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ON CERTAIN UNIVALENT CLASS ASSOCIATED WITH FIRST ORDER DIFFERENTIAL SUBORDINATIONS

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Abstract. In this paper, we consider certain differential inequalities and first order differential subordinations. As their applications, we obtain some sufficient conditions for univalence, which generalize and refine some previous results.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $U = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the class of functions f, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let f be analytic in U, g analytic and univalent in U and $f(0) = g(0)$. Then, by the symbol $f(z) \prec g(z)$ (f subordinate to g) in U, we shall mean $f(U) \subset g(U)$.

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

The function $f \in \mathcal{A}$ is called $\Phi-$like if

$$\Re\{\frac{zf'(z)}{\Phi(f(z))}\} > 0, \ z \in U.$$

This concept was introduced by Brickman [1] and established that a function $f \in \mathcal{A}$ is univalent if and only if f is $\Phi-$like for some Φ.

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Definition 1.1. Let Φ be analytic function in a domain containing $f(U)$, $\Phi(0)=0$, $\Phi'(0)=1$ and $\Phi(\omega)\neq 0$ for $\omega\in f(U)-0$. Let $q(z)$ be a fixed analytic function in U, $q(0)=1$. The function $f\in\mathscr{A}$ is called $\Phi-$like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))}\prec q(z),\ z\in U.$$

Ruscheweyh [2] investigated this general class of Φ-like functions.

In the present paper, we consider another new class $H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$ involving two different types of $\Phi-$like functions, Φ_1 and Φ_2, which defined by

$$\frac{zf'(z)}{\Phi_1(f(z))}\Big\{(1-\alpha)\frac{zf'(z)}{\Phi_2(f(z))}+\alpha\Big(1+\frac{\lambda zf''(z)}{f'(z)}\Big)\Big\}\prec F(z), \tag{1.1}$$

where $\alpha\in[0,1]$, $\lambda\in\mathbb{R}$, F is the conformal mapping of the unit disk U with $F(0)=1$ and Φ_1 and Φ_2 satisfy Definition 1.1.

Remark 1. As special cases of the class $H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$ are the following well known classes: $H\Big(0;\Phi(f(z)),zf'(z)\Big)$ (see [2]); $H\Big(\alpha,1;zf'(z),z\Big)$ (see [3-5]); $H\Big(1,\lambda;f(z)\Big)$ (see [6-16]). Also this class reduces to the classes of starlike functions, convex functions and close-to-convex functions.

In order to obtain our results, we need the following lemmas.

Lemma 1.([17]) *Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then*

$$z_0w'(z_0)=kw(z_0), \tag{1.2}$$

where k is a real number and $k\geq 1$.

Lemma 2.([18]) *Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set $Q(z):=zq'(z)\phi(q(z))$, $h(z):=\theta(q(z))+Q(z)$. Suppose that*

1. *$Q(z)$ is starlike univalent in U, and*
2. $\Re\{\frac{zh'(z)}{Q(z)}\}>0$ *for $z\in U$.*

If $\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z))$ then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

2. The class $H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$

Let us consider the sufficient condition for $f(z)\in\mathscr{A}$ to be in $H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$. Our first result is contained in

Theorem 1. *Let* $p_1(z):=\frac{zf'(z)}{\Phi_1(f(z))}$ *and* $p_2=\frac{zf'(z)}{\Phi_2(f(z))}$. *If* $f\in\mathscr{A}$ *satisfies*

$$\begin{aligned}\Re\Big\{(1-\alpha)z(p_1(z)p_2(z))'+\alpha z\Big[p_1'(z)+\lambda\Big(&\frac{zf''(z)+p_2'(z)\Phi_2'(z)+p_2''(z)\Phi_2(z)}{f'(z)}\\ &-\frac{p_2'(z)\Phi_2(z)f''(z)}{(f'(z))^2}\Big)\Big]\Big\}\\ &<\frac{2\alpha}{(1-\alpha)^2},\ (z\in U)\end{aligned}\tag{1.3}$$

for some $\alpha\neq1,\ \lambda\in\mathbb{R}$ *then* $f\in H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$.

Proof Let $w(z)$ defined by

$$H(z):=\frac{zf'(z)}{\Phi_1(f(z))}\Big\{(1-\alpha)\frac{zf'(z)}{\Phi_2(f(z))}+\alpha\Big(1+\frac{\lambda zf''(z)}{f'(z)}\Big)\Big\}=\frac{\alpha+w(z)}{\alpha-w(z)},\ (\alpha\neq w(z)).$$

Then

$$w(z)=\frac{\alpha\Big(H(z)-1\Big)}{1+H(z)}$$

is analytic in U with $w(0)=0$. It follows that

$$\Re\{zH'(z)\}=\Re\Big\{\frac{2\alpha zw'(z)}{(\alpha-w(z))^2}\Big\}<\frac{2\alpha}{(1-\alpha)^2},\ \alpha\neq1.$$

Now we proceed to prove that $|w(z)|<1$. Suppose that there exists a point $z_0\in U$ such that

$$max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1.\tag{1.4}$$

Then, using the Lemma 1 and letting $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=ke^{i\theta},\ k\geq1$ yields

$$\begin{aligned}\Re\{z_0H'(z_0)\}&=\Re\Big\{\frac{2\alpha z_0w'(z_0)}{(\alpha-w(z_0))^2}\Big\}\\ &=\Re\Big\{\frac{2\alpha ke^{i\theta}}{(\alpha-e^{i\theta})^2}\Big\}\\ &\geq\frac{2\alpha}{(1-\alpha)^2}.\end{aligned}$$

Thus we have

$$\begin{aligned}\Re\{z_0H'(z_0)\}=\Re\Big\{&(1-\alpha)z_0(p_1(z_0)p_2(z_0))'\\ &+\alpha z_0\Big[p_1'(z_0)+\lambda\Big(\frac{z_0f''(z_0)+p_2'(z_0)\Phi_2'(z_0)+p_2''(z_0)\Phi_2(z_0)}{f'(z_0)}\\ &-\frac{p_2'(z_0)\Phi_2(z_0)f''(z_0)}{(f'(z_0))^2}\Big)\Big]\Big\}\\ &\geq\frac{2\alpha}{(1-\alpha)^2},\ (z\in U)\end{aligned}\tag{1.5}$$

which contradicts the hypothesis (3). Therefore, we conclude that $|w(z)|<1$ for all $z\in U$ that is $f\in H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$. This completes the proof. □

Corollary 1. *If $f(z)\in\mathscr{A}$ satisfies the condition in Theorem 1, then for $\alpha\in[0,1)$*

$$\Big|H(z)-\frac{1+\alpha}{1-\alpha}\Big|<\frac{1+\alpha}{1-\alpha}. \tag{1.6}$$

Proof. Since $f\in H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$ yields

$$|w(z)|=\Big|\frac{\alpha\Big(H(z)-1\Big)}{H(z)+1}\Big|<1$$

we obtain (6). □

Next results show the starlikeness ($\mathscr{S}^*$), convexity ($\mathscr{C}$) and close to convex ($\mathscr{K}$) for different order.

By letting $\alpha=0,\Phi_1(z)=f(z)$ and $\Phi_2(z)=zf'(z)$ we have the following result

Corollary 2. *If $f(z)\in\mathscr{A}$ satisfies the condition in Theorem 1, then for $\alpha\in[0,1)$*

$$\Big|\frac{zf'(z)}{f(z)}-1\Big|<1. \tag{1.7}$$

This implies that $f(z)\in\mathscr{S}^*$ and $\int_0^z\frac{f(t)}{t}dt\in\mathscr{C}$.

By setting $\alpha=0,\Phi_1(z)=g(z)$ where g is starlike and satisfies $g(0)=0$ and $g'(0)=1$ and $\Phi_2(z)=zf'(z)$ we have the following result

Corollary 3. *If $f(z)\in\mathscr{A}$ satisfies the condition in Theorem 1, then for $\alpha\in[0,1)$*

$$\Big|\frac{zf'(z)}{g(z)}-1\Big|<1. \tag{1.8}$$

This implies that $f(z)\in\mathscr{K}$.

Note that Corollary 2 and Corollary 3 implies the univalence of the class $H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$.

3. The region of variability

In this section, we show that for $\alpha\in[0,1)$ and $f\in H\Big(\alpha,\lambda;\Phi_1(f(z)),\Phi_2(f(z))\Big)$ then f is univalent in U. Moreover, we estimate the region of variability. We prove a subordination theorem by using Lemma 2 and as applications of this result, we find the sufficient conditions for $f \in \mathscr{A}$ to be univalent.

Theorem 2. *Let $q, q(z) \neq 0$, be a univalent function in U, and $g(z) \neq 0$ be analytic in $\mathbb{C}$ such that for nonnegative real numbers μ and ν*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\} > \max\left\{0, \left(\frac{\mu}{\nu}\right)\Re\left(q(z)[1+\frac{g'(z)}{g(z)}(\frac{q(z)}{q'(z)}+\frac{\nu z}{\mu q(z)})]\right)\right\}. \tag{1.9}$$

If $p(z) \neq 0$, $z \in U$ satisfies the differential subordination

$$g(z)\left[\mu p(z)+\nu\frac{zp'(z)}{p(z)}\right] \prec g(z)\left[\mu q(z)+\nu\frac{zq'(z)}{q(z)}\right], \tag{1.10}$$

then $p \prec q$ and q is the best dominant.

Proof. Define the functions θ and ϕ as follows:

$$\theta(w(z)) := \mu w(z) g(z) \quad \text{and} \quad \phi(w(z)) := \frac{\nu g(z)}{w(z)}.$$

Obviously, the functions θ and ϕ are analytic in domain $D = \mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$ in D. Now, define the functions Q and h as follows:

$$Q(z) := zq'(z)\phi(q(z)) = \nu g(z)\frac{zq'(z)}{q(z)},$$

$$h(z) := \theta(q(z)) + Q(z) = \mu q(z) g(z) + \nu g(z)\frac{zq'(z)}{q(z)}.$$

Then in view of condition (9), we obtain Q is starlike in U and $\Re\{\frac{zh'(z)}{Q(z)}\} > 0$ for $z \in U$. Furthermore, in view of condition (10) we have

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)).$$

Therefore, the proof follows from Lemma 2.

By letting $\mu = 1, \nu = \alpha, g(z) := \frac{zf'(z)}{\Phi_1(z)}$ and $p = \frac{zf'(z)}{f(z)}$ in Theorem 2 we have

Corollary 4. *Let $q, q(z) \neq 0$, be a univalent function in U, and $g(z) \neq 0$ be analytic in U satisfy (9). If $\frac{zf'(z)}{f(z)} \neq 0$, $z \in U$ and*

$$\frac{zf'(z)}{\Phi_1(z)}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{zf'(z)}{\Phi_1(z)}\left[q(z)+\alpha\frac{zq'(z)}{q(z)}\right], \tag{1.11}$$

then $\frac{zf'(z)}{f(z)} \prec q$ and q is the best dominant.

By setting $\mu = 1-\alpha, \nu = \alpha, g(z) := 1$ and $p = \frac{zf'(z)}{f(z)}$ in Theorem 2 we obtain the following result which can be found in [5, Theorem 3.2].

Corollary 5. *Let $q, q(z) \neq 0$, be a univalent function in U, and $g(z) \neq 0$ be analytic in U satisfy (9). If $\frac{zf'(z)}{f(z)} \neq 0$, $z \in U$ and*

$$(1-2\alpha)\frac{zf'(z)}{f(z)} + \alpha\Big(1 + \frac{zf''(z)}{f'(z)}\Big) \prec (1-\alpha)q(z) + \alpha\frac{zq'(z)}{q(z)}, \qquad (1.12)$$

then $\frac{zf'(z)}{f(z)} \prec q$ and q is the best dominant.

By assuming $\mu = 1-\alpha, \nu = \alpha, g(z) := 1$ and $p(z) = \frac{zf'(z)}{\Phi(f(z))}$ in Theorem 2 we obtain the following result which can be found in [5, Theorem 3.3].

Corollary 6. *Let $q, q(z) \neq 0$, be a univalent function in U, and $g(z) \neq 0$ be analytic in U satisfy (9). If $\frac{zf'(z)}{\Phi(f(z))} \neq 0$, $z \in U$ and*

$$(1-\alpha)\frac{zf'(z)}{\Phi(f(z))} + \alpha\Big(1 + \frac{zf''(z)}{f'(z)} - \frac{z\Phi(f(z))}{\Phi'(f(z))}\Big) \prec (1-\alpha)q(z) + \alpha\frac{zq'(z)}{q(z)}, \qquad (1.13)$$

then $\frac{zf'(z)}{\Phi(f(z))} \prec q$ and q is the best dominant.

Finally, by assuming $\mu = 1-\alpha, \nu = \alpha, g(z) := \frac{zf'(z)}{\Phi_1 f((z))}$ and $p(z) = \frac{zf'(z)}{\Phi_2(f(z))}$ in Theorem 2 we obtain the following result:

Corollary 7. *Let $q, q(z) \neq 0$, be a univalent function in U, and $g(z) \neq 0$ be analytic in U satisfy (9). If $\frac{zf'(z)}{\Phi_2(f(z))} \neq 0$, $z \in U$ and*

$$\frac{zf'(z)}{\Phi_1 f((z))}\Big[(1-\alpha)\frac{zf'(z)}{\Phi_2(f(z))} + \alpha\Big(1 + \frac{zf''(z)}{f'(z)} - \frac{z\Phi_2'(f(z))}{\Phi_2(f(z))}\Big)\Big] \prec \frac{zf'(z)}{\Phi_1(z)}\Big[(1-\alpha)q(z) + \alpha\frac{zq'(z)}{q(z)}\Big], \qquad (1.14)$$

then $\frac{zf'(z)}{\Phi_2(f(z))} \prec q$ and q is the best dominant.

References

[1] L. Brickman, *Φ−like analytic functions, I*, Bull. Amer. Math. Soc., **79**(1973), 555–558.

[2] St. Ruscheweyh, *A subordination theorem for Φ-like functions*, J. London Math. Soc., **2**(1976), 275–280.

[3] S. Singh, S. Gupta and S. Singh, *On a problem of univalence of functions satisfying a differential inequality*, Mathematical Inequalities and Applications, **10**(2007), 95–98.

[4] V. Singh, S. Singh and S. Gupta, *A problem in the theory of univalent functions*, Integral Transforms and Special Functions, **16**(2005), 179–186.

[5] S. Singh, S. Gupta and S. Singh, *An extension of the region of variability of a subclass of univalent functions*, J. Inequal. Pure and Appl. Math., **10**(2009), Art. 113, 7 pp.

[6] Z. Lewandowski, S. S. Miller and E. Zlotkiewicz, *Generating functions for classes of uni- valent functions*, Proc. Amer. Math. Soc., **56**(1976), 111–117.

[7] J.-L. Li and S. Owa, *Sufficient conditions for starlikeness*, Indian J. Pure Appl. Math., **33**(2002), 313–318.

[8] C. Ramesha, S. Kumar and K. S. Padmanabhan, *A sufficient condition for starlikeness*, Chinese J. Math., **23**(1995), 167–171.

[9] Oh Sang Kwon, *Sufficient conditions for starlikeness and strongly-starlikeness*, Commun. Korean Math. Soc., **22**(2007), 19–26.

[10] V. Ravichandran, C. Selvaraj and R. Rajalakshmi, *Sufficient conditions for starlike functions of order α*, J. Inequal. Pure and Appl. Math., **3**(2002), 1–6.(Art.81).

[11] K. S. Padmanabhan, *On sufficient conditions for starlikeness*, Indian J. Pure Appl. Math., **32**(2001), 543–550.

[12] M. Obradovic and S. B. Joshi, *On certain classes of strongly starlike functions*, Taiwanese J. Math., **2**(1998), 297–302.

[13] M. Obradovic, S. B. Joshi and I. Jovanovic, *On certain sufficient conditions for starlike- ness and convexity*, Indian J. Pure Appl. Math., **29**(1998), 271-275.

[14] S. Singh and S. Gupta, *First order differential subordination and starlikeness of ana- lytic maps in unit disc*, Kyungpook Math.J. **45**(2005), 395-404.

[15] S. Singh and S. Gupta, *A differential subordination and starlikeness of analytic func- tions*, Appl. Math. lett., **19**(2006), 618–627.

[16] N. Xu and D.-G. Yang, *Some criteria for starlikeness and strongly starlikeness*, Bull. Korean Math. Soc., **42** (2005), 579–590.

[17] I. S. Jack, *Functions starlike and convex of order K*, J. London Math. Soc., **3**(1971), 469–474 .

[18] S. S. Miller and P. T. Mocanu, Differential Subordinantions: Theory and Applications. Pure and Applied Mathematics No.225 Dekker, New York, 2000.

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