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UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we present a different and very simple technique to handle various uniqueness problems involving three small entire functions. It also gives a new additional insight into such problems.

1. Introduction and the Main Results

In this paper, the term 'meromorphic' will always mean meromorphic function in the whole complex plane $\mathbb{C}$. Let $f(z)$ be a non-constant meromorphic function.We shall use the following standard notations of value distribution theory $T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \ldots$ (see [6], [8]). We denote by $S(r,f)$ any function satisfying $S(r,f) = o\{T(r,f)\}$, as $r \to +\infty$, possibly outside of finite measure. Let a be an arbitrary complex number and k be a positive integer. We denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$ or $N_{k)}(r,a,f)$ the counting function for the zeros of $f(z)-a$ with multiplicity $\le k$ and by $\overline{N}_{k)}(r,a,f)$ the counting function for the zeros of $f(z)-a$ with multiplicity $\le k$, counting only once. We use $\overline{E}_{k)}(a,f) = \{z/$ zeros of $f(z)-a$ with multiplicity $\le k$, where each zero being counted only once$\}$. Further we denote by $N_2\left(r, \frac{1}{f-a}\right)$ the counting function for the zeros of $f(z)-a$ where a simple zero is counted once and a mutiple zero is counted twice. Similarly, we can define $N_2(r,f)$. We denote by $\overline{N}_2\left(r, \frac{1}{f-a}\right)$ the counting function for the zeros of $f(z)-a$ where a simple zero and multiple zero is counted only once. Similarly, we can define $\overline{N}_2(r,f)$.

We say f and g share the value a CM if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities. We define

$$\Theta_2(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_2(r,a,f)}{T(r,f)}, \delta_2(a,f) = 1 - \limsup_{r\to\infty} \frac{N_2(r,a,f)}{T(r,f)}$$

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Definition 1. Any expression of the type

$$P(f)=\sum_{i=1}^{n}\alpha_i(z)f^{n_{i_0}}(f')^{n_{i_1}}(f'')^{n_{i_2}}\cdots(f^{(m)})^{n_{i_m}},$$

is called a differential polynomial in f of degree $\overline{d}(P)$, lower degree $\underline{d}(P)$ and weight Γ_P where for each $i=1,2\ldots n$, $n_{i_0},n_{i_1}\ldots,n_{i_m}$ are non-negative integers, $\alpha_i=\alpha_i(z)$ are meromorphic functions satisfying $T(r,\alpha_i)=S(r,f)$ and

$$\overline{d}(P)=\max\Big\{\sum_{j=0}^{m}n_{i_j}:1\le i\le n\Big\},\quad \underline{d}(P)=\min\Big\{\sum_{j=0}^{m}n_{i_j}:1\le i\le n\Big\}$$

and

$$\Gamma_P=\max\Big\{\sum_{j=0}^{m}(j+1)n_{i_j}:1\le i\le n\Big\}.$$

In 1989, H. X. Yi [5] proved the following theorem .

Theorem A. *Let $f_1(z)$ and $f_2(z)$ be non-constant meromorphic functions, $b_j(j=1,2,3)$ be three distinct finite non-zero complex numbers, k be a positive integer or ∞, and n be a positive integer satisfying*

$$\overline{E}_{k)}(b_j,f_1^{(n)})=\overline{E}_{k)}(b_j,f_2^{(n)})$$

Furthermore, let
$C_i=3(k+1)\delta(0,f_i)+(2nk+3n+k+1)\Theta(\infty,f_i)-(2nk+3n+3k+4)\ (i=1,2).$
If

$$\min\{C_1,C_2\}\ge 0,$$
$$\max\{C_1,C_2\}>0$$

then $f_1(z)\equiv f_2(z)$.

In 2007, Anupama J. Patil [1] proved the following theorem which generalises the above result to differential polynomials in f and also improve the conditions in the above theorem.

Theorem B. *Let $f_1(z)$ and $f_2(z)$ be two non-constant meromorphic functions and $P(f_1)$ and $P(f_2)$ be non-constant differential polynomials in f_1 and f_2 respectively. Let $\alpha_j(\not\equiv 0,\infty)(j=1,2,3)$ be three non-zero distinct entire small functions of $P(f_1)$ and $P(f_2)$, $k_1\ge k_2\ge k_3$ be positive integers or ∞ and n be a positive integer satisfying*

$$\overline{E}_{k_i)}(\alpha_j,P(f_1))=\overline{E}_{k_i)}(\alpha_j,P(f_2)),\quad i,j=1,2,3.$$

Furthermore, let
$D_i=3(k_1+1)(k_3+1)\delta_{m+1}(a,f_i)-(\overline{d}+Qc_i)[6(k_1+1)+4k_1(k_3+1)]\quad (i=1,2),$

where $\overline{d} = \overline{d}(P(f_1)(z)) = \overline{d}(P(f_2)(z)) = \max\left\{\sum_{j=0}^{m} n_{i_j}; 1 \le i \le n\right\}$,
$Q = \max\{n_{i_1} + 2n_{i_2} + 3n_{i_3} + \cdots\cdots + mn_{i_m}; 1 \le i \le n\}$, *$m$ = order of the highest derivative of f occurring in P and* $c_i = 1 - \Theta(\infty, f_i)$. *If*

$$\min(D_1, D_2) \ge 0 \tag{1}$$

$$\max(D_1, D_2) > 0 \tag{2}$$

then $P(f_1)(z) \equiv P(f_2)(z)$.

As a corollary to the above theorem, Anupama J. Patil [1] improve Theorem A by considering three non-zero small entire functions of $f^{(k)}$ instead of three non-zero complex numbers.

Corollary B. *Let $f_1(z)$ and $f_2(z)$ be non-constant meromorphic functions, $b_j (j = 1,2,3)$ be three non-zero entire small functions of $f_1^{(m)}$ and $f_2^{(m)}$, k be a positive integer or ∞, and m be a positive integer satisfying*

$$\overline{E}_{k)}(b_j, f_1^{(m)}) = \overline{E}_{k)}(b_j, f_2^{(m)})$$

Furthermore, let $C_i = 3(k+1)\delta(0, f_i) + (2mk+3m+k+1)\Theta(\infty, f_i) - (2mk+3m+3k+4)$ $(i = 1,2)$. *If*

$$\min\{C_1, C_2\} \ge 0$$

$$\max\{C_1, C_2\} > 0$$

then $f_1(z) \equiv f_2(z)$.

In this paper,we study the uniqueness problems on meromorphic functions concerning differential polynomials that share three entire small functions as an application of Theorem B. Here our techniques employed are much different and relatively simple and lead to several significant results.They also throw new light on such topics.

MAIN RESULTS

Theorem 1. *Let f and g be two non-constant meromorphic functions and $\alpha_j (\not\equiv 0, \infty) (j = 1,2,3)$ be three non-zero distinct entire small functions, k be a positive integer or ∞ satisfying*

$$\overline{E}_{k)}(\alpha_j, f^n(f^p - 1)f') = \overline{E}_{k)}(\alpha_j, g^n(g^p - 1)g'), \quad j = 1,2,3,$$

where n and p are positive integers, then either $f \equiv g$ or

$$g = \left[\frac{(n+p+1)(h^{n+1} - 1)}{(n+1)(h^{n+p+1} - 1)}\right]^{1/p}, \quad f = \left[\frac{(n+p+1)h^p(h^{n+1} - 1)}{(n+1)(h^{n+p+1} - 1)}\right]^{1/p}$$

where h is a non-constant meromorphic function.

Letting $k \to \infty$ and $p = 1$ in the above theorem, we have the following result.

Corollary 1.1. *Let f and g be two non-constant meromorphic functions and $\alpha_j (\not\equiv 0, \infty)(j = 1,2,3)$ be three non-zero distinct entire small functions satisfying*

$$\overline{E}(\alpha_j, f^n(f-1)f') = \overline{E}(\alpha_j, g^n(g-1)g'), \quad j = 1,2,3,$$

where n is a positive integer, then either $f \equiv g$ or $g = [(n+2)(h^{n+1}-1)]/[(n+1)(h^{n+2}-1)]$, $f = [(n+2)h(h^{n+1}-1)]/[(n+1)(h^{n+2}-1)]$, where h is a non-constant meromorphic function.

Theorem 2. *Let f and g be two non-constant meromorphic functions and $\alpha_j (\not\equiv 0, \infty)(j = 1,2,3)$ be three non-zero distinct entire small functions, k be a positive integer or ∞ and p is a positive integer satisfying*

$$\overline{E}_{k)}(\alpha_j, f^n(f-1)^p f') = \overline{E}_{k)}(\alpha_j, g^n(g-1)^p g'), \quad j = 1,2,3,$$

(i) *if $p = 1$ and n is a positive integer, then either $f \equiv g$ or $g = [(n+2)(h^{n+1}-1)]/[(n+1)(h^{n+2}-1)]$, $f = [(n+2)h(h^{n+1}-1)]/[(n+1)(h^{n+2}-1)]$, where h is a non-constant meromorphic function.*

(ii) *if $p = 2$ and $n \geq 3$, then $f \equiv g$.*

(iii) *if $p > 2$ and n is a positive integer, then*

$$f^{n+1} \sum_{l=0}^{p} \frac{(-1)^l C_p^l}{n+p-l+1} f^{p-l} \equiv g^{n+1} \sum_{l=0}^{p} \frac{(-1)^l C_p^l}{n+p-l+1} g^{p-l}.$$

Theorem 3. *Let f and g be two non-constant meromorphic functions and $\alpha_j (\not\equiv 0, \infty)(j = 1,2,3)$ be three non-zero distinct entire small functions, k be a positive integer or ∞ satisfying*

$$\overline{E}_{k)}(\alpha_j, f^n f') = \overline{E}_{k)}(\alpha_j, g^n g'), \quad j = 1,2,3$$

where n is positive integer, then either $f \equiv g$ or $h^{n+1} - 1 = 0$, where h is a non-constant meromorphic function.

Letting $k \to \infty$, we have the following result.

Corollary 3.1. *Let f and g be two non-constant meromorphic functions and $\alpha_j (\not\equiv 0, \infty)(j = 1,2,3)$ be three non-zero distinct entire small functions satisfying*

$$\overline{E}(\alpha_j, f^n f') = \overline{E}(\alpha_j, g^n g'), \quad j = 1,2,3$$

where n is a positive integer, then either $f \equiv g$ or $h^{n+1} - 1 = 0$, where h is a non-constant meromorphic function.

2. Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1.([2]) *Let $f(z)$ be a non-constant meromorphic function and let*

$$R(f)=\sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r,R(f)) = dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.2.([6]) *Suppose that $f(z)$ is a non-constant meromorphic function in the complex plane and $a\,(\in \mathbb{C}\cup\{\infty\})$ is any complex number. Then*

$$\sum_{a}\Theta(a,f)\leq 2.$$

Lemma 2.3.([4]) *Let*

$$Q(\omega)=(n-1)^2(\omega^n-1)(\omega^{n-2}-1)-n(n-2)(\omega^{n-1}-1)^2,$$

then

$$Q(\omega)=(\omega-1)^4(\omega-\beta_1)(\omega-\beta_2)\ldots(\omega-\beta_{2n-6}).$$

where $\beta_j \in C-[0,1]$ $(j=1,2,\ldots,2n-6)$, which are distinct respectively.

Lemma 2.4.([3]) *Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_0$, where $a_n(\neq 0)$, $a_{n-1},\ldots,a_0$ are constants. If $f(z)$ is a meromorphic function, then*

$$T(r,P(f))=nT(r,f)+S(r,f).$$

Lemma 2.5.([7]) *Let f be a meromorphic function and $P(f)$ be a differential polynomial in f. Then*

$$T(r,P(f))=Q\overline{N}(r,f)+\overline{d}(P)T(r,f)+S(r,f)$$

$\overline{d}(P)=\max\left\{\sum_{j=0}^{m} n_{i_j};1\leq i\leq n\right\}, Q=\max\left\{n_{i_1}+2n_{i_2}+3n_{i_3}+\cdots\cdots+mn_{i_m};1\leq i\leq n\right\}.$

Theorem 2.1.([7]) *Let f be a meromorphic function and let $P(f)$ be a non-constant differential polynomial in f such that every term in $P(f)$ contains at least one of the derivatives of f.Let m*

be the order of the highest derivative of f occuring in $P(f)$. If k is a positive integer $\le m$ such that every term in $P(f)$ contains atleast one of $f', f'', \dots, f^{(k)}$, then

$$d_k T(r,f) \le d_k N_{k+1}(r,a,f) + \overline{N}\left(r, \frac{1}{P(f)-b}\right) + \overline{N}\left(r, \frac{1}{P(f)-c}\right) + S(r,f)$$

where $a \in C$, $b(\not\equiv 0)$, $c(\not\equiv 0)$ are distinct meromorphic functions satisfying $T(r,b) = S(r,P)$, $T(r,c) = S(r,P)$ and $d_k = \min\left\{\sum_{j=1}^{k} n_{i_j} : 1 \le i \le n\right\}$.

3. Proof of Theorems

The proof of Theorem B is in [7]. However, we give the proof for the sake of completeness.

Proof of Theorem B. Let

$$P(f_1)(z) = P_1(z)$$
$$P(f_2)(z) = P_2(z).$$

From the Theorem 2.1, we have for the meromorphic function f_1, its differential polynomial P_1, any element $a \in C$ and two distinct non-zero small functions α, β of $P(f_1)$,

$$d_k T(r,f_1) \le d_k N_{m+1}(r,a,f_1) + \overline{N}\left(r, \frac{1}{P_1-\alpha}\right) + \overline{N}\left(r, \frac{1}{P_1-\beta}\right) + S(r,f_1)$$

Since $d_k \ge 1$,

$$T(r,f_1) \le N_{m+1}(r,a,f_1) + \overline{N}\left(r, \frac{1}{P_1-\alpha}\right) + \overline{N}\left(r, \frac{1}{P_1-\beta}\right) + S(r,f_1)$$

where m is the order of the highest derivative of f occuring in P_1. Since there are C_2^3 ways of selecting one element a and two elements from $\alpha_1, \alpha_2, \alpha_3$. For a given $a \in C$ and three non-zero distinct entire small functions $\alpha_1, \alpha_2, \alpha_3$ of P_1 and P_2,we have

$$T(r,f_1) \le N_{m+1}(r,a,f_1) + \overline{N}\left(r, \frac{1}{P_1-\alpha_1}\right) + \overline{N}\left(r, \frac{1}{P_1-\alpha_2}\right) + S(r,f_1)$$
$$T(r,f_1) \le N_{m+1}(r,a,f_1) + \overline{N}\left(r, \frac{1}{P_1-\alpha_2}\right) + \overline{N}\left(r, \frac{1}{P_1-\alpha_3}\right) + S(r,f_1)$$
$$T(r,f_1) \le N_{m+1}(r,a,f_1) + \overline{N}\left(r, \frac{1}{P_1-\alpha_1}\right) + \overline{N}\left(r, \frac{1}{P_1-\alpha_3}\right) + S(r,f_1)$$

adding all the above equations, we get

$$\begin{aligned} 3T(r,f_1) &\le 3N_{m+1}(r,a,f_1) + 2\sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{P_1-\alpha_i}\right) + S(r,f_1) \\ &\le 3\left(1-\delta_{m+1}(a,f_1)\right) T(r,f_1) + 2\sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{P_1-\alpha_i}\right) + S(r,f_1) \end{aligned}$$

i.e.,

$$\begin{aligned}
3\delta_{m+1}(a,f_1)T(r,f_1) &\le 2\sum_{i=1}^{3}\overline{N}\left(r,\frac{1}{P_1-\alpha_i}\right)+S(r,f_1)\\
&\le 2\sum_{i=1}^{3}\left[\frac{k_i}{k_i+1}\overline{N}_{k_i)}\left(r,\frac{1}{P_1-\alpha_i}\right)+\frac{1}{k_i+1}N\left(r,\frac{1}{P_1-\alpha_i}\right)\right]+S(r,f_1)\\
&\le 2\sum_{i=1}^{3}\frac{k_i}{k_i+1}\overline{N}_{k_i)}\left(r,\frac{1}{P_1-\alpha_i}\right)+2\sum_{i=1}^{3}\frac{1}{k_i+1}N\left(r,\frac{1}{P_1-\alpha_i}\right)+S(r,f_1)\\
&\le 2\frac{k_1}{k_1+1}\sum_{i=1}^{3}\overline{N}_{k_i)}\left(r,\frac{1}{P_1-\alpha_i}\right)+\frac{6}{k_3+1}T(r,P_1)+S(r,f_1).
\end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned}
T(r,P_1) &= Q\overline{N}(r,f_1)+\overline{d}(P_1)T(r,f_1)+S(r,f_1)\\
&= Q\big(1-\Theta(\infty,f_1)\big)\,T(r,f_1)+\overline{d}(P_1)T(r,f_1)+S(r,f_1).
\end{aligned}$$

Therefore

$$3\delta_{m+1}(a,f_1)T(r,f_1)\le 2\frac{k_1}{k_1+1}\sum_{i=1}^{3}\overline{N}_{k_i)}\left(r,\frac{1}{P_1-\alpha_i}\right)+\frac{6(\overline{d}+Q(1-\Theta(\infty,f_1)))}{k_3+1}T(r,f_1)+S(r,f_1)$$

i.e., $$d_1T(r,f_1)\le 2k_1(k_3+1)\sum_{i=1}^{3}\overline{N}_{k_i)}\left(r,\frac{1}{P_1-\alpha_i}\right)+S(r,f_1)$$

where

$$d_i=3\delta_{m+1}(a,f_i)(k_1+1)(k_3+1)-6(\overline{d}+Q(1-\Theta(\infty,f_i)))(k_1+1),\quad (i=1,2).$$

Similarly, we have

$$d_2T(r,f_2)\le 2k_1(k_3+1)\sum_{i=1}^{3}\overline{N}_{k_i)}\left(r,\frac{1}{P_2-\alpha_i}\right)+S(r,f_2).$$

Adding the above two equations, we get

$$d_1T(r,f_1)+d_2T(r,f_2)\le 2k_1(k_3+1)\sum_{i=1}^{3}\left[\overline{N}_{k_i)}\left(r,\frac{1}{P_1-\alpha_i}\right)+\overline{N}_{k_i)}\left(r,\frac{1}{P_2-\alpha_i}\right)\right]+S(r,f_1)+S(r,f_2)$$

Since by hypothesis,

$$\overline{E}_{k_i)}\big(\alpha_j,P(f_1)\big)=\overline{E}_{k_i)}\big(\alpha_j,P(f_2)\big)\qquad (i,j=1,2,3)$$

so that

$$\overline{N}_{k_i)}\left(r,\frac{1}{P_1-\alpha_i}\right)=\overline{N}_{k_i)}\left(r,\frac{1}{P_2-\alpha_i}\right)=N_0^{k_i)}(r,\alpha_i).$$

Thus

$$d_1 T(r,f_1)+d_2 T(r,f_2) \le 2k_1(k_3+1)\sum_{i=1}^{3}\left[\overline{N}_{k_i)}\left(r,\frac{1}{P_1-\alpha_i}\right)+\overline{N}_{k_i)}\left(r,\frac{1}{P_2-\alpha_i}\right)\right]+S(r,f_1)+S(r,f_2)$$

$$d_1 T(r,f_1)+d_2 T(r,f_2) \le 4k_1(k_3+1)\sum_{i=1}^{3} N_0^{k_i)}(r,\alpha_i)+S(r,f_1)+S(r,f_2). \quad (3)$$

Suppose,

$$P_1(z) \not\equiv P_2(z). \quad (4)$$

Then under the assumption that P_1 and P_2 are distinct, it follows that for any $\alpha \in S(P_1)\bigcap S(P_2)-\{0,\infty\}$, each common zero of $P_1-\alpha$ and $P_2-\alpha$ is a zero of P_1-P_2. Since $\alpha_1,\alpha_2,\alpha_3$ are distinct, we have

$$\begin{aligned}\sum_{i=1}^{3} N_0^{k_i)}(r,\alpha_i) &\le N\left(r,\frac{1}{P_1-P_2}\right) \le T(r,P_1-P_2)\\ &\le T(r,P_1)+T(r,P_2)+O(1)\\ \sum_{i=1}^{3} N_0^{k_i)}(r,\alpha_i) &\le (\overline{d}+Q(1-\Theta(\infty,f_1)))T(r,f_1)+S(r,f_1)\\ &\quad +(\overline{d}+Q(1-\Theta(\infty,f_2)))T(r,f_2)+S(r,f_2) \quad (5)\end{aligned}$$

From equations (3) and (5) , we get

$$D_1 T(r,f_1)+D_2 T(r,f_2) \le S(r,f_1)+S(r,f_2) \quad (6)$$

where for $i=1,2$, $D_i = d_i - 4k_1(k_3+1)(\overline{d}+Q(1-\Theta(\infty,f_i)))$ which is required D_i mentioned earlier. From conditions (1), (2) the above inequility (6) is not possible. Thus our assumption (4) is not true and hence we must have

$$P_1(z) \equiv P_2(z)$$
$$i.e., P(f_1)(z) \equiv P(f_2)(z).$$

Proof of Theorem 1. First, we need to show that $f^n(f^p-1)f' \equiv g^n(g^p-1)g'$. Consider $k_1 = k_2 = k_3 = k$ and $a=0$. We have $\overline{d}\{f^n(f^p-1)f'\} = \overline{d}\{g^n(g^p-1)g'\} = n+p+1$, $Q=1$ and $m=1$. Therefore

$$\min\{D_f,D_g\} \ge 0, \quad \max\{D_f,D_g\} > 0 \quad ,$$

where

$$D_f = 3(k+1)^2\delta_2(0,f)-(n+p+2)(k+1)(6+4k)+(k+1)(6+4k)\Theta(\infty,f)$$

and

$$D_g = 3(k+1)^2\delta_2(0,g) - (n+p+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,g).$$

By Theorem B, we get

$$f^n(f^p-1)f' \equiv g^n(g^p-1)g'$$
$$(F^*)' \equiv (G^*)'$$

Then

$$F^* \equiv G^* + c, \quad c \text{ is a constant} \tag{7}$$

where

$$F^* = \frac{f^{n+p+1}}{n+p+1} - \frac{f^{n+1}}{n+1}, \quad G^* = \frac{g^{n+p+1}}{n+p+1} - \frac{g^{n+1}}{n+1}.$$

By Lemma 2.1, we have $T(r,F^*) = (n+p+1)T(r,f) + S(r,f)$. Note that

$$\begin{aligned}\overline{N}_2\left(r,\frac{1}{F^*}\right) &= \overline{N}_2\left(r,\frac{1}{f}\right) + \overline{N}_2\left(r,\frac{1}{\frac{f^p}{n+p+1}-\frac{1}{n+1}}\right)\\ &\le \overline{N}_2\left(r,\frac{1}{f}\right) + T\left(r,\frac{1}{\frac{f^p}{n+p+1}-\frac{1}{n+1}}\right)\\ &= \overline{N}_2\left(r,\frac{1}{f}\right) + pT(r,f) + S(r,f).\end{aligned}$$

So,

$$\frac{\overline{N}_2\left(r,\frac{1}{F^*}\right)}{T(r,F^*)} \le \frac{\overline{N}_2\left(r,\frac{1}{f}\right)}{(n+p+1)T(r,f)+S(r,f)} + \frac{pT(r,f)}{(n+p+1)T(r,f)+S(r,f)}$$

Therefore,

$$\begin{aligned}\limsup_{r\to\infty}\frac{\overline{N}_2\left(r,\frac{1}{F^*}\right)}{T(r,F^*)} &\le \frac{\limsup_{r\to\infty}\overline{N}_2(r,\frac{1}{f})/T(r,f)}{n+p+1} + \frac{p}{n+p+1}\\ 1-\Theta_2(0,F^*) &\le \frac{1-\Theta_2(0,f)}{n+p+1} + \frac{p}{n+p+1}\\ i.e, \qquad \Theta_2(0,F^*) &\ge \frac{n}{n+p+1} + \frac{\Theta_2(0,f)}{n+p+1}\end{aligned}$$

Similarly, we have

$$\Theta_2(0,G^*) \ge \frac{n}{n+p+1} + \frac{\Theta_2(0,g)}{n+p+1}$$

Note that $\overline{N}_2(r,F^*) = \overline{N}_2(r,f)$. So,

$$\frac{\overline{N}_2(r,F^*)}{T(r,F^*)} = \frac{\overline{N}_2(r,f)}{(n+p+1)T(r,f)+S(r,f)}$$

$$\limsup_{r\to\infty}\frac{\overline{N}_2(r,F^*)}{T(r,F^*)} = \frac{\limsup_{r\to\infty}\frac{\overline{N}_2(r,f)}{T(r,f)}}{(n+p+1)}$$
$$1-\Theta_2(\infty,F^*) = \frac{1-\Theta_2(\infty,f)}{n+p+1}$$
$$\Theta_2(\infty,F^*) = \frac{n+p}{n+p+1}+\frac{\Theta_2(\infty,f)}{n+p+1}$$

And, by the definition we have,

$$\Theta_2(c,F^*) = 1-\limsup_{r\to\infty}\frac{\overline{N}_2\left(r,\frac{1}{F^*-c}\right)}{T(r,F^*)}$$
$$= 1-\limsup_{r\to\infty}\frac{\overline{N}_2\left(r,\frac{1}{G^*}\right)}{T(r,F^*)}, \qquad \text{by (7)}$$

Since $F^*\equiv G^*+c$, c a constant, so $T(r,F^*)=T(r,G^*)$. Therefore,

$$\Theta_2(c,F^*)=1-\limsup_{r\to\infty}\frac{\overline{N}_2\left(r,\frac{1}{G^*}\right)}{T(r,G^*)}=\Theta_2(0,G^*)$$

We now show that $c=0$ in (7) . Suppose that $c\neq 0$, then

$$\begin{aligned}
&\Theta_2(0,F^*)+\Theta_2(\infty,F^*)+\Theta_2(c,F^*)\\
&\geq \frac{n}{n+p+1}+\frac{\Theta_2(0,f)}{n+p+1}+\frac{n+p}{n+p+1}+\frac{\Theta_2(\infty,f)}{n+p+1}+\frac{n}{n+p+1}+\frac{\Theta_2(0,g)}{n+p+1}\\
&\geq \frac{n}{n+p+1}+\frac{\delta_2(0,f)}{n+p+1}+\frac{n+p}{n+p+1}+\frac{\Theta(\infty,f)}{n+p+1}+\frac{n}{n+p+1}+\frac{\delta_2(0,g)}{n+p+1}, \qquad (8)
\end{aligned}$$

because $\overline{N}_2(r,f)=\overline{N}(r,f)$ and hence

$$\limsup_{r\to\infty}\frac{\overline{N}_2(r,f)}{T(r,f)}=\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}\Rightarrow\Theta_2(\infty,f)=\Theta(\infty,f).$$

Since $\min\{D_f,D_g\}\geq 0$,

$$3(k+1)^2\delta_2(0,f)-(n+p+2)(k+1)(6+4k)+(k+1)(6+4k)\Theta(\infty,f)\geq 0$$

and

$$3(k+1)^2\delta_2(0,g)-(n+p+2)(k+1)(6+4k)+(k+1)(6+4k)\Theta(\infty,g)\geq 0.$$

Therefore

$$\delta_2(0,f)\geq\frac{(n+p+2)(6+4k)}{3(k+1)}-\frac{(6+4k)}{3(k+1)}\Theta(\infty,f) \qquad (9)$$
$$\delta_2(0,g)\geq\frac{(n+p+2)(6+4k)}{3(k+1)}-\frac{(6+4k)}{3(k+1)}\Theta(\infty,g) \qquad (10)$$

Substituting (9) and (10) in (8) , we get

$$\begin{aligned}
&\Theta_2(0,F^*)+\Theta_2(\infty,F^*)+\Theta_2(c,F^*)\\
&\geq \frac{3n+p}{n+p+1}+\frac{1}{n+p+1}\left\{\frac{(n+p+2)(6+4k)}{3(k+1)}-\frac{(6+4k)}{3(k+1)}\Theta(\infty,f)\right\}+\frac{\Theta(\infty,f)}{n+p+1}\\
&\quad+\frac{1}{n+p+1}\left\{\frac{(n+p+2)(6+4k)}{3(k+1)}-\frac{(6+4k)}{3(k+1)}\Theta(\infty,g)\right\}\\
&=\frac{3n+p}{n+p+1}+\frac{2(n+p+2)(6+4k)}{3(k+1)(n+p+1)}-\frac{(k+3)}{3(k+1)(n+p+1)}\Theta(\infty,f)-\frac{(6+4k)}{3(k+1)(n+p+1)}\Theta(\infty,g)\\
&\geq \frac{3n+p}{n+p+1}+\frac{2(n+p+2)(6+4k)}{3(k+1)(n+p+1)}-\frac{(k+3)}{3(k+1)(n+p+1)}-\frac{(6+4k)}{3(k+1)(n+p+1)}\\
&=\frac{17kn+11kp+21n+15p+11k+15}{3(k+1)(n+p+1)}>2.
\end{aligned}$$

Because, let

$$H_n=\frac{17kn+11kp+21n+15p+11k+15}{3(k+1)(n+p+1)},\quad k>0,\quad p>0.$$

Then

$$H_n'=\frac{6kp+6p+6k+6}{3(k+1)(n+p+1)^2}>0\quad for\quad k>0,\quad p>0.$$

Thus H_n is an increasing function and

$$H_n\quad at\quad \{p=1,n=1\}=\frac{26k+62}{9k+9}\quad and\quad \lim_{k\to\infty}\left\{\frac{26k+62}{9k+9}\right\}=\frac{26}{9}=2.888...$$

This shows that H_n always exceeds the value 2, which contradicts Lemma 2.2. Hence $c=0$. Therefore

$$\begin{aligned}
F^*&\equiv G^*\\
i.e,\quad \frac{f^{n+p+1}}{n+p+1}-\frac{f^{n+1}}{n+1}&=\frac{g^{n+p+1}}{n+p+1}-\frac{g^{n+1}}{n+1}\\
f^{n+1}\left\{\frac{f^p}{n+p+1}-\frac{1}{n+1}\right\}&=g^{n+1}\left\{\frac{g^p}{n+p+1}-\frac{1}{n+1}\right\}
\end{aligned}$$

Now, let $h=\frac{f}{g}$. If $h\equiv 1$, then $f\equiv g$. Suppose $h\not\equiv 1$, then

$$\left(\frac{f}{g}\right)^{n+1}\left\{\frac{f^p}{n+p+1}-\frac{1}{n+1}\right\}=\frac{g^p}{n+p+1}-\frac{1}{n+1}$$

$$h^{n+1}\left\{\frac{(hg)^p}{n+p+1}-\frac{1}{n+1}\right\}=\frac{g^p}{n+p+1}-\frac{1}{n+1}$$

$$h^{n+1}\left\{\frac{(hg)^p(n+1)-(n+p+1)}{(n+1)(n+p+1)}\right\} = \frac{g^p(n+1)-(n+p+1)}{(n+1)(n+p+1)}$$

$$\begin{aligned} h^{n+p+1}g^p(n+1)-h^{n+1}(n+p+1) &= g^p(n+1)-(n+p+1) \\ h^{n+p+1}g^p(n+1)-g^p(n+1) &= h^{n+1}(n+p+1)-(n+p+1) \\ g^p(n+1)\left[h^{n+p+1}-1\right] &= (n+p+1)\left(h^{n+1}-1\right) \end{aligned}$$

$$g^p = \frac{(n+p+1)(h^{n+1}-1)}{(n+1)(h^{n+p+1}-1)}.$$

Similarly

$$f^p = \frac{(n+p+1)h^p(h^{n+1}-1)}{(n+1)(h^{n+p+1}-1)}.$$

Therefore,

$$f = \left[\frac{(n+p+1)h^p(h^{n+1}-1)}{(n+1)(h^{n+p+1}-1)}\right]^{1/p}, \quad g = \left[\frac{(n+p+1)(h^{n+1}-1)}{(n+1)(h^{n+p+1}-1)}\right]^{1/p},$$

which proves the Theorem 1. □

Proof of Theorem 2. First, we show that $f^n(f-1)^p f' \equiv g^n(g-1)^p g'$, where

$$(f-1)^p = f^p - pf^{p-1} + \frac{p(p-1)}{2}f^{p-2} - \cdots\cdots + (-1)^p$$

So,

$$f^n(f-1)^p f' = f^{n+p}f' - pf^{n+p-1}f' + \frac{p(p-1)}{2}f^{n+p-2}f' - \cdots\cdots + (-1)^p f^n f'$$

$$g^n(g-1)^p g' = g^{n+p}g' - pg^{n+p-1}g' + \frac{p(p-1)}{2}g^{n+p-2}g' - \cdots\cdots + (-1)^p g^n g'$$

Consider $k_1 = k_2 = k_3 = k$ and $a = 0$. We have, $\overline{d}\left\{f^n(f-1)^p f'\right\} = \overline{d}\left\{g^n(g-1)^p g'\right\} = n+p+1$, $Q = 1$ and $m = 1$. Therefore

$$\min\left\{D_f, D_g\right\} \geq 0 \quad , \quad \max\left\{D_f, D_g\right\} > 0 \quad ,$$

where

$$D_f = 3(k+1)^2\delta_2(0,f) - (n+p+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,f)$$

and

$$D_g = 3(k+1)^2\delta_2(0,g) - (n+p+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,g)$$

By Theorem B, we obtain

$$f^n(f-1)^p f' \equiv g^n(g-1)^p g'$$

$$(F^*)' \equiv (G^*)'$$

Then

$$F^* \equiv G^* + c \quad , \quad c \text{ is a constant} \tag{11}$$

where

$$F^* = \frac{f^{n+p+1}}{n+p+1} - p\frac{f^{n+p}}{n+p} + \frac{p(p-1)}{2}\frac{f^{n+p-1}}{n+p-1} - \frac{p(p-1)(p-2)}{6}\frac{f^{n+p-2}}{n+p-2} + \cdots + (-1)^p\frac{f^{n+1}}{n+1}$$

$$G^* = \frac{g^{n+p+1}}{n+p+1} - p\frac{g^{n+p}}{n+p} + \frac{p(p-1)}{2}\frac{g^{n+p-1}}{n+p-1} - \frac{p(p-1)(p-2)}{6}\frac{g^{n+p-2}}{n+p-2} + \cdots + (-1)^p\frac{g^{n+1}}{n+1}$$

By Lemma 2.1, we have $T(r, F^*) = (n+p+1)T(r, f) + S(r, f)$. Note that

$$\begin{aligned}
\overline{N}_2\left(r, \frac{1}{F^*}\right) &= \overline{N}_2\left(r, \frac{1}{f}\right) + \overline{N}_2\left(r, \frac{1}{\frac{f^p}{n+p+1} - p\frac{f^{p-1}}{n+p} + \frac{p(p-1)}{2}\frac{f^{p-2}}{n+p+1} - \frac{p(p-1)(p-2)}{6}\frac{f^{p-3}}{n+p-2}\cdots\cdots + \frac{(-1)^p}{n+1}}\right) \\
&\leq \overline{N}_2\left(r, \frac{1}{f}\right) + T\left(r, \frac{f^p}{n+p+1} - p\frac{f^{p-1}}{n+p} + \frac{p(p-1)}{2}\frac{f^{p-2}}{n+p+1}\cdots + \frac{(-1)^p}{n+1}\right) \\
&= \overline{N}_2\left(r, \frac{1}{f}\right) + pT(r, f) + S(r, f) \qquad \text{(using Lemma 2.4)}
\end{aligned}$$

So,

$$\frac{\overline{N}_2\left(r, \frac{1}{F^*}\right)}{T(r, F^*)} \leq \frac{\overline{N}_2\left(r, \frac{1}{f}\right)}{(n+p+1)T(r, f) + S(r, f)} + \frac{pT(r, f)}{(n+p+1)T(r, f) + S(r, f)}$$

Therefore,

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\overline{N}_2\left(r, \frac{1}{F^*}\right)}{T(r, F^*)} &\leq \frac{\limsup_{r\to\infty} \overline{N}_2(r, \frac{1}{f})/T(r, f)}{n+p+1} + \frac{p}{n+p+1} \\
1 - \Theta_2(0, F^*) &\leq \frac{1 - \Theta_2(0, f)}{n+p+1} + \frac{p}{n+p+1}
\end{aligned}$$

i.e,

$$\Theta_2(0, F^*) \geq \frac{n}{n+p+1} + \frac{\Theta_2(0, f)}{n+p+1}$$

Similarly, we have

$$\Theta_2(0, G^*) \geq \frac{n}{n+p+1} + \frac{\Theta_2(0, g)}{n+p+1}$$

Note that $\overline{N}_2(r, F^*) = \overline{N}_2(r, f)$. So,

$$\begin{aligned}
\frac{\overline{N}_2(r, F^*)}{T(r, F^*)} &= \frac{\overline{N}_2(r, f)}{(n+p+1)T(r, f) + S(r, f)} \\
\limsup_{r\to\infty} \frac{\overline{N}_2(r, F^*)}{T(r, F^*)} &= \frac{\limsup_{r\to\infty} \frac{\overline{N}_2(r, f)}{T(r, f)}}{(n+p+1)}
\end{aligned}$$

$$1-\Theta_2(\infty,F^*) = \frac{1-\Theta_2(\infty,f)}{n+p+1}$$
$$\Theta_2(\infty,F^*) = \frac{n+p}{n+p+1}+\frac{\Theta_2(\infty,f)}{n+p+1}$$

And, by the definition we have,

$$\Theta_2(c,F^*) = 1-\limsup_{r\to\infty}\frac{\overline{N}_2\left(r,\frac{1}{F^*-c}\right)}{T(r,F^*)}$$
$$= 1-\limsup_{r\to\infty}\frac{\overline{N}_2\left(r,\frac{1}{G^*}\right)}{T(r,F^*)}. \qquad \text{by (11)}$$

Since $F^*\equiv G^*+c$, c a constant, so $T(r,F^*)=T(r,G^*)$. Therefore,

$$\Theta_2(c,F^*) = 1-\limsup_{r\to\infty}\frac{\overline{N}_2\left(r,\frac{1}{G^*}\right)}{T(r,G^*)} = \Theta_2(0,G^*)$$

We now show that $c=0$. Proceeding as in the proof of Theorem 1, we obtain

$$F^*\equiv G^*$$

(i) If $p=1$, then

$$F^*=\frac{f^{n+2}}{n+2}-\frac{f^{n+1}}{n+1} \quad , \quad G^*=\frac{g^{n+2}}{n+2}-\frac{g^{n+1}}{n+1}$$

Then, we can write

$$f^{n+1}\left\{\frac{f}{n+2}-\frac{1}{n+1}\right\}=g^{n+1}\left\{\frac{g}{n+2}-\frac{1}{n+1}\right\}$$

Now, let $h=\frac{f}{g}$. If $h\equiv 1$, then $f\equiv g$. Suppose $h\not\equiv 1$, then we easily obtain

$$g=\frac{(n+2)(h^{n+1}-1)}{(n+1)(h^{n+2}-1)}, \quad f=\frac{(n+2)h(h^{n+1}-1)}{(n+1)(h^{n+2}-1)}.$$

(ii) If $p=2$, then

$$F^*\equiv G^*$$

i.e.,

$$\frac{f^{n+3}}{n+3}-2\frac{f^{n+2}}{n+2}+\frac{f^{n+1}}{n+1}=\frac{g^{n+3}}{n+3}-2\frac{g^{n+2}}{n+2}+\frac{g^{n+1}}{n+1}. \tag{12}$$

Set $h=\frac{f}{g}$. Substitute $f=hg$ in (12) , we obtain

$$(n+2)(n+1)g^2(h^{n+3}-1)-2(n+3)(n+1)g(h^{n+2}-1)+(n+2)(n+3)(h^{n+1}-1)=0. \tag{13}$$

If h is not a constant, then

$$\left[(n+2)(n+1)g(h^{n+3}-1)-(n+3)(n+1)(h^{n+2}-1)\right]^2$$

$$= (n+1)^2(n+2)^2g^2(h^{n+3}-1)^2 - 2(n+1)(n+2)(n+3)(n+1)g(h^{n+3}-1)(h^{n+2}-1)$$
$$+(n+3)^2(n+1)^2(h^{n+2}-1)^2$$
$$= (n+2)(n+1)(h^{n+3}-1)\left[(n+2)(n+1)g^2(h^{n+3}-1) - 2(n+3)(n+1)g(h^{n+2}-1)\right]$$
$$+(n+3)^2(n+1)^2(h^{n+2}-1)^2$$
$$= (n+2)(n+1)(h^{n+3}-1)\left\{-(n+2)(n+3)(h^{n+1}-1)\right\} + (n+3)^2(n+1)^2(h^{n+2}-1)^2 \text{ [by (13)]}$$
$$= -(n+3)(n+1)\left\{(n+2)^2(h^{n+3}-1)(h^{n+1}-1) - (n+3)(n+1)(h^{n+1}-1)^2\right\}$$

using Lemma 2.3, we get

$$\left[(n+2)(n+1)g(h^{n+3}-1) - (n+3)(n+1)(h^{n+2}-1)\right]^2 = -(n+3)(n+1)Q(h),$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\ldots\ldots(h-\beta_{2n})$, $\beta_j \in C - \{0,1\}$ $(j = 1,2,\ldots,2n)$, which are pairwise distinct.

This implies that every zero of $(h-\beta_j)$ $(j = 1,2,\ldots,2n)$ has a multiplicity of atleast 2. By the second fundamental theorem we obtain $n \le 2$, which is again a contradiction. Therefore, h is a constant. We have from (13) that $h^{n+1}-1 = 0$ and $h^{n+2}-1 = 0$, which imply $h = 1$ and hence $f \equiv g$.

(iii) If $p \ge 2$, we get

$$\frac{f^{n+p+1}}{n+p+1} - p\frac{f^{n+p}}{n+p} + \frac{p(p-1)}{2}\frac{f^{n+p-1}}{n+p-1} - \frac{p(p-1)(p-2)}{6}\frac{f^{n+p-2}}{n+p-2} + \cdots + (-1)^p\frac{f^{n+1}}{n+1}$$
$$= \frac{g^{n+p+1}}{n+p+1} - p\frac{g^{n+p}}{n+p} + \frac{p(p-1)}{2}\frac{g^{n+p-1}}{n+p-1} - \frac{p(p-1)(p-2)}{6}\frac{g^{n+p-2}}{n+p-2} + \cdots + (-1)^p\frac{g^{n+1}}{n+1}$$

above equality can be represented as

$$f^{n+1}\sum_{l=0}^{p}\frac{(-1)^l C_p^l}{n+p-l+1}f^{p-l} \equiv g^{n+1}\sum_{l=0}^{p}\frac{(-1)^l C_p^l}{n+p-l+1}g^{p-l}.$$

This completes the proof of the theorem. □

Proof of Theorem 3. First we show that $f^n f' \equiv g^n g'$. Consider $k_1 = k_2 = k_3 = k$ and $a = 0$.We have $\overline{d}\{f^n f'\} = \overline{d}\{g^n g'\} = n+1$, $Q = 1$ and $m = 1$. Therefore

$$\min\{D_f, D_g\} \ge 0 \quad , \quad \max\{D_f, D_g\} > 0 \quad ,$$

where

$$D_f = 3(k+1)^2\delta_2(0,f) - (n+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,f)$$

and

$$D_g = 3(k+1)^2\delta_2(0,g) - (n+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,g)$$

By Theorem B, we get

$$f^n f' \equiv g^n g'$$
$$(F^*)' \equiv (G^*)'$$

Then

$$F^* \equiv G^* + c, \quad c \text{ is a constant.} \tag{14}$$

where

$$F^* = \frac{f^{n+1}}{n+1}, \quad G^* = \frac{g^{n+1}}{n+1}$$

By Lemma 2.1, we have $T(r,F^*) = (n+1)T(r,f) + S(r,f)$. Note that

$$\overline{N_2}\left(r,\frac{1}{F^*}\right) = \overline{N_2}\left(r,\frac{1}{f}\right)$$
$$\overline{N_2}\left(r,F^*\right) = \overline{N_2}\left(r,f\right)$$

Similarly, as in the Theorem 1, we obtain

$$\Theta_2(0,F^*) = \frac{n}{n+1} + \frac{\Theta_2(0,f)}{n+1}$$
$$\Theta_2(\infty,F^*) = \frac{n}{n+1} + \frac{\Theta_2(\infty,f)}{n+1}$$
$$\Theta_2(c,F^*) = \Theta_2(0,G^*)$$

and

$$\Theta_2(0,G^*) = \frac{n}{n+1} + \frac{\Theta_2(0,g)}{n+1}$$

We now show that $c = 0$ in (14) .Suppose that $c \neq 0$, then

$$\begin{aligned}\Theta_2(0,F^*) + \Theta_2(\infty,F^*) + \Theta_2(c,F^*) &= \frac{n}{n+1} + \frac{\Theta_2(0,f)}{n+1} + \frac{n}{n+1} + \frac{\Theta_2(\infty,f)}{n+1} + \frac{n}{n+1} + \frac{\Theta_2(0,g)}{n+1}\\ &= \frac{3n}{n+1} + \frac{\delta_2(0,f)}{n+1} + \frac{\Theta(\infty,f)}{n+1} + \frac{\delta_2(0,g)}{n+1}\end{aligned} \tag{15}$$

Since $\min\{D_f, D_g\} \geq 0$,

$$D_f = 3(k+1)^2\delta_2(0,f) - (n+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,f) \geq 0$$
$$D_f = 3(k+1)^2\delta_2(0,f) - (n+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,f) \geq 0$$

Therefore

$$\delta_2(0,f) \geq \frac{(n+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)}\Theta(\infty,f) \tag{16}$$

$$\delta_2(0,g) \geq \frac{(n+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)}\Theta(\infty,g) \tag{17}$$

Substituting (16) and (17) in (15) , we get

$$\begin{aligned}
&\Theta_2(0,F^*)+\Theta_2(\infty,F^*)+\Theta_2(c,F^*)\\
&\geq \frac{3n}{n+1}+\frac{1}{n+1}\left\{\frac{(n+2)(6+4k)}{3(k+1)}-\frac{(6+4k)}{3(k+1)}\Theta(\infty,f)\right\}+\frac{\Theta(\infty,f)}{n+1}\\
&\quad+\frac{1}{n+1}\left\{\frac{(n+2)(6+4k)}{3(k+1)}-\frac{(6+4k)}{3(k+1)}\Theta(\infty,g)\right\}\\
&=\frac{3n}{n+1}+\frac{(2n+4)(6+4k)}{3(k+1)(n+1)}-\frac{(k+3)}{3(k+1)(n+1)}\Theta(\infty,f)-\frac{(6+4k)}{3(k+1)(n+1)}\Theta(\infty,g)\\
&\geq \frac{3n}{n+1}+\frac{(2n+4)(6+4k)}{3(k+1)(n+1)}-\frac{(k+3)}{3(k+1)(n+1)}-\frac{(6+4k)}{3(k+1)(n+1)}\\
&=\frac{17nk+21n+11k+15}{3(k+1)(n+1)}\geq 4.666..
\end{aligned}$$

Because, let

$$Q_n=\frac{17nk+21n+11k+15}{3(k+1)(n+1)}\quad,\quad k>0$$

Then

$$Q_n'=\frac{18k^2+36k+18}{9(k+1)^2(n+1)^2}>0\quad for\quad k>0$$

Thus Q_n is an increasing function and

$$Q_n\quad at\quad n=1=\frac{28k+36}{6k+6}\quad,\quad \lim_{k\to\infty}\left\{\frac{28k+36}{6k+6}\right\}=\frac{28}{6}=4.6666.....$$

This shows that Q_n always exceeds the value 4.666...which contradicts Lemma 2.2. Hence $c=0$. Therefore

$$F^*\equiv G^*$$
$$\frac{f^{n+1}}{n+1}=\frac{g^{n+1}}{n+1}$$

Let $h=\frac{f}{g}$. If $h\equiv 1$ then $f\equiv g$. Suppose $h\not\equiv 1$, then $h^{n+1}-1=0$. This completes the proof of the theorem. □

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