le k}\left\{\inf_{t\ge 0}\tau_i(t)\right\}\right\},$$
$$\hat{t}_{-1} = \min\left\{0, \min_{1\le i\le l}\left\{\inf_{t\ge 0}\rho_i(t)\right\}\right\},$$
$$\tilde{t}_{-1} = \min\left\{0, \min_{1\le i\le m}\left\{\inf_{t\ge 0}\sigma_i(t)\right\}\right\}.$$

Definition 2. The vector solution $u(x,t) = \{u_1(x,t),u_2(x,t),\dots,u_N(x,t)\}^T$ is said to be oscillatory in Ω if at least one of its nontrivial component has arbitrarily zeros. Otherwise, the vector solution $u(x,t)$ is said to be nonoscillatory.

There is much interest in oscillation problems for systems of parabolic equations with functional arguments. In 1990, Gopalsamy [9] introduced the approach of oscillation criteria for systems of parabolic equations with neutral terms. There are several papers dealing with the same approach in [9], see, for example [3–8]. However, it seems that there does not exist known oscillation results for systems of nonlinear parabolic equations.

The purpose of this paper is to obtain oscillation criteria for solution of the boundary value problems for (E), (B_i) $(i = 1,2)$ by referring results of paper [10–12].

2. Reduction to scalar functional differential inequalities

In this section we reduce the multi-dimensional oscillation problems to certain one-dimensional oscillation problems for scalar nonlinear functional differential inequalities.

Theorem 1. *Assume that* (H1)–(H4) *hold. If the differential inequality*

$$\frac{d}{dt}\left(y(t)+\sum_{i=1}^{l}h_i(t)y(\rho_i(t))\right)+\sum_{i=1}^{m}q_i(t)\varphi_i(y(\sigma_i(t)))\le 0 \tag{1}$$

has no eventually positive solution, then every solution u of the problem (E), (B_1) *is oscillatory in* Ω.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t),u_2(x,t),\dots,u_N(x,t)\}^T$ of the problem (E), (B_1). We assume that $|u_r(x,t)|>0$ in $G\times[t_0,\infty)$ for some $t_0>0$. Set

$$\theta_r=\operatorname{sgn} u_r(x,t)$$

and

$$z_r(x,t)=\theta_r u_r(x,t),$$

then we see that $z_r(x,t)>0$ in $G\times[t_0,\infty)$. The hypothesis (H2) implies that $u_r(x,\rho_i(t))>0$ $(i=1,2,\dots,l)$, $u_r(x,\tau_i(t))>0$ $(i=1,2,\dots,k)$ and $u_r(x,\sigma_i(t))>0$ $(i=1,2,\dots,m)$ in $G\times[t_1,\infty)$ for some $t_1\ge t_0$. Therefore, we have

$$\begin{aligned}\frac{\partial}{\partial t}&\left(\theta_r z_r(x,t)+\sum_{i=1}^{l}h_i(t)\theta_r z_r(x,\rho_i(t))\right)\\ &-\theta_r a_r(t)\Delta z_r(x,t)-\theta_r\sum_{i=1}^{k}b_{ri}(t)\Delta z_r(x,\tau_i(t))\\ &\quad+\sum_{i=1}^{m}\sum_{j=1}^{N}\theta_j q_{rji}(x,t)\varphi_i(z_j(x,\sigma_i(t)))=0,\ t\ge t_1.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\frac{\partial}{\partial t}&\left(z_r(x,t)+\sum_{i=1}^{l}h_i(t)z_r(x,\rho_i(t))\right)\\ &-a_r(t)\Delta z_r(x,t)-\sum_{i=1}^{k}b_{ri}(t)\Delta z_r(x,\tau_i(t))\\ &\quad+\sum_{i=1}^{m}\sum_{j=1}^{N}\frac{\theta_j}{\theta_r}q_{rji}(x,t)\varphi_i(z_j(x,\sigma_i(t)))=0,\ t\ge t_1,\end{aligned}$$

and so

$$\frac{\partial}{\partial t}\left(z_r(x,t)+\sum_{i=1}^{l}h_i(t)z_r(x,\rho_i(t))\right)$$
$$-a_r(t)\Delta z_r(x,t)-\sum_{i=1}^{k}b_{ri}(t)\Delta z_r(x,\tau_i(t))$$
$$+\sum_{i=1}^{m}\left\{q_{rri}(x,t)\varphi_i(z_r(x,\sigma_i(t)))-\sum_{j=1,j\neq r}^{N}q_{rji}(x,t)\varphi_i(z_j(x,\sigma_i(t)))\right\}\le 0,\ t\ge t_1. \quad (2)$$

Dividing (2) by N and summing both sides of (2) for $r=1,2,\dots,N$, we obtain

$$\frac{\partial}{\partial t}\left(z(x,t)+\sum_{i=1}^{l}h_i(t)z(x,\rho_i(t))\right)$$
$$-\sum_{r=1}^{N}\frac{a_r(t)}{N}\Delta z_r(x,t)-\sum_{r=1}^{N}\sum_{i=1}^{k}\frac{b_{ri}(t)}{N}\Delta z_r(x,\tau_i(t))$$
$$+\sum_{i=1}^{m}\left\{\frac{1}{N}\sum_{r=1}^{N}\left[q_{rri}(x,t)\varphi_i(z_r(x,\sigma_i(t)))\right.\right.$$
$$\left.\left.-\sum_{j=1,j\neq r}^{N}q_{rji}(x,t)\varphi_i(z_j(x,\sigma_i(t)))\right]\right\}\le 0,\ t\ge t_1, \quad (3)$$

where

$$z(x,t)=\frac{\sum_{r=1}^{N}z_r(x,t)}{N}.$$

We note that

$$\sum_{r=1}^{N}\left[q_{rri}(x,t)\varphi_i(z_r(x,\sigma_i(t)))-\sum_{j=1,j\neq r}^{N}q_{rji}(x,t)\varphi_i(z_j(x,\sigma_i(t)))\right]$$
$$=\left[q_{11i}(x,t)\varphi_i(z_1(x,\sigma_i(t)))-\sum_{j=1,j\neq 1}^{N}q_{1ji}(x,t)\varphi_i(z_j(x,\sigma_i(t)))\right]$$
$$+\left[q_{22i}(x,t)\varphi_i(z_2(x,\sigma_i(t)))-\sum_{j=1,j\neq 2}^{N}q_{2ji}(x,t)\varphi_i(z_j(x,\sigma_i(t)))\right]$$
$$+\cdots+\left[q_{NNi}(x,t)\varphi_i(z_N(x,\sigma_i(t)))-\sum_{j=1,j\neq N}^{N}q_{Nji}(x,t)\varphi_i(z_j(x,\sigma_i(t)))\right]$$

$$
\begin{aligned}
&= \left[q_{11i}(x,t) - \sum_{j=1, j\neq 1}^{N} q_{j1i}(x,t) \right] \varphi_i(z_1(x,\sigma_i(t))) \\
&\quad + \left[q_{22i}(x,t) - \sum_{j=1, j\neq 2}^{N} q_{j2i}(x,t) \right] \varphi_i(z_2(x,\sigma_i(t))) \\
&\quad + \cdots + \left[q_{NNi}(x,t) - \sum_{j=1, j\neq N}^{N} q_{jNi}(x,t) \right] \varphi_i(z_N(x,\sigma_i(t))) \\
&\geq \min_{1\leq r\leq N} \left\{ q_{rri}(x,t) - \sum_{j=1, j\neq r}^{N} q_{jri}(x,t) \right\} \sum_{r=1}^{N} \varphi_i(z_r(x,\sigma_i(t))) \\
&= q_i(x,t) \sum_{r=1}^{N} \varphi_i(z_r(x,\sigma_i(t))),\ t\geq t_1.
\end{aligned}
$$

Applying Jensen's inequality, we obtain

$$
\begin{aligned}
\sum_{r=1}^{N} \frac{1}{N} \varphi_i(z_r(x,\sigma_i(t))) &\geq \sum_{r=1}^{N} \frac{1}{N} \cdot \varphi_i \left(\frac{\displaystyle\sum_{r=1}^{N} \frac{1}{N} z_r(x,\sigma_i(t))}{\displaystyle\sum_{r=1}^{N} \frac{1}{N}} \right) = \varphi_i \left(\frac{\sum_{r=1}^{N} z_r(x,\sigma_i(t))}{N} \right) \\
&= \varphi_i(z(x,\sigma_i(t))),\ t\geq t_1. \qquad (4)
\end{aligned}
$$

Combining (3) with (4) yields

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(z(x,t) + \sum_{i=1}^{l} h_i(t) z(x,\rho_i(t)) \right) \\
&\quad - \sum_{r=1}^{N} \frac{a_r(t)}{N} \Delta z_r(x,t) - \sum_{r=1}^{N}\sum_{i=1}^{k} \frac{b_{ri}(t)}{N} \Delta z_r(x,\tau_i(t)) \\
&\quad + \sum_{i=1}^{m} q_i(x,t) \varphi_i(z(x,\sigma_i(t))) \leq 0,\ t\geq t_1. \qquad (5)
\end{aligned}
$$

Multiplying (5) by $\Phi(x)(\int_G \Phi(x))^{-1}$ and then integrating over G, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(Z(t) + \sum_{i=1}^{l} h_i(t) Z(\rho_i(t)) \right) - \sum_{r=1}^{N} \frac{a_r(t)}{N} K_\Phi \int_G \Delta z_r(x,t)\Phi(x)dx \\
&\quad - \sum_{r=1}^{N}\sum_{i=1}^{k} \frac{b_{ri}(t)}{N} K_\Phi \int_G \Delta z_r(x,\tau_i(t))\Phi(x)dx \\
&\quad + \sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(z(x,\sigma_i(t)))\Phi(x)dx \leq 0,\ t\geq t_1, \qquad (6)
\end{aligned}
$$

where

$$\begin{aligned} Z(t) &= K_\Phi \int_G z(x,t)\Phi(x)dx, \\ K_\Phi &= \left(\int_G \Phi(x)dx\right)^{-1}. \end{aligned} \tag{7}$$

From Green's formula it follows that

$$\begin{aligned} \int_G \Delta z_r(x,t)\Phi(x)dx &= \int_G z_r(x,t)\Delta\Phi(x)dx \\ &= -\lambda_1 \int_G z_r(x,t)\Phi(x)dx \le 0,\ t \ge t_1. \end{aligned} \tag{8}$$

Analogously we obtain

$$\int_G \Delta z_r(x,\tau_i(t))\Phi(x)dx = -\lambda_1 \int_G z_r(x,\tau_i(t))\Phi(x)dx \le 0,\ t \ge t_1. \tag{9}$$

An application of Jensen's inequality shows that

$$\begin{aligned} K_\Phi \int_G q_i(x,t)\varphi_i(z(x,\sigma_i(t)))\Phi(x)dx &\ge q_i(t)K_\Phi \int_G \varphi_i(z(x,\sigma_i(t)))\Phi(x)dx \\ &\ge q_i(t)\varphi_i\Big(Z(\sigma_i(t))\Big),\ t \ge t_1. \end{aligned} \tag{10}$$

Combining (6)–(10) yields

$$\frac{d}{dt}\left(Z(t) + \sum_{i=1}^{l} h_i(t)Z(\rho_i(t))\right) + \sum_{i=1}^{m} q_i(t)\varphi_i\Big(Z(\sigma_i(t))\Big) \le 0,\ t \ge t_1. \tag{11}$$

Hence, $Z(t)$ is a positive solution of (1) on $[t_1,\infty)$. This contradicts the hypothesis and completes the proof.

Theorem 2. *Assume that* (H1)–(H4) *hold. If the differential inequality* (1) *has no eventually positive solution, then every solution u of the problem* (E), (B_2) *is oscillatory in Ω.*

Proof. Suppose that there exists a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t),\dots,u_N(x,t)\}^T$ of the problem (E), (B_2). We assume that $|u_r(x,t)| > 0$ in $G \times [t_0,\infty)$ for some $t_0 > 0$. By the same arguments as were used in Theorem 1, we obtain the inequality (5). Dividing (5) by $|G|$ and then integrating over G yields

$$\begin{aligned} &\frac{d}{dt}\left(\tilde{Z}(t) + \sum_{i=1}^{l} h_i(t)\tilde{Z}(\rho_i(t))\right) \\ &\quad - \sum_{r=1}^{N} \frac{a_r(t)}{N|G|}\int_G \Delta z_r(x,t)dx - \sum_{r=1}^{N}\sum_{i=1}^{k} \frac{b_{ri}(t)}{N|G|}\int_G \Delta z_r(x,\tau_i(t))dx \\ &\quad + \sum_{i=1}^{m} \frac{1}{|G|}\int_G q_i(x,t)\varphi_i(z(x,\sigma_i(t)))dx \le 0,\ t \ge t_1, \end{aligned} \tag{12}$$

where

$$\tilde{Z}(t) = \frac{1}{|G|}\int_G z(x,t)dx, \tag{13}$$
$$|G| = \int_G dx$$

for some $t_1 \geq t_0$. From Green's formula it follows that

$$\int_G \Delta z_r(x,t)dx = \int_{\partial G}\left(\frac{\partial z_r(x,t)}{\partial \nu}\right)dS$$
$$= -\int_{\partial G}\Big(\mu_r(x,t)z_r(x,t)\Big)dS \leq 0,\ t \geq t_1. \tag{14}$$

Analogously we obtain

$$\int_G \Delta z_r(x,\tau_i(t)) = -\int_{\partial G}\Big(\mu_r(x,\tau_i(t))z_r(x,\tau_i(t))\Big)dS \leq 0,\ t \geq t_1. \tag{15}$$

Applying of Jensen's inequality, we have

$$\frac{1}{|G|}\int_G q_i(x,t)\varphi_i(z(x,\sigma_i(t))) \geq q_i(t)\varphi_i\Big(\tilde{Z}(\sigma_i(t))\Big),\ t \geq t_1. \tag{16}$$

Combining (12)–(16) yields

$$\frac{d}{dt}\left(\tilde{Z}(t) + \sum_{i=1}^{l} h_i(t)\tilde{Z}(\rho_i(t))\right) + \sum_{i=1}^{m} q_i(t)\varphi_i\Big(\tilde{Z}(\sigma_i(t))\Big) \leq 0,\ t \geq t_1.$$

Hence, $\tilde{Z}(t)$ is a positive solution of (1) on $[t_1,\infty)$. This contradicts the hypothesis and completes the proof.

Applying the results of [10, 11], we obtain the following corollaries.

The following notation will be used :

$$U_1(t) = U(t) + \sum_{i=1}^{l} h_i(t)U(\rho_i(t)),$$
$$U_2(t) = \tilde{U}(t) + \sum_{i=1}^{l} h_i(t)\tilde{U}(\rho_i(t)), \tag{17}$$

where $U(t) = K_\Phi \int_G u(x,t)\Phi(x)dx$ and $\tilde{U}(t) = \frac{1}{|G|}\int_G u(x,t)dx$.

Corollary 1. *Assume that* (H1)–(H4) *hold, and that* :

(H5) $\sum_{i=1}^{l} h_i(t) \leq 1$;

(H6) $t \le \rho_i(t)$ $(i = 1, 2, \ldots, l)$;

(H7) *there is a integer* $j_0 \in \{1, 2, \ldots, m\}$ *such that* $\varphi_{j_0}(s_1 s_2) \ge \tilde{\varphi}_{j_0 1}(s_1)\tilde{\varphi}_{j_0 2}(s_2)$ *for* $s_1 \ge 0$, $s_2 > 0$, *where* $\tilde{\varphi}_{j_0 1}(s_1) \ge 0$, $\tilde{\varphi}_{j_0 2}(s_2) > 0$ *and* $\tilde{\varphi}_{j_0 2}(s_2)$ *is nondecreasing for* $s_2 > 0$.

If every eventually positive solution $y(t)$ *of the differential inequality*

$$y'(t) + q_{j_0}(t)\tilde{\varphi}_{j_0 1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_{j_0}(t))\right)\tilde{\varphi}_{j_0 2}(y(\sigma_{j_0}(t))) \le 0 \tag{18}$$

satisfies $\lim_{t\to\infty} y(t) = 0$, *then every solution* u *of the problem* (E), (B_1) *is oscillatory in* Ω *or satisfies*

$$\lim_{t\to\infty} U_1(t) = 0. \tag{19}$$

Proof. Suppose that the assertion is not true, that is, that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t), \ldots, u_N(x,t)\}^T$ which does not satisfy (19). Arguing as in the proof of Theorem 1, we observe that the inequality (11) holds for some $t_1 \ge t_0$. Setting

$$Y(t) = Z(t) + \sum_{i=1}^{l} h_i(t) Z(\rho_i(t)),$$

then we see that

$$Y'(t) \le -q_{j_0}(t)\varphi_{j_0}\Big(Z(\sigma_{j_0}(t))\Big) \le 0, \ t \ge t_1$$

for some $j_0 \in \{1, 2, \ldots, m\}$ and hence $Y(t)$ is nonincreasing and $Y(t) > 0$ for $t \ge t_1$. Therefore we obtain

$$Z(t) \ge \left[1 - \sum_{i=1}^{l} h_i(t)\right] Y(t), \ t \ge t_1.$$

We easily see that

$$Y'(t) + q_{j_0}(t)\varphi_{j_0}\left(\left(1 - \sum_{i=1}^{l} h_i(\sigma_{j_0}(t))\right) Y(\sigma_{j_0}(t))\right) \le 0, \ t \ge t_1.$$

Using the hypothesis (H7), we have

$$Y'(t) + q_{j_0}(t)\tilde{\varphi}_{j_0 1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_{j_0}(t))\right)\tilde{\varphi}_{j_0 2}\Big(Y(\sigma_{j_0}(t))\Big) \le 0, \ t \ge t_1.$$

From (17) it follows that

$$0 \le |U_1(t)| \le K_\Phi \int_G |u(x,t)|\Phi(x)dx + \sum_{i=1}^{l} h_i(t) K_\Phi \int_G |u(x,\rho_i(t))|\Phi(x)dx$$
$$= Z(t) + \sum_{i=1}^{l} h_i(t) Z(\rho_i(t)) = Y(t).$$

Hence, $Y(t)$ is a positive solution of (18) on $[t_1,\infty)$ which does not satisfy $\lim_{t\to\infty} Y(t) = 0$. This contradicts the hypothesis and completes the proof.

Corollary 2. *Assume that* (H1)–(H7) *hold. If every eventually positive solution* $y(t)$ *of the differential inequality* (18) *satisfies* $\lim_{t\to\infty} y(t) = 0$*, then every solution* u *of the problem* (E), (B$_2$) *is oscillatory in* Ω *or satisfies*

$$\lim_{t\to\infty} U_2(t) = 0. \tag{20}$$

In the linear case we consider the system

$$(\mathrm{E_L})\quad \frac{\partial}{\partial t}\left(u_r(x,t) + \sum_{i=1}^{l} h_i(t) u_r(x,\rho_i(t))\right)$$
$$-a_r(t)\Delta u_r(x,t) - \sum_{i=1}^{k} b_{ri}(t)\Delta u_r(x,\tau_i(t))$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{N} q_{rji}(x,t) u_j(x,\sigma_i(t)) = 0,$$
$$(x,t)\in\Omega,\ r = \{1,2,\dots,N\}.$$

By the same arguments as were used in Theorems 1–2 and Corollaries 1–2, we obtain the following theorems.

Theorem 3.(Linear case) *Assume that* (H1)–(H3), (H5) *and* (H6) *hold. If the differential inequality*

$$y'(t) + q_{j_0}(t)\left(1 - \sum_{i=1}^{l} h_i(\sigma_{j_0}(t))\right) y(\sigma_{j_0}(t)) \le 0 \tag{21}$$

has no eventually positive solution, then every solution u *of the problem* (E$_\mathrm{L}$), (B$_1$) *is oscillatory in* Ω.

Theorem 4. (Linear case) *Assume that* (H1)–(H3), (H5) *and* (H6) *hold. If the differential inequality* (21) *has no eventually positive solution, then every solution* u *of the problem* (E$_\mathrm{L}$), (B$_2$) *is oscillatory in* Ω.

3. Oscillation criteria for the system

In this section we can derive the oscillation results for the systems (E), (B_i) ($i = 1, 2$) and (E_L), (B_i) ($i = 1, 2$).

By combining the results obtained in Section 2 and Kitamura and Kusano [1], we obtain the following theorems.

Theorem 5. *Assume that* (H1)–(H7) *hold. Every solution u of the problem* (E), (B_1) *is oscilllatory in* Ω *or satisfies* (19) *if*

$$\int_{R[\sigma_{j_0}]} q_{j_0}(t)\tilde{\varphi}_{j_0 1}\left(1 - \sum_{i=1}^{l} h_i(\sigma_{j_0}(t))\right) dt = \infty, \tag{22}$$

where $R[\sigma_{j_0}] = \{t \in [0, \infty); 0 \le \sigma_{j_0}(t) \le t\}$.

Theorem 6. *Assume that* (H1)–(H7) *hold. If* (22) *holds, then every solution u the problem* (E), (B_2) *is oscilllatory in* Ω *or satisfies* (20).

Using the results of Section 2 and Koplatadze and Čanturija [2], we establish the following.

Theorem 7. (Linear case) *Assume that* (H1)-(H3), (H5), (H6) *and the following:*

(H8) $\sigma_{j_0}(t) \le t$ *and* $\sigma_{j_0}(t)$ *is nondecreasing on* $[t_0, \infty)$ *for some* $t_0 > 0$ *and some* $j_0 \in \{1, 2, \ldots, m\}$.

Every solution u of the problem (E_L), (B_1) *is oscillatory in* Ω *if*

$$\liminf_{t\to\infty} \int_{\sigma_{j_0}(t)}^{t} q_{j_0}(s)\left(1 - \sum_{i=1}^{l} h_i(\sigma_{j_0}(t))\right) ds > \frac{1}{e}. \tag{23}$$

Theorem 8. (Linear case) *Assume that* (H1)–(H3), (H5), (H6) *and* (H8) *hold. If* (23) *holds, then every solution u of the problem* (E_L), (B_2) *is oscillatory in* Ω.

A special case of the system (E), (B_1) is

$$\frac{\partial}{\partial t}\Big(u_r(x,t) + hu_r(x,t+\rho)\Big) - a_r(t)\Delta u_r(x,t) \tag{24}$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{N} q_{rji}(x,t)\Big(u_j(x,t-\sigma)\Big)^{\gamma_j} = 0,$$

$$(x,t) \in (0,L) \times (0,\infty),$$

$$u_r(0,t) = u_r(L,t) = 0,\ t > 0,\ r = \{1, 2, \ldots, N\}, \tag{25}$$

where $h(<1)$, ρ, σ are positive constants and γ_j $(j=1,2,\dots,N)$ are the quotients of odd integers.

Corollary 3. *If*

$$\int^{\infty} q_{j_0}(t)dt = \infty,$$

then every solution of the problem (24), (25) *is oscillatory in* $(0,L)\times(0,\infty)$ *or satisfies* $\lim_{t\to\infty} U_1(t) = 0$.

Example 1. We consider the system of parabolic equations

$$\begin{cases} \dfrac{\partial}{\partial t}\Big(u_1(x,t) + hu_1(x,t+1)\Big) - \left(\dfrac{L}{\pi}\right)^2 \Delta u_1(x,t) \\ \qquad\qquad + \dfrac{3}{4}he^{-\sigma-1}u_1(x,t-\sigma) + \dfrac{1}{4}he^{-\sigma}u_2(x,t-\sigma) = 0, \\ \dfrac{\partial}{\partial t}\Big(u_2(x,t) + hu_2(x,t+1)\Big) - \left(\dfrac{L}{\pi}\right)^2 \Delta u_2(x,t) \\ \qquad\qquad + \dfrac{1}{4}he^{-\sigma-2}u_1(x,t-\sigma) + \dfrac{3}{4}he^{-\sigma-1}u_2(x,t-\sigma) = 0, \\ (x,t)\in(0,L)\times(0,\infty), \end{cases} \tag{26}$$

Here $n=l=m=1$, $N=2$, $h_1(t)=h<1$, $\rho_1(t)=t+1$, $a_1(t)=\left(\frac{L}{\pi}\right)^2$, $q_{111}(x,t)=\frac{3}{4}he^{-\sigma-1}$, $q_{121}(x,t)=\frac{1}{4}he^{-\sigma}$, $\sigma_1(t)=t-\sigma$, $a_2(t)=\left(\frac{L}{\pi}\right)^2$, $q_{211}(x,t)=\frac{1}{4}he^{-\sigma-2}$, $q_{221}(x,t)=\frac{3}{4}he^{-\sigma-1}$ and $\gamma_1=\gamma_2=1$. It is easy to see that $q(t)=\frac{h}{4}e^{-\sigma}\left(\frac{3}{e}-1\right)$. It is readily seen that

$$\int_{t-\sigma}^{t} \frac{h}{4}e^{-\sigma}\left(\frac{3}{e}-1\right)(1-h)ds \le \frac{3h}{4e}\cdot\frac{\sigma}{e^{\sigma}} \le \frac{1}{e},$$

and therefore (23) does not hold. Hence, Theorem 7 is not applicable to (26). Since

$$\int^{\infty} \frac{h}{4}e^{-\sigma}\left(\frac{3}{e}-1\right)dt = \infty,$$

from Corollary 3 it follows that every nonoscillatory solution of the problem (25), (26) satisfies (19). In fact $u_1(x,t)=e^{-t}\sin\left(\frac{\pi}{L}\right)x$, $u_2(x,t)=e^{-t-1}\sin\left(\frac{\pi}{L}\right)x$ are nonoscillatory solutions which satisfy (19).

Example 2. Consider the system of parabolic equations

$$\begin{cases} \dfrac{\partial}{\partial t}\Big(u_1(x,t)+\dfrac{1}{2}u_1(x,t+\pi)\Big)-2\Delta u_1(x,t)-\Delta u_1\left(x,t-\dfrac{3}{2}\pi\right) \\ \qquad\qquad +u_1(x,t-\pi)+u_2(x,t-\pi) \\ \qquad\qquad\qquad\qquad +\dfrac{5}{2}u_1\left(x,t-\dfrac{\pi}{2}\right)+u_2\left(x,t-\dfrac{\pi}{2}\right)=0, \\ \dfrac{\partial}{\partial t}\Big(u_2(x,t)+\dfrac{1}{2}u_2(x,t+\pi)\Big)-\Delta u_2(x,t)-4\Delta u_2\left(x,t-\dfrac{3}{2}\pi\right) \\ \qquad\qquad +\dfrac{1}{2}u_1(x,t-\pi)+3u_2(x,t-\pi) \\ \qquad\qquad\qquad\qquad +2u_1\left(x,t-\dfrac{\pi}{2}\right)+4u_2\left(x,t-\dfrac{\pi}{2}\right)=0, \\ (x,t)\in(0,\pi)\times(0,\infty), \end{cases} \tag{27}$$

$$u_i(0,t)=u_i(\pi,t)=0,\ t>0,\ i=1,2. \tag{28}$$

Here $n=1$, $l=k=1$, $m=N=2$, $h_1(t)=\frac{1}{2}$, $\rho_1(t)=t+\pi$, $a_1(t)=2$, $b_{11}(t)=1$, $\tau_1(t)=t-\frac{3}{2}\pi$, $q_{111}(x,t)=1$, $q_{121}(x,t)=1$, $q_{112}(x,t)=\frac{5}{2}$, $q_{122}(x,t)=1$, $\sigma_1(t)=t-\pi$, $\sigma_2(t)=t-\frac{\pi}{2}$, $a_2(t)=1$, $b_{21}(t)=4$, $q_{211}(x,t)=\frac{1}{2}$, $q_{221}(x,t)=3$, $q_{212}(x,t)=2$, $q_{222}(x,t)=4$. It is easy to see that $q_1(t)=q_2(t)=\frac{1}{2}$, and the conditions of Theorem 5 are fullfilled. Thus every solutions of the problem (27), (28) are oscillatory in $(0,\pi)\times(0,\infty)$. In fact, $u_1(x,t)=\sin x\cos t$, $u_2(x,t)=\sin x\sin t$ are such solutions.

Example 3. Consider the system of parabolic equations

$$\begin{cases} \dfrac{\partial}{\partial t}\Big(u_1(x,t)+\dfrac{e}{3}u_1(x,t+1)\Big)-\dfrac{1}{3}\Delta u_1(x,t) \\ \qquad\qquad\qquad +\dfrac{2}{3e}u_1(x,t-1)+\dfrac{2}{3e}u_2(x,t-1)=0, \\ \dfrac{\partial}{\partial t}\Big(u_2(x,t)+\dfrac{e}{3}u_2(x,t+1)\Big)-\dfrac{1}{6}\Delta u_2(x,t) \\ \qquad\qquad\qquad +\dfrac{1}{3e}u_1(x,t-1)+\dfrac{1}{e}u_2(x,t-1)=0, \\ (x,t)\in(0,\pi)\times(0,\infty), \end{cases} \tag{29}$$

$$\frac{\partial}{\partial x}u_i(0,t)=\frac{\partial}{\partial x}u_i(\pi,t)=0,\ t>0,\ i=1,2. \tag{30}$$

Here $n=1$, $l=k=m=1$, $N=2$, $h_1(t)=\frac{e}{3}$, $\rho_1(t)=t+1$, $a_1(t)=\frac{1}{3}$, $a_2(t)=\frac{1}{6}$, $q_{111}(x,t)=\frac{2}{3e}$, $q_{121}(x,t)=\frac{2}{3e}$, $\sigma_1(t)=t-1$, $q_{211}(x,t)=\frac{1}{3e}$ and $q_{221}(x,t)=\frac{1}{e}$. Since

$$\int_{t-1}^{t}\frac{1}{3e}\left(1-\frac{e}{3}\right)ds=\frac{1}{3e}\left(1-\frac{e}{3}\right)\le\frac{1}{e},$$
$$\int^{\infty}\frac{1}{3e}\left(1-\frac{e}{3}\right)dt=\infty,$$

Theorem 8 does not apply but Theorem 6 does. Theorefore every solutions of the problem (29), (30) are oscillatory in $(0,\pi)\times(0,\infty)$ or satisfy (20). For example $u_1(x,t)=e^{-t}\cos^2 x$, $u_2(x,t)=e^{-t}\sin^2 x$ are such solutions.

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