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A REACTION-DIFFUSION SYSTEM AND ITS SHADOW SYSTEM DESCRIBING HARMFUL ALGAL BLOOMS

SHINTARO KONDO AND MASAYASU MIMURA

Abstract. The occurrence of harmful algal blooms (HAB) in ecosystems is a worldwide environmental issue that currently needs to be addressed. An attempt to theoretically understand the mechanism behind the formation of HAB has led to the proposal of a reaction-diffusion model of the Lotka–Volterra type. In particular, a shadow system, as a limiting system of the model in which the diffusion rate tends to infinity, has been proposed to study whether or not stable nonconstant equilibrium solutions of the system exist, because these solutions are mathematically associated with HAB. In this paper, we discuss the convergence property between solutions of the full system and its shadow system from the point of view of an evolutional problem.

1. Introduction

It is known that an algal bloom has a negative impact on other organisms via the production of toxins, mechanical damage, or by other means. In recent years, the occurrence of toxic blooms of cyanobacteria in lakes and rivers has been causing increasing concern from an ecological viewpoint. Therefore, a theoretical understanding of the mechanism behind the formation of spatial blooms on toxic plankton is one of the important subjects in mathematical ecology.

The study of this problem has recently led to the proposal of the following three-component reaction-diffusion system of the Lotka–Volterra type ([17]):

$$\begin{cases} \dfrac{\partial u}{\partial t} = r_1 u\left(1 - \dfrac{u + av}{K_1} - w\right) + d_1 \Delta u, \\[2ex] \dfrac{\partial v}{\partial t} = r_2 v\left(1 - \dfrac{v + bu}{K_2} - d(\mu) w\right) + d_2 \Delta v, \\[2ex] \dfrac{\partial w}{\partial t} = w\left(u - \mu v - 1\right) + d_3 \Delta w, \end{cases} \tag{1.1}$$

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where $u = u(t,x)$, $v = v(t,x)$, and $w = w(t,x)$ represent the population densities of the non-toxic phytoplankton, the toxic phytoplankton, and the zooplankton for time t and position x, respectively. The parameters a, b, μ, r_i, K_i $(i = 1,2)$, and d_i $(i = 1,2,3)$ are all positive constants and $d(\mu)$ is a positive and monotone decreasing function of μ with $d(0) = 1$. It is noted that μ, which we will refer to as "toxicity", is an important parameter in (1.1). The ecological explanation of (1.1) is stated in [17]. Here we simply assume $r_1 = r_2 = r$, $K_1 = K_2 = K$, $d_1 = d_2 = d$, and $D = d_3/d$ so that (1.1) is rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t} = ru\left(1 - \dfrac{u+av}{K} - w\right) + \Delta u, \\ \dfrac{\partial v}{\partial t} = rv\left(1 - \dfrac{v+bu}{K} - d(\mu)w\right) + \Delta v, \\ \dfrac{\partial w}{\partial t} = w\left(u - \mu v - 1\right) + D\Delta w. \end{cases} \tag{1.2}$$

From ecological viewpoints, we may assume that D is rather large, because it was reported that swimming speed of some species of the zooplankton is the order of mm/s and that of Cyanobacteria which is one of the phytoplankton is the order of μm/s ([17]).

We consider (1.2) in a bounded domain Ω in $\mathbf{R}^n$ $(n = 1,2,3)$ with the zero-flux boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega, \tag{1.3}$$

where $\partial\Omega$ is the smooth boundary of Ω and $\frac{\partial}{\partial \nu}$ is the outward unit normal derivative on $\partial\Omega$ and the initial conditions

$$\begin{cases} u(0,x) = u_0(x), \\ v(0,x) = v_0(x), \quad x \in \Omega, \\ w(0,x) = w_0(x), \end{cases} \tag{1.4}$$

where $u_0(x)$, $v_0(x)$ and $w_0(x)$ are non-negative smooth functions. We first not that in the absence of the nontoxic prey, it is obvious to see that the predator fades out; that is, when $u \equiv 0$,

$$\lim_{t\to\infty} (v(t,x), w(t,x)) = (K,0), \quad x \in \Omega$$

holds if $v_0(x)$ is not identically zero ([16]). We now impose the following two assumptions for (1.2):

$(A1)$ $a < 1 < b$,

which implies that, in the absence of the predator (w), the nontoxic prey (u) is a competitor who is absolutely stronger than the toxic prey (v) in terms of common resources ([11]); that is, when $w \equiv 0$,

$$\lim_{t\to\infty} (u(t,x), v(t,x)) = (K, 0), \quad x \in \Omega$$

holds if $u_0(x)$ is not identically zero ([10]), and

$(A2)$ $K > 1$,

which implies that, in the absence of toxic prey, the predator and nontoxic prey coexist ([8]); that is, when $v \equiv 0$,

$$\lim_{t\to\infty} (u(t,x), w(t,x)) = \left(1, \frac{K-1}{K}\right), \quad x \in \Omega$$

holds if $u_0(x)$ and $w_0(x)$ are both not identically zero.

These results indicate that in order to understand the occurrence of harmful algal blooms (HAB), the analysis of the full three-component reaction-diffusion system (1.2) for (u, v, w) is required.

First, noting that $E_3 = \left(1, 0, \frac{K-1}{K}\right)$ is a constant equilibrium solution of (1.2) and (1.3), which exists for any $\mu > 0$, we study the stability of E_3. Instead of $(A2)$, we assume

$(A3)$ $K > b$.

By $(A3)$, E_3 is stable for $\mu < \mu_c = d^{-1}(\frac{K-b}{K-1}) > 0$, while it is unstable for $\mu_c < \mu$ and there exists a positive constant equilibrium solution $E_4 = \left(\overline{u_\mu}, \overline{v_\mu}, \overline{w_\mu}\right)$ of (1.2) and (1.3). When we simply specify $d(\mu)$ as $d(\mu) = \frac{1}{1+\mu}$ ([5]), E_4 is given by

$$\begin{cases} \overline{u_\mu} = \dfrac{1-a+\mu+K\mu^2}{1-a+b\mu+b\mu^2}, \\[2ex] \overline{v_\mu} = \dfrac{1-b-b\mu+K\mu}{1-a+b\mu+b\mu^2} = \dfrac{(K-b)(\mu-\mu_c)}{1-a+b\mu+b\mu^2}, \\[2ex] \overline{w_\mu} = 1 + \dfrac{a}{K\mu} - \dfrac{(a+\mu)(1-a+\mu+K\mu^2)}{K\mu(1-a+b\mu+b\mu^2)}. \end{cases}$$

This implies that E_4 bifurcates super-critically from E_3 at $\mu = \mu_c$ when μ increases. However, it is not necessarily stable, that is, the stability of E_4 depends on the parameters μ, r, K, and D. If the one-dimensional problem of (1.2) and (1.3) is considered in the interval $(0, L)$, then the local stability of E_4 can be easily studied. As an example, if $d(\mu) = \frac{1}{1+\mu}$ and $r = 2.3$, $a = 0.95$, $b = 1.2$, $K = 2.9$, and $L = 30$, and μ and D are free parameters, the stable and unstable regions of E_4 can be drawn in (D, μ) space, as shown in Figure 1. This suggests the following results:

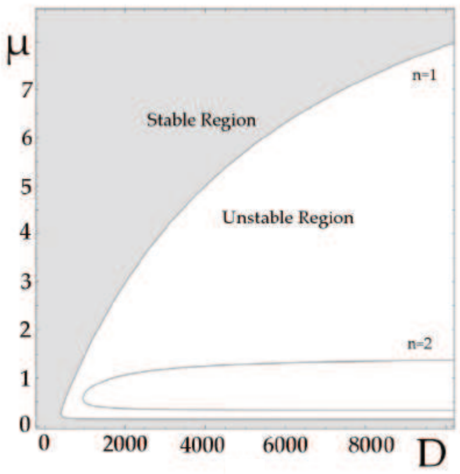


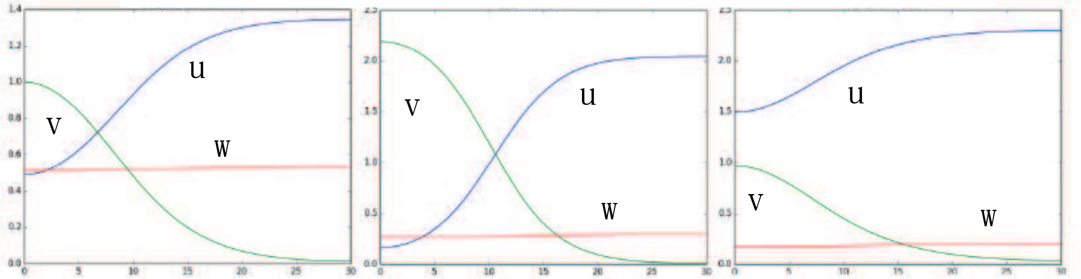
Figure 1: Stable and unstable regions of E_4 of the one dimensional problem of (1.2) with $d(\mu) = \frac{1}{1+\mu}$ and (1.3) in the (D,μ)-plane where $r = 2.3,\ a = 0.95,\ b = 1.2,\ K = 2.9,\ \mu_c = \frac{2}{17}$ and $L = 30$. The curve indicated by n corresponds to the n-mode bifurcation where the zero solution of the linearized problem of (1.2) and (1.3) around E_4 is destabilized under the nth eigenmode $\cos(\frac{n\pi x}{L})$ perturbation.

(i) When μ is greater than $\mu_c = \frac{2}{17} = 0.117\cdots$ but is relatively small, E_4 is stable for any D. Conversely when D is rather small, it is also stable for any $\mu > 0$.

(ii) When μ is in the intermediate range, E_4 is destabilized, as D increases. It implies Turing's diffusion induced instability ([18]). In fact, when $D = 2500$, we can see the existence of one-dimensional *stable* equilibrium solutions of (1.2) and (1.3) in Figure 2, where the predator (w) is almost spatially homogeneous, because D is rather large, while the non-toxic (u) and toxic (v) planktons of the prey exhibit large spatial heterogeneity, which ecologically indicates the occurrence of HAB.

Ecologically speaking, the results (i) and (ii) are stated in more detail as follows: If the predator consumes nontoxic and toxic planktons of prey with the same predation rates ($\mu = 0$), HAB does not occur (see the case (i)), whereas if the predator does not prefer to intake the toxic prey rather than the nontoxic prey ($\mu > 1$), then HAB possibly occurs when the diffusion rate of the predator is considerably larger than that of the prey, that is, D is relatively large (see case (ii)).

These numerical results arise the following mathematical question: *Can the existence and stability of such nonconstant equilibrium solutions of* (1.2) *and* (1.3) *be discussed analytically?* For this question, the "shadow system" approach can be applied to study nonconstant equilibrium solutions of (1.2) and (1.3), assuming that D is sufficiently large ([9]). Let us first introduce the shadow system, which is derived from (1.2). From the third equation of (1.2),



$\mu=0.15$ $\mu=0.5$ $\mu=3.1$

Figure 2: Spatial profiles of one-dimensional stable equilibrium solutions of (1.2) and (1.3) where $D = 2500$, $\mu = 0.15$, 0.5, and 3.1 and the other parameters are the same as those in Figure 1.

we obtain

$$\frac{1}{|\Omega|}\frac{d}{dt}\int_\Omega w\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega w\left(u-\mu v-1\right)\mathrm{d}x, \quad t>0, \tag{1.5}$$

where the symbol $|\Omega|$ denotes the measure of Ω. If $D\to\infty$, then $w(t,x)\to\xi(t)$ in the third equation of (1.2), so that (1.5) is rewritten as

$$\frac{d}{dt}\xi = \xi\left(\frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x - \frac{\mu}{|\Omega|}\int_\Omega v\,\mathrm{d}x - 1\right).$$

Consequently, as $D\to\infty$, (1.2)-(1.4) formally reduces to the following limiting system for $(u(t,x), v(t,x), \xi(t))$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = ru\left(1-\dfrac{u+av}{K}-\xi\right)+\Delta u, \\ \dfrac{\partial v}{\partial t} = rv\left(1-\dfrac{v+bu}{K}-d(\mu)\xi\right)+\Delta v, \quad t>0,\ x\in\Omega, \\ \dfrac{\partial \xi}{\partial t} = \xi\left(\dfrac{1}{|\Omega|}\displaystyle\int_\Omega u\,\mathrm{d}x - \dfrac{\mu}{|\Omega|}\displaystyle\int_\Omega v\,\mathrm{d}x - 1\right), \end{cases} \tag{1.6}$$

which is termed a *shadow system* of (1.2). The zero-flux boundary and initial conditions to (1.6) are

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad t>0, \qquad x\in\partial\Omega \tag{1.7}$$

and

$$\begin{cases} u(0,x) = u_0(x), \\ v(0,x) = v_0(x), \\ \xi(0) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega w_0(x)\,\mathrm{d}x, \end{cases} \qquad x\in\Omega, \tag{1.8}$$

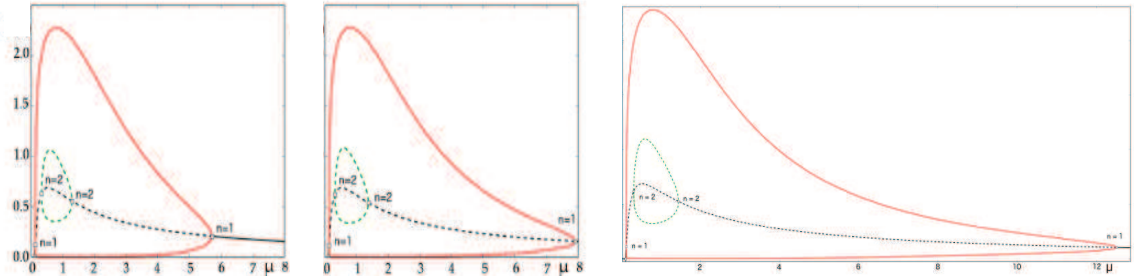
respectively. For the problems (1.2)-(1.4) and (1.6)-(1.8), the following questions naturally arise:

(i) What is the asymptotical behavior of $(u^\infty(t,x), v^\infty(t,x), \xi(t))$ of the limiting problem (1.6)-(1.8) ? Especially, how are the existence and stability of nonconstant equilibrium solutions $(\overline{u^\infty}(x), \overline{v^\infty}(x), \overline{\xi})$ of (1.6) and (1.7) ?
(ii) How is the convergence of the solution $\left(u^D(t,x), v^D(t,x), w^D(t,x)\right)$ of the full problem (1.2)-(1.4) to the one $(u^\infty(t,x), v^\infty(t,x), \xi(t))$ of (1.6)-(1.8) as $D \to \infty$?

For the question in (i), Ikeda, Mimura, and Scotti have recently discussed the one-dimensional stationary problem of (1.6) with $d(\mu) = \frac{1}{1+\mu}$ and (1.7) by using the numerical continuation software AUTO ([5], for instance) ([9]). Let us show an example. Assume that $L = 30$, for instance, and μ is a free parameter. Figure 3 demonstrates the global structures of equilibrium solutions of (1.2) with (1.3) with $D = 5000$ and 10000 and the shadow system (1.6) with (1.7). It indicates that there are stable non-constant equilibrium solutions of (1.2) with (1.3) for a suitable range of μ, and that when D is very large, the global structure of equilibrium solutions of (1.2) and (1.3) is *qualitatively* similar to that of (1.6) and (1.7), that is, the shadow system (1.6) with (1.7) would be a good approximation to (1.2) with (1.3) to study the existence and stability of equilibrium solutions of (1.2) and (1.3) if D is very large. Here we remark the following: When ξ is assumed to be a known constant, it is well known that non-constant equilibrium solutions $(\overline{u^\infty}(x;\xi), \overline{v^\infty}(x;\xi))$ of the first two equations of (1.6) and (1.7) are *unstable*, if Ω is convex ([12]). However, when $\overline{\xi}$ is an unknown variable, the situation is drastically changed, that is, there occur *stable* non-constant equilibrium solutions $(\overline{u^\infty}(x;\overline{\xi}), \overline{v^\infty}(x;\overline{\xi}), \overline{\xi})$, of (1.6) and (1.7) as shown in Figure 2. In relation to this problem, we refer the papers by Nishiura who discusses the shadow system of two-component reaction-diffusion systems with an activator-inhibitor type ([14]), and by Miyamoto who discusses the relation between global attractors for the Gierer–Meinhardt model and its shadow system when one of the diffusion rates is rather large ([13]).

In this paper, we focus on the question (ii). In other words, we study the problem whether "the shadow system (1.6) is an approximation to the full system (1.2) when D is very large." First, we make one remark to (1.1) where all of the diffusion rates d_i $(i = 1,2,3)$ in (1.1) are suitably large enough. Then, as one could expect, any solution of (1.1), (1.3), and (1.4) decays to be spatially homogeneous, and as d_i $(i = 1,2,3) \to \infty$, the ODEs corresponding to (1.1) is derived as its limiting system ([4]). Our situation is different from the above in a sense that only D is very large in (1.2).

We begin by showing *a priori* estimates for solutions of the problems (1.2)-(1.4) and (1.6)-(1.8), which assure the global existence of nonnegative solutions with respect to time, and then discuss the convergence problem of solutions of (1.2)-(1.4) and (1.6)-(1.8).



(a) $D = 5000$ (b) $D = 10000$ (c) Shadow system

Figure 3: Global structures of equilibrium solutions $(\overline{u^D}(x), \overline{v^D}(x), \overline{w^D}(x))$ of (1.2) and (1.3) for different values of large D and $(\overline{u^\infty}(x), \overline{v^\infty}(x), \overline{\xi})$ of (1.6) and (1.7) where the other parameters are the same as those in Figure 1 except for μ. The horizontal axis is μ and the vertical axis is $\overline{v^D}(0)$ ((a), (b)) and $\overline{v^\infty}(0)$ ((c)). Solid and dashed lines represent stable and unstable equilibrium solutions, respectively ([9]).

Theorem 1. *Let* $\left(u^D(t,x), v^D(t,x), w^D(t,x)\right)$ *be the nonnegative global smooth solution of* (1.2) $-$(1.4) *and let* λ *be the smallest (positive) eigenvalue of* $-\Delta$ *on* Ω *with zero-flux boundary conditions. Then there exist positive constants* c_1, c_2, *and* c_3 *independent of* D *such that*

$$\begin{aligned} 0 \le u^D(t,x) \le c_1, \\ 0 \le v^D(t,x) \le c_2, \end{aligned} \qquad \textit{for } t > 0,\ x \in \Omega \tag{1.9}$$

hold, where $c_1 = \max\left\{\|u_0\|_{L^\infty(\Omega)}, K\right\}$ *and* $c_2 = \max\left\{\|v_0\|_{L^\infty(\Omega)}, K\right\}$, *and that if* $D \ge \frac{2c_1}{\lambda}$ *is assumed,*

$$0 \le w^D(t,x) \le c_3 \quad \textit{for } t > 0,\ x \in \Omega \tag{1.10}$$

holds, where

$$c_3 = \begin{cases} c_3(u_0, v_0, w_0, \nabla u_0, \nabla v_0, \nabla w_0) & \textit{if } n = 1, \\ c_3(u_0, v_0, w_0, \Delta u_0, \Delta v_0, \Delta w_0) & \textit{if } n = 2, 3. \end{cases}$$

In the following we denote $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega f(x)\,dx$ by $\overline{f}$ simply.

Theorem 2. *Let* $(u^\infty(t,x), v^\infty(t,x), \xi(t))$ *be the nonnegative global smooth solution of* (1.6) $-$(1.8). *Then there exists a positive constant* c_4 *such that*

$$\begin{cases} 0 \le u^\infty(t,x) \le c_1, \\ 0 \le v^\infty(t,x) \le c_2, & \textit{for } t > 0,\ x \in \Omega, \\ 0 \le \xi(t) \le c_4 \end{cases} \tag{1.11}$$

hold, where c_1 *and* c_2 *are the same as those in Theorem 1 and* $c_4 = \frac{1}{r}\left(\overline{u_0} + r\overline{w_0}\right) + \frac{K}{4}\left(1 + \frac{1}{r}\right)^2$.

The proofs of Theorems 1 and 2 will be shown in Section 3.

We now show our main results as Theorems 3 and 4.

Theorem 3. *Let $\left(u^D(t,x), v^D(t,x), w^D(t,x)\right)$ be the nonnegative global smooth solution of* (1.2) $-$(1.4)*. If $D \geq \frac{2c_1}{\lambda}$ is assumed, then there exist positive constants c_5 independent of D and $T(D)$ such that*

$$\|(w^D - \overline{w^D})(t)\|_{L^\infty(\Omega)} \leq c_5 \sqrt{\frac{1}{D}} \quad \text{for } t \in [T(D), \infty) \tag{1.12}$$

holds, where

$$c_5 = \begin{cases} c_5(u_0, v_0, w_0, \nabla u_0, \nabla v_0, \nabla w_0) & \text{if } n = 1, \\ c_5(u_0, v_0, w_0, \Delta u_0, \Delta v_0, \Delta w_0) & \text{if } n = 2, 3 \end{cases}$$

and

$$T(D) = \begin{cases} \max\left\{0, \dfrac{4\log\left(\|\nabla w_0\|^2_{L^2(\Omega)} D\right)}{D\lambda}\right\} & \text{if } n = 1, \\ \max\left\{0, \dfrac{4\log\left(\|\Delta w_0\|^2_{L^2(\Omega)} D\right)}{D\lambda}\right\} & \text{if } n = 2, 3. \end{cases}$$

By noting that $\lim_{D\to\infty} T(D) = 0$, this theorem indicates that if D is sufficiently large, then $w^D(t,x)$ of (1.2)$-$(1.4) becomes almost spatially homogeneous in time. This result supports the behavior of $w^D(t,x)$ in Figure 2. However, this theorem does not imply that $w^D(t,x)$ converges to $\overline{w^D}(x)$ as $t \to \infty$, even if D is large enough.

Theorem 4. *Let $\left(u^D(t,x), v^D(t,x), w^D(t,x)\right)$ and $(u^\infty(t,x), v^\infty(t,x), \xi(t))$ be the nonnegative global smooth solutions of* (1.2)-(1.4) *and* (1.6)-(1.8)*, respectively. If $D \geq \frac{2c_1}{\lambda}$ is assumed, then for any fixed $T > 0$, there exists a positive constant $c_6(T)$ independent of D such that*

$$\begin{aligned} &\|(u^D - u^\infty)(t)\|_{L^\infty(\Omega)} + \|(v^D - v^\infty)(t)\|_{L^\infty(\Omega)} + \|(w^D - \xi)(t)\|_{L^\infty(\Omega)} \\ &\qquad \leq c_6(T)\sqrt{\frac{1}{D}} \quad \text{for } t \in [T(D), T] \end{aligned} \tag{1.13}$$

holds, where $T(D)$ is the same as that in Theorems 3 and

$$c_6(T) = \begin{cases} c_6(T, u_0, v_0, w_0, \nabla u_0, \nabla v_0, \nabla w_0) & \text{if } n = 1, \\ c_6(T, u_0, v_0, w_0, \Delta u_0, \Delta v_0, \Delta w_0) & \text{if } n = 2, 3. \end{cases}$$

This theorem indicates that, if $T > 0$ is arbitrarily fixed, the solution of (1.2)$-$(1.4) converges to that of (1.6)$-$(1.8) in $(0, T]$ as $D \to \infty$.

The proofs of Theorems 3 and 4 will be stated in Section 4.

In the following sections, we use the Banach space $L^p(\Omega)$ with the norm

$$\|u\|_p = \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$, and

$$\|u\|_\infty = \|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |u(x)|$$

for $p = \infty$. In addition, we use Sobolev spaces $W_2^l(\Omega)$ $(l = 1,2,3)$ of which the elements are functions u such that $D_x^k u$ belong to $L^2(\Omega)$ for any k $(|k| \le l)$, where $D_x^k u = \partial^{|k|} u / \partial x_1^{k_1} \ldots \partial x_n^{k_n}$ is the generalized derivative of order $|k| = k_1 + k_2 + \ldots + k_n$ for a multi-index $k = (k_1, k_2, \ldots, k_n)$. $W_2^l(\Omega)$ are endowed with the finite norm $\|u\|^2_{W_2^l(\Omega)} = \sum_{|k|\le l} \|D_x^k u\|^2_{L^2(\Omega)}$.

Throughout the proofs of the theorems which will be stated later, we use c, c_i, c_i', c_i'' and c_i''' $(i = 1,2,\ldots)$ as positive constants independent of D.

2. Preliminaries

Before proceeding to the next section, we recall the well-known inequalities, which are used in the proofs of Theorems 1–4.

(i) Young's inequality ([7]): For any positive constants a, b, p, and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \tag{2.1}$$

holds.

(ii) Let f be a suitably smooth function defined in Ω with the zero-flux boundary conditions at $\partial\Omega$.

(1) The elliptic estimate ([2]): For some c

$$\begin{cases} \|f\|_{W_2^2(\Omega)} \le c(\|\Delta f\|_2 + \|f\|_2), \\ \|f\|_{W_2^3(\Omega)} \le c(\|\nabla(\Delta f)\|_2 + \|f\|_2) \end{cases} \tag{2.2}$$

hold.

(2) ([4]):

$$\begin{cases} \lambda\|f - \overline{f}\|_2^2 \le \|\nabla f\|_2^2, \\ \lambda\|\nabla f\|_2^2 \le \|\Delta f\|_2^2 \end{cases} \tag{2.3}$$

hold.

(iii) The Gagliardo–Nirenberg inequality ([3], [6]): Let $\alpha = \alpha(n)$ be a constant satisfying $\frac{1}{2} < \alpha < 1$ $(n = 1, 2, 3)$. For some c

$$\|\nabla f\|_4 \le c\|f\|^{\alpha}_{W_2^2(\Omega)} \|f\|_2^{1-\alpha} \tag{2.4}$$

holds.

(iv) The Sobolev embedding theorem ([1]): For some c

$$\|f\|_\infty \le \begin{cases} c\|f\|_{W_2^1(\Omega)} & \text{if } n = 1, \\ c\|f\|_{W_2^2(\Omega)} & \text{if } n = 2, 3 \end{cases} \tag{2.5}$$

holds.

3. Proofs of Theorems 1 and 2

In this section, we obtain *a priori* estimates for solutions of the problems (1.2)−(1.4) and (1.6)−(1.8).

3.1. Proof of Theorem 1

First, it is obvious to see that

$$\begin{aligned} & 0 \le u^D(t,x) \le \max\{\|u_0\|_\infty, K\} \\ \text{and} \quad & 0 \le v^D(t,x) \le \max\{\|v_0\|_\infty, K\} \end{aligned} \qquad \text{for } t > 0,\ x \in \Omega.$$

So we only show (1.10). Since (2.2) and (2.5) lead to

$$\|w^D(t)\|_\infty^2 \le \begin{cases} c\left(\|w^D(t)\|_2^2 + \|\nabla w^D(t)\|_2^2\right) & \text{if } n = 1, \\ c\left(\|w^D(t)\|_2^2 + \|\Delta w^D(t)\|_2^2\right) & \text{if } n = 2, 3 \end{cases} \tag{3.1}$$

for some c, therefore we obtain *a priori* estimates for $\|w^D(t)\|_2$, $\|\nabla w^D(t)\|_2$ and $\|\Delta w^D(t)\|_2$ as follows:

Lemma 3.1. *If $D \ge \frac{2c_1}{\lambda}$, then there exist c_7, c_8 and c_9 such that*

$$\|w^D(t)\|_2^2 \le c_7, \tag{3.2}$$

$$\|\nabla w^D(t)\|_2^2 \le c_8 \qquad \textit{for } t > 0 \tag{3.3}$$

$$\textit{and} \qquad \|\Delta w^D(t)\|_2^2 \le c_9 \tag{3.4}$$

hold, where $c_7 = c_7(u_0, v_0, w_0)$, $c_8 = c_8(u_0, v_0, w_0, \nabla u_0, \nabla v_0, \nabla w_0)$ and $c_9 = c_9(u_0, v_0, w_0, \Delta u_0, \Delta v_0, \Delta w_0)$.

Proof. (i) Proof of (3.2): Define $\widetilde{w^D}(t,x)$ by $w^D(t,x)=\overline{w^D}(t)+\widetilde{w^D}(t,x)$. Then we know

$$\|w^D(t)\|_2^2 \le 2(|\Omega||\overline{w^D}(t)|^2+\|\widetilde{w^D}(t)\|_2^2). \tag{3.5}$$

We first show *a priori* estimate for $\overline{w^D}(t)$ in a similar way to the ones in [15] and [19]. Adding the first equation and the third one of (1.2) multiplied by $\frac{1}{|\Omega|}$ and by $\frac{r}{|\Omega|}$, respectively, and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\overline{u^D}+r\overline{w^D}\right)+\frac{ar}{K|\Omega|}\int_\Omega u^D v^D\,\mathrm{d}x+\frac{\mu r}{|\Omega|}\int_\Omega w^D v^D\,\mathrm{d}x\\
&\quad=\frac{1}{|\Omega|}\int_\Omega r u^D\left(1-\frac{u^D}{K}\right)\mathrm{d}x-r\overline{w^D}\le\frac{Kr}{4}\left(1+\frac{1}{r}\right)^2-\left(\overline{u^D}+r\overline{w^D}\right).
\end{aligned}$$

Here we used

$$r u^D\left(1-\frac{u^D}{K}\right)\le\frac{Kr}{4}\left(1+\frac{1}{r}\right)^2-u^D.$$

Then by Gronwall's lemma, we obtain

$$\overline{u^D}+r\overline{w^D}\le \mathrm{e}^{-t}\left(\overline{u_0}+r\overline{w_0}\right)+\frac{Kr}{4}\left(1+\frac{1}{r}\right)^2$$

and by $\overline{u^D}>0$,

$$0<\overline{w^D}(t)\le\frac{1}{r}\mathrm{e}^{-t}\left(\overline{u_0}+r\overline{w_0}\right)+\frac{K}{4}\left(1+\frac{1}{r}\right)^2 \quad \text{for } t>0. \tag{3.6}$$

We next show *a priori* estimate for $\|\widetilde{w^D}(t)\|_2$. Define $\mathcal{M}$ by $\mathcal{M}f=\overline{f}$. Subtracting $\mathcal{M}$ {the third equation of (1.2)} from the third equation of (1.2), and multiplying it by $\widetilde{w^D}$ and integrating over Ω, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widetilde{w^D}(t)\|_2^2+\|\widetilde{w^D}(t)\|_2^2+D\|\nabla w^D(t)\|_2^2\\
&\quad=\int_\Omega\widetilde{w^D}\left[w^D\left(u^D-\mu v^D\right)-\mathcal{M}\left\{w^D(u^D-\mu v^D)\right\}\right]\mathrm{d}x\\
&\quad=\int_\Omega\widetilde{w^D}\left(\overline{w^D}+\widetilde{w^D}\right)\left(u^D-\mu v^D\right)\mathrm{d}x\\
&\quad\le c_1\left(\sqrt{|\Omega|}\|\widetilde{w^D}(t)\|_2|\overline{w^D}|+\|\widetilde{w^D}(t)\|_2^2\right)-\mu\int_\Omega\widetilde{w^D}\left(\overline{w^D}+\widetilde{w^D}\right)v^D\,\mathrm{d}x\\
&\quad\le(c_1+1)\|\widetilde{w^D}(t)\|_2^2+\frac{|\Omega|(c_1^2+2\mu c_2)}{4}|\overline{w^D}|^2-\frac{\mu}{2}\int_\Omega|\widetilde{w^D}|^2v^D\,\mathrm{d}x.
\end{aligned}$$

Here we used (1.9) and the following inequalities:

$$-\mu\int_\Omega\widetilde{w^D}\,\overline{w^D}v^D\,\mathrm{d}x\le\frac{\mu}{2}\int_\Omega|\widetilde{w^D}|^2v^D\,\mathrm{d}x+\frac{\mu|\Omega|c_2}{2}|\overline{w^D}|^2$$

and

$$c_1\sqrt{|\Omega|}\|\widetilde{w^D}(t)\|_2|\overline{w^D}| \le \|\widetilde{w^D}(t)\|_2^2 + \frac{|\Omega|c_1^2}{4}|\overline{w^D}|^2$$

which was obtained from (2.1) with $a=\sqrt{2}\|\widetilde{w^D}(t)\|_2$, $b=c_1\sqrt{\frac{|\Omega|}{2}}|\overline{w^D}|$ and $p=q=2$. By using (2.3) and (3.6), we obtain

$$\frac{d}{dt}\|\widetilde{w^D}(t)\|_2^2 \le 2(c_1-D\lambda)\|\widetilde{w^D}(t)\|_2^2 + \frac{|\Omega|(c_1^2+2\mu c_2)c_7'^2}{2}, \tag{3.7}$$

where $c_7' = \frac{1}{r}\left(\overline{u_0}+r\overline{w_0}\right)+\frac{K}{4}\left(1+\frac{1}{r}\right)^2$. Therefore, since $2(c_1-D\lambda)\le -\frac{D\lambda}{4}$ by $D\ge\frac{2c_1}{\lambda}$, (3.7) is rewritten as

$$\frac{d}{dt}\left(\|\widetilde{w^D}(t)\|_2^2 e^{\frac{D\lambda}{4}t}\right) \le \frac{|\Omega|(c_1^2+2\mu c_2)c_7'^2}{2}e^{\frac{D\lambda}{4}t}.$$

Thus, putting $c_7''=\frac{3|\Omega|}{8\lambda}(c_1^2+2\mu c_2)c_7'^2$, we obtain

$$\|\widetilde{w^D}(t)\|_2^2 \le \|\widetilde{w_0}\|_2^2 e^{-\frac{D\lambda}{4}t} + \frac{c_7''}{D}\left(1-e^{-\frac{D\lambda}{4}t}\right) \quad \text{for } t>0. \tag{3.8}$$

Then, if (3.6) and (3.8) are used in (3.5), (3.2) is proved, where c_7 is simply taken as $c_7 = 2|\Omega|c_7'^2 + 2(\|\widetilde{w_0}\|_2^2+\frac{\lambda}{3c_1}c_7'')$.

(ii) Proof of (3.3): Multiplying the third equation of (1.2) by Δw^D and integrating over Ω, by $D\ge\frac{2c_1}{\lambda}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla w^D(t)\|_2^2 + \|\nabla w^D(t)\|_2^2 + \mu\int_\Omega |\nabla w^D|^2 v^D\,\mathrm{d}x + \frac{2c_1}{\lambda}\|\Delta w^D(t)\|_2^2 \\ \le \|w^D(t)\|_2\|\nabla(u^D-\mu v^D)(t)\|_4\|\nabla w^D(t)\|_4 + c_1\|\nabla w^D(t)\|_2^2.\end{aligned}$$

Since we need to obtain *a priori* estimates for u^D and v^D, we use $\mathbf{s}^D = {}^T(u^D, v^D, w^D)$ and define $\|\mathbf{s}^D_\nabla(t)\|_2$ and $\|\mathbf{s}^D_\Delta(t)\|_2$ by

$$\|\mathbf{s}^D_\nabla(t)\|_2^2 = \|\nabla u^D(t)\|_2^2 + \|\nabla v^D(t)\|_2^2 + \|\nabla w^D(t)\|_2^2$$

and

$$\|\mathbf{s}^D_\Delta(t)\|_2^2 = \|\Delta u^D(t)\|_2^2 + \|\Delta v^D(t)\|_2^2 + \|\Delta w^D(t)\|_2^2,$$

respectively. Then we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla w^D(t)\|_2^2 + \|\nabla w^D(t)\|_2^2 + \frac{2c_1}{\lambda}\|\Delta w^D(t)\|_2^2 \le \delta\|\mathbf{s}^D_\Delta(t)\|_2^2 + c_8'\delta^{-\frac{\alpha}{1-\alpha}} \tag{3.9}$$

for any constant δ satisfying $0<\delta\le 1$, where $c_8' = c_8'(c_1,c_2,c_7)$. Here we used (1.9), (2.2), (2.4), (3.2) and the inequality

$$\|w^D(t)\|_2\|\nabla(u^D-\mu v^D)(t)\|_4\|\nabla w^D(t)\|_4$$

$$\le c\sqrt{c_7}\left(\|\Delta u^D(t)\|_2+\|\Delta v^D(t)\|_2+\sqrt{|\Omega|}(c_1+c_2)\right)^{\alpha}\left(\sqrt{|\Omega|}(c_1+c_2)\right)^{1-\alpha}$$
$$\times\left(\|\Delta w^D(t)\|_2+\sqrt{c_7}\right)^{\alpha}(\sqrt{c_7})^{1-\alpha}$$
$$\le \delta\|\mathbf{s}_\Delta^D(t)\|_2^2+c_8''\delta^{-\frac{\alpha}{1-\alpha}}$$

for $c_8''=c_8''(c_1,c_2,c_7)$, which was obtained from (2.1) with

$$a=\delta^{\alpha}\left(\|\Delta u^D(t)\|_2+\|\Delta v^D(t)\|_2+\sqrt{|\Omega|}(c_1+c_2)\right)^{\alpha}\left(\|\Delta w^D(t)\|_2+\sqrt{c_7}\right)^{\alpha},$$
$$b=\delta^{-\alpha}c\sqrt{c_7}\left(\sqrt{|\Omega|}(c_1+c_2)\sqrt{c_7}\right)^{1-\alpha},\quad p=\frac{1}{\alpha}\quad\text{and}\quad q=\frac{1}{1-\alpha}.$$

Similarly to (3.9), we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^D(t)\|_2^2+\|\Delta u^D(t)\|_2^2\le\delta\|\mathbf{s}_\Delta^D(t)\|_2^2+c_8'''\delta^{-\frac{\alpha}{1-\alpha}} \tag{3.10}$$

for $c_8'''=c_8'''(c_1,c_2,c_7)$ and

$$\frac{1}{2}\frac{d}{dt}\|\nabla v^D(t)\|_2^2+\|\Delta v^D(t)\|_2^2\le\delta\|\mathbf{s}_\Delta^D(t)\|_2^2+c_8''''\delta^{-\frac{\alpha}{1-\alpha}} \tag{3.11}$$

for $c_8''''=c_8''''(c_1,c_2,c_7)$. Adding (3.9) and (3.10) and (3.11), we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{s}_\nabla^D(t)\|_2^2+\left(\frac{\min\{\lambda,2c_1\}}{\lambda}-3\delta\right)\|\mathbf{s}_\Delta^D(t)\|_2^2\le(c_8'+c_8'''+c_8'''')\delta^{-\frac{\alpha}{1-\alpha}}.$$

Since (2.3) leads to $\lambda\|\mathbf{s}_\nabla^D(t)\|_2^2\le\|\mathbf{s}_\Delta^D(t)\|_2^2$, if δ is chosen as $\delta=\frac{\min\{\lambda,2c_1\}}{6\lambda}$, we obtain

$$\|\mathbf{s}_\nabla^D(t)\|_2^2\le\|\mathbf{s}_\nabla^D(0)\|_2^2\mathrm{e}^{-\min\{\lambda,2c_1\}t}+\frac{2(c_8'+c_8'''+c_8'''')}{\min\{\lambda,2c_1\}}\left(\frac{\min\{\lambda,2c_1\}}{6\lambda}\right)^{-\frac{\alpha}{1-\alpha}}. \tag{3.12}$$

Consequently when c_8 is simply taken as

$$c_8=\|\mathbf{s}_\nabla^D(0)\|_2^2+\frac{2(c_8'+c_8'''+c_8'''')}{\min\{\lambda,2c_1\}}\left(\frac{\min\{\lambda,2c_1\}}{6\lambda}\right)^{-\frac{\alpha}{1-\alpha}},$$

(3.3) is proved.

(iii) Proof of (3.4): Define $\|\mathbf{s}_{\nabla\Delta}^D(t)\|_2$ by

$$\|\mathbf{s}_{\nabla\Delta}^D(t)\|_2^2=\|\nabla(\Delta u^D(t))\|_2^2+\|\nabla(\Delta v^D(t))\|_2^2+\|\nabla(\Delta w^D(t))\|_2^2.$$

In a similar way to (3.9), we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta w^D(t)\|_2^2+\|\Delta w^D(t)\|_2^2+\mu\int_\Omega|\Delta w^D|^2v^D\,\mathrm{d}x+\frac{2c_1}{\lambda}\|\nabla(\Delta w^D(t))\|_2^2$$
$$\le\delta\|\mathbf{s}_{\nabla\Delta}^D(t)\|_2^2+c_9'\delta^{-\frac{\alpha}{1-\alpha}}$$

for any constant δ satisfying $0<\delta\le 1$, where $c_9' = c_9'(c_1,c_2,c_7)$. Here we used

$$\frac{\partial \Delta \mathbf{s}^D}{\partial \nu} = {}^T(0,0,0) \quad \text{for } t>0,\ x\in\partial\Omega, \tag{3.13}$$

which was obtained from (1.2) and (1.3). Similarly to (3.10), (3.11) and (3.12), we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta u^D(t)\|_2^2 + \|\nabla(\Delta u^D(t))\|_2^2 \le \delta\|\mathbf{s}^D_{\nabla\Delta}(t)\|_2^2 + c_9''\delta^{-\frac{\alpha}{1-\alpha}},$$
$$\frac{1}{2}\frac{d}{dt}\|\Delta v^D(t)\|_2^2 + \|\nabla(\Delta v^D(t))\|_2^2 \le \delta\|\mathbf{s}^D_{\nabla\Delta}(t)\|_2^2 + c_9'''\delta^{-\frac{\alpha}{1-\alpha}}$$

and

$$\|\mathbf{s}^D_\Delta(t)\|_2^2 \le \|\mathbf{s}^D_\Delta(0)\|_2^2 \mathrm{e}^{-\min\{\lambda,2c_1\}t} + \frac{2(c_9'+c_9''+c_9''')}{\min\{\lambda,2c_1\}}\left(\frac{\min\{\lambda,2c_1\}}{6\lambda}\right)^{-\frac{\alpha}{1-\alpha}}, \tag{3.14}$$

respectively, where $c_9'' = c_9''(c_1,c_2,c_7)$ and $c_9''' = c_9'''(c_1,c_2,c_7)$. Then if c_9 is simply taken as $c_9 = \|\mathbf{s}^D_\Delta(0)\|_2^2 + \frac{2(c_9'+c_9''+c_9''')}{\min\{\lambda,2c_1\}}\left(\frac{\min\{\lambda,2c_1\}}{6\lambda}\right)^{-\frac{\alpha}{1-\alpha}}$, (3.4) is proved. □

Consequently, by using (3.1) and Lemma 3.1, Theorem 1 can be proved.

3.2. Proof of Theorem 2

The first two inequalities of (1.11) are obvious. For the proof of the third inequality of (1.11), we apply a similar way used in the proof of (3.6) to the third equation of (1.6), and obtain

$$0<\xi(t)\le \frac{1}{r}\mathrm{e}^{-t}\left(\overline{u_0}+r\overline{w_0}\right)+\frac{K}{4}\left(1+\frac{1}{r}\right)^2 \quad \text{for } t>0.$$

Therefore, by putting $c_4 = \frac{1}{r}\left(\overline{u_0}+r\overline{w_0}\right)+\frac{K}{4}\left(1+\frac{1}{r}\right)^2$, Theorem 2 can be proved.

4. Proofs of Theorems 3 and 4

4.1. Proof of Theorem 3

Since $w^D - \overline{w^D} = \widetilde{w^D}$, we derive the uniform estimate for $\|\widetilde{w^D}(t)\|_\infty$ with respect to D. By using (2.2) and (2.5), we find that for some c,

$$\|\widetilde{w^D}(t)\|_\infty^2 \le \begin{cases} c(\|\widetilde{w^D}(t)\|_2^2+\|\nabla\widetilde{w^D}(t)\|_2^2) & \text{if } n=1,\\ c(\|\widetilde{w^D}(t)\|_2^2+\|\Delta\widetilde{w^D}(t)\|_2^2) & \text{if } n=2,3.\end{cases} \tag{4.1}$$

Since the uniform estimate for $\|\widetilde{w^D}(t)\|_2$ is already known in (3.8), we will obtain the uniform estimates for $\|\nabla\widetilde{w^D}(t)\|_2$ $(=\|\nabla w^D(t)\|_2)$ and $\|\Delta\widetilde{w^D}(t)\|_2$ $(=\|\Delta w^D(t)\|_2)$ with respect to D.

Lemma 4.1. *If $D\ge\frac{2c_1}{\lambda}$, then there exist c_{10} and c_{11} such that*

$$\|\nabla w^D(t)\|_2^2 \le \|\nabla w_0\|_2^2\mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_{10}}{D}\left(1-\mathrm{e}^{-\frac{D\lambda}{4}t}\right) \tag{4.2}$$

$$\textit{and} \qquad \|\Delta w^D(t)\|_2^2 \le \|\Delta w_0\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_{11}}{D}\left(1-\mathrm{e}^{-\frac{D\lambda}{4}t}\right) \qquad \textit{for } t>0 \tag{4.3}$$

hold, respectively, where $c_{10} = c_{10}(u_0, v_0, w_0, \nabla u_0, \nabla v_0, \nabla w_0)$ *and* $c_{11} = c_{11}(u_0, v_0, w_0, \Delta u_0, \Delta v_0, \Delta w_0)$.

Proof. Multiplying the third equation of (1.2) by Δw^D and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla w^D(t)\|_2^2 + \|\nabla w^D(t)\|_2^2 + \mu\int_\Omega |\nabla w^D|^2 v^D\,\mathrm{d}x + D\|\Delta w^D(t)\|_2^2 \\
&\quad\le \|w^D(t)\|_\infty \|\nabla(u^D - \mu v^D)(t)\|_2 \|\nabla w^D(t)\|_2 + c_1\|\nabla w^D(t)\|_2^2 \\
&\quad\le c(\sqrt{c_7}+\sqrt{c_9})\sqrt{c_8}\|\nabla w^D(t)\|_2 + c_1\|\nabla w^D(t)\|_2^2 \\
&\quad\le (c_1+1)\,\|\nabla w^D(t)\|_2^2 + c_{10}',
\end{aligned}$$

where $c_{10}' = c_{10}'(c_7, c_8, c_9)$. Here we used (2.2), (2.5), (3.2), (3.12) and (3.14). Hence, by (2.3) we have

$$\frac{d}{dt}\|\nabla w^D(t)\|_2^2 \le 2(c_1 - D\lambda)\,\|\nabla w^D(t)\|_2^2 + 2c_{10}',$$

and by (3.13)

$$\frac{d}{dt}\|\Delta w^D(t)\|_2^2 \le 2(c_1 - D\lambda)\,\|\Delta w^D(t)\|_2^2 + 2c_{11}',$$

where $c_{11}' = c_{11}'(c_1, c_2, c_7, c_8, c_9)$. Noting that $D \ge \frac{2c_1}{\lambda}$ leads $2(c_1 - D\lambda) \le -\frac{D\lambda}{4}$, (4.2) and (4.3) are proved. □

Using (3.8) and (4.2) in (4.1) and applying the first inequality of (2.3) to $\|\widetilde{w_0}\|_2^2$, we obtain

$$\begin{aligned}
\|\widetilde{w^D}(t)\|_\infty^2 &\le c\left(\|\widetilde{w_0}\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_7''}{D} + \|\nabla w_0\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_{10}}{D}\right) \\
&\le c\left(\|\nabla w_0\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_7''+c_{10}}{D}\right)
\end{aligned}$$

for $n = 1$, and

$$\begin{aligned}
\|\widetilde{w^D}(t)\|_\infty^2 &\le c\left(\|\widetilde{w_0}\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_7''}{D} + \|\Delta w_0\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_{11}}{D}\right) \\
&\le c\left(\|\Delta w_0\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_7''+c_{11}}{D}\right)
\end{aligned}$$

for $n = 2, 3$. Then if $T(D)$ is defined by

$$T(D) = \begin{cases} 0 & \text{if } \|\nabla w_0\|_2^2 D \le 1, \\[2ex] \dfrac{4\log\left(\|\nabla w_0\|_2^2 D\right)}{D\lambda} & \text{if } \|\nabla w_0\|_2^2 D > 1, \end{cases}$$

Theorem 3 is proved.

4.2. Proof of Theorem 4

Define U^D, V^D and W^D by

$$U^D = u^D - u^\infty, \quad V^D = v^D - v^\infty \quad \textit{and} \quad W^D = w^D - \xi,$$

respectively, and obtain the uniform estimates for $\|U^D(t)\|_\infty$, $\|V^D(t)\|_\infty$ and $\|W^D(t)\|_\infty$ with respect to D. First, by $\widetilde{W^D} = \widetilde{w^D}$, we note

$$\|W^D(t)\|_\infty \le \|\overline{W^D}(t)\|_\infty + \|\widetilde{w^D}(t)\|_\infty. \tag{4.4}$$

We therefore obtain the uniform estimates for $\|U^D(t)\|_\infty$, $\|V^D(t)\|_\infty$ and $\|\overline{W^D}(t)\|_\infty$, because the uniform estimate for $\|\widetilde{w^D}(t)\|_\infty$ is already obtained in Theorem 3.

Put $\overline{\mathbf{S}^D} = {}^T(U^D, V^D, \overline{W^D})$ and define

$$\begin{aligned}
\|\overline{\mathbf{S}^D}(t)\|_\infty &= \|U^D(t)\|_\infty + \|V^D(t)\|_\infty + \|\overline{W^D}(t)\|_\infty,\\
\|\overline{\mathbf{S}^D}(t)\|_2^2 &= \|U^D(t)\|_2^2 + \|V^D(t)\|_2^2 + \|\overline{W^D}(t)\|_2^2,\\
\|\overline{\mathbf{S}^D_\nabla}(t)\|_2^2 &= \|\nabla U^D(t)\|_2^2 + \|\nabla V^D(t)\|_2^2
\end{aligned}$$

and

$$\|\overline{\mathbf{S}^D_\Delta}(t)\|_2^2 = \|\Delta U^D(t)\|_2^2 + \|\Delta V^D(t)\|_2^2.$$

Then from (2.2) and (2.5),

$$\|\overline{\mathbf{S}^D}(t)\|_\infty \le c\sqrt{\|\overline{\mathbf{S}^D}(t)\|_2^2 + \|\overline{\mathbf{S}^D_\nabla}(t)\|_2^2} \quad \text{if } n = 1 \tag{4.5}$$

and

$$\|\overline{\mathbf{S}^D}(t)\|_\infty \le c\sqrt{\|\overline{\mathbf{S}^D}(t)\|_2^2 + \|\overline{\mathbf{S}^D_\Delta}(t)\|_2^2} \quad \text{if } n = 2,3 \tag{4.6}$$

hold for some c.

We now obtain the uniform estimates for $\|\overline{\mathbf{S}^D}(t)\|_2^2$, $\|\overline{\mathbf{S}^D_\nabla}(t)\|_2^2$ and $\|\overline{\mathbf{S}^D_\Delta}(t)\|_2^2$ with respect to D. In order to obtain these estimates, we first derive the initial and boundary value problem for $\overline{\mathbf{S}^D}$. The equations for both U^D and V^D can be written as

$$\frac{\partial U^D}{\partial t} = A^D U^D - \frac{ar}{K} u^\infty V^D - r u^\infty W^D - r u^\infty \widetilde{w^D} + \Delta U^D$$

and

$$\frac{\partial V^D}{\partial t} = B^D V^D - \frac{br}{K} v^\infty U^D - d(\mu) r v^\infty W^D - d(\mu) r v^\infty \widetilde{w^D} + \Delta V^D,$$

respectively, where A^D and B^D are given by

$$A^D = r\left(1 - \frac{u^D + u^\infty + a v^D}{K} - w^D\right)$$

and

$$B^D = r\left(1 - \frac{v^D + v^\infty + bu^D}{K} - d(\mu)w^D\right),$$

respectively. In order to derive the equation for $\overline{W^D}$, we obtain the following equation for $\overline{w^D}$:

$$\begin{aligned}\frac{\partial \overline{w^D}}{\partial t} &= \frac{1}{|\Omega|}\int_\Omega w^D\left(u^D - \mu v^D - 1\right)\mathrm{d}x \\ &= \overline{w^D}\left(\frac{1}{|\Omega|}\int_\Omega u^D\,\mathrm{d}x - \frac{\mu}{|\Omega|}\int_\Omega v^D\,\mathrm{d}x - 1\right) + g^D(t),\end{aligned}$$

where $g^D(t)$ is given by

$$g^D(t) = \frac{1}{|\Omega|}\int_\Omega \widetilde{w^D}\left(u^D - \mu v^D - 1\right)\mathrm{d}x.$$

Then, subtracting the third equation of (1.6) from it, we obtain

$$\frac{\partial \overline{W^D}}{\partial t} = C^D\overline{W^D} + \xi\left(\frac{1}{|\Omega|}\int_\Omega U^D\,\mathrm{d}x - \frac{\mu}{|\Omega|}\int_\Omega V^D\,\mathrm{d}x\right) + g^D(t),$$

where C^D is given by

$$C^D = \frac{1}{|\Omega|}\int_\Omega u^D\,\mathrm{d}x - \frac{\mu}{|\Omega|}\int_\Omega v^D\,\mathrm{d}x - 1$$

and by (1.9) and (3.8), $g^D(t)$ satisfies

$$\|g^D(t)\|_\infty^2 \le c\|\widetilde{w_0}\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c}{D}$$

for some c. From the above, we now obtain the following initial and boundary value problem for $\overline{\mathbf{S}^D} = {}^T(U^D, V^D, \overline{W^D})$:

$$\begin{cases} \dfrac{\partial \overline{\mathbf{S}^D}}{\partial t} = L^D\overline{\mathbf{S}^D} + R(D), & t > 0,\ x \in \Omega, \\ \overline{\mathbf{S}^D}(0,x) = {}^T(0,0,0), & x \in \Omega, \\ \dfrac{\partial \overline{\mathbf{S}^D}}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{4.7}$$

where L^D and $R(D)$ are given by

$$L^D = \begin{pmatrix} \Delta + A^D & -\dfrac{ar}{K}u^\infty & -ru^\infty \\ -\dfrac{br}{K}v^\infty & \Delta + B^D & -d(\mu)rv^\infty \\ 0 & 0 & C^D \end{pmatrix}$$

and

$$R(D)=\begin{pmatrix} -ru^{\infty}\widetilde{w^D} \\ -d(\mu)rv^{\infty}\widetilde{w^D} \\ \xi\left(\frac{1}{|\Omega|}\int_\Omega U^D\,\mathrm{d}x-\frac{\mu}{|\Omega|}\int_\Omega V^D\,\mathrm{d}x\right)+g^D(t)\end{pmatrix},$$

respectively.

We first obtain the uniform estimate for $\|\overline{\mathbf{S}^D}(t)\|_2$, $\|\overline{\mathbf{S}^D_\nabla}(t)\|_2$ and $\|\overline{\mathbf{S}^D_\Delta}(t)\|_2$.

Lemma 4.2. *If $D\ge\frac{2c_1}{\lambda}$, then for any fixed $T>0$, there exist $c_{12}(T)$, $c_{13}(T)$ and $c_{14}(T)$ depending on T but independent of D, such that*

$$\|\overline{\mathbf{S}^D}(t)\|_2^2\le\frac{c_{12}(T)}{D},\tag{4.8}$$

$$\|\overline{\mathbf{S}^D_\nabla}(t)\|_2^2\le\frac{c_{13}(T)}{D}\qquad \text{for } t\in[0,T]\tag{4.9}$$

and

$$\|\overline{\mathbf{S}^D_\Delta}(t)\|_2^2\le\frac{c_{14}(T)}{D}\tag{4.10}$$

hold, where $c_{12}(T)=c_{12}(T,u_0,v_0,w_0)$, $c_{13}(T)=c_{13}(T,u_0,v_0,w_0,\Delta u_0,\Delta v_0,\Delta w_0)$ and $c_{14}(T)=c_{14}(T,u_0,v_0,w_0,\Delta u_0,\Delta v_0,\Delta w_0)$.

Proof. Multiplying the first equation of (4.7) by $\overline{\mathbf{S}^D}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d\|\overline{\mathbf{S}^D}(t)\|_2^2}{dt}=\int_\Omega L^D\overline{\mathbf{S}^D}\cdot\overline{\mathbf{S}^D}\,\mathrm{d}x+\int_\Omega R(D)\cdot\overline{\mathbf{S}^D}\,\mathrm{d}x.\tag{4.11}$$

Here using (1.9)-(1.11), two terms in the right hand side of (4.11) can be estimated as follows: There is some c such that

$$\begin{aligned}\int_\Omega L^D\overline{\mathbf{S}^D}\cdot\overline{\mathbf{S}^D}\,\mathrm{d}x\le{}&-\|\nabla U^D(t)\|_2^2-\|\nabla V^D(t)\|_2^2\\ &-r\int_\Omega |U^D|^2\left(\frac{u^D+u^\infty+av^D}{K}+w^D\right)\mathrm{d}x\\ &-r\int_\Omega |V^D|^2\left(\frac{v^D+v^\infty+bu^D}{K}+d(\mu)w^D\right)\mathrm{d}x\\ &-\left(\frac{\mu}{|\Omega|}\int_\Omega v^D\,\mathrm{d}x+1\right)|\overline{W^D}(t)|^2+c\|\overline{\mathbf{S}^D}(t)\|_2^2\le c\|\overline{\mathbf{S}^D}(t)\|_2^2\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}\int_\Omega R(D)\cdot\overline{\mathbf{S}^D}\,\mathrm{d}x\le{}& c\left(\|U^D(t)\|_2+\|V^D(t)\|_2\right)\|\widetilde{w^D}(t)\|_2\\ &+\left\{c\left(\|U^D(t)\|_2+\|V^D(t)\|_2\right)+|g^D(t)|\right\}|\overline{W^D}(t)|\\ \le{}& c(\|\overline{\mathbf{S}^D}(t)\|_2^2+\|\widetilde{w^D}(t)\|_2^2),\end{aligned}\tag{4.13}$$

respectively. Hence, substituting (4.12) and (4.13) in the right hand side of (4.11) and using (3.8), then we have

$$\frac{1}{2}\frac{d\|\overline{\mathbf{S}^D}(t)\|_2^2}{dt} \le c\left(2\|\overline{\mathbf{S}^D}(t)\|_2^2 + \|\widetilde{w_0}\|_2^2 \mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_7''}{D}\right).$$

Noting $\|\overline{\mathbf{S}^D}(0)\|_2^2 = 0$, Gronwall's lemma leads to

$$\|\overline{\mathbf{S}^D}(t)\|_2^2 \le \left(\frac{3c\|\widetilde{w_0}\|_2^2}{6c+2D\lambda} + \frac{c_7''}{2D}\right)\mathrm{e}^{4ct}. \tag{4.14}$$

Thus putting $c_{12}(t) = \left(\frac{3c}{2\lambda}\|\widetilde{w_0}\|_2^2 + \frac{c_7''}{2}\right)\mathrm{e}^{4ct}$, (4.8) can be derived.

We next consider (4.9). In a similar way to (4.11)-(4.13), we have

$$\frac{1}{2}\frac{d\|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2}{dt} = \int_\Omega L^D\overline{\mathbf{S}^D}\cdot\Delta\overline{\mathbf{S}^D}\,\mathrm{d}x + \int_\Omega R(D)\cdot\Delta\overline{\mathbf{S}^D}\,\mathrm{d}x. \tag{4.15}$$

Using (2.3), (2.4), (3.12), (3.14) and $\|\nabla\widetilde{w^D}(t)\|_2 = \|\nabla w^D(t)\|_2$, two terms in the right hand sides of (4.15) can be estimated as follows: For some c,

$$\begin{aligned}\int_\Omega L^D\overline{\mathbf{S}^D}\cdot\Delta\overline{\mathbf{S}^D}\,\mathrm{d}x &\le c\|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2 + \left\|\nabla\left(\frac{u^D+u^\infty+av^D}{K}+w^D\right)(t)\right\|_4\|U^D(t)\|_4\|\nabla U^D(t)\|_2\\ &\quad+\left\|\nabla\left(\frac{v^D+v^\infty+bu^D}{K}+d(\mu)w^D\right)(t)\right\|_4\|V^D(t)\|_4\|\nabla V^D(t)\|_2\\ &\le c\left(\|\overline{\mathbf{S}^D}(t)\|_2^2+\|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2\right)\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}\int_\Omega R(D)\cdot\Delta\overline{\mathbf{S}^D}\,\mathrm{d}x &\le c\left(\|\nabla u^\infty(t)\|_4\|\widetilde{w^D}(t)\|_4 + \|u^\infty(t)\|_\infty\|\nabla\widetilde{w^D}(t)\|_2\right)\|\nabla U^D(t)\|_2\\ &\quad+c\left(\|\nabla v^\infty(t)\|_4\|\widetilde{w^D}(t)\|_4 + \|v^\infty(t)\|_\infty\|\nabla\widetilde{w^D}(t)\|_2\right)\|\nabla V^D(t)\|_2\\ &\le c\left(\|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2 + \|\nabla w^D(t)\|_2^2\right),\end{aligned} \tag{4.17}$$

respectively hold. Here we used

$$\begin{aligned}&\|\nabla u^\infty(t)\|_2^2 + \|\nabla v^\infty(t)\|_2^2 \le c_{15},\\ &\|\Delta u^\infty(t)\|_2^2 + \|\Delta v^\infty(t)\|_2^2 \le c_{15}',\end{aligned} \qquad \text{for } t>0,$$

where $c_{15} = c_{15}(u_0, v_0, w_0, \nabla u_0, \nabla v_0, \nabla w_0)$ and $c_{15}' = c_{15}'(u_0, v_0, w_0, \Delta u_0, \Delta v_0, \Delta w_0)$. which were obtained by using the similar way to the proofs of (3.3) and (3.4). Then we have

$$\frac{1}{2}\frac{d\|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2}{dt} \le c\left(2\|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2 + \|\nabla w_0\|_2^2\mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_{10}}{D} + \frac{c_{12}(t)}{D^2}\right),$$

where $c_{12}(t) = \left(\frac{3c}{2\lambda}\|\widetilde{w_0}\|_2^2 + \frac{c_7''}{2}\right)\mathrm{e}^{4ct}$. Noting $\|\overline{\mathbf{S}_\nabla^D}(0)\|_2^2 = 0$, Gronwall's lemma leads to

$$\|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2 \le \left(\frac{6c\,\|\nabla w_0\|_2^2}{12c + D\lambda} + \frac{c_{10}}{2D}\right)\mathrm{e}^{4ct} + \frac{2ctc_{12}(t)}{D^2}. \tag{4.18}$$

Thus putting $c_{13}(t) = \left(\frac{6c}{\lambda}\|\nabla w_0\|_2^2 + \frac{c_{10}}{2}\right)\mathrm{e}^{4ct} + \frac{2ctc_{12}(t)}{D}$, (4.9) can be derived.

Finally we obtain (4.10). Similarly to (4.11)–(4.13), we have

$$\frac{1}{2}\frac{d\|\overline{\mathbf{S}_\Delta^D}(t)\|_2^2}{dt} = \int_\Omega \Delta L^D\overline{\mathbf{S}^D}\cdot\Delta\overline{\mathbf{S}^D}\,\mathrm{d}x + \int_\Omega \Delta R(D)\cdot\Delta\overline{\mathbf{S}^D}\,\mathrm{d}x, \tag{4.19}$$

where, two terms in the right hand sides of (4.19) can be estimated as follows: For some c,

$$\begin{aligned}
\int_\Omega \Delta L^D\overline{\mathbf{S}^D}\cdot\Delta\overline{\mathbf{S}^D}\,\mathrm{d}x \le\ & c\|\overline{\mathbf{S}_\Delta^D}(t)\|_2^2 \\
&+ \left\|\Delta\left(\frac{u^D + u^\infty + av^D}{K} + w^D\right)(t)\right\|_2 \|U^D(t)\|_\infty\|\Delta U^D(t)\|_2 \\
&+ \left\|\Delta\left(\frac{v^D + v^\infty + bu^D}{K} + d(\mu)w^D\right)(t)\right\|_2 \|V^D(t)\|_\infty\|\Delta V^D(t)\|_2 \\
&+ \left\|\nabla\left(\frac{u^D + u^\infty + av^D}{K} + w^D\right)(t)\right\|_4 \|\nabla U^D(t)\|_4\|\Delta U^D(t)\|_2 \\
&+ \left\|\nabla\left(\frac{v^D + v^\infty + bu^D}{K} + d(\mu)w^D\right)(t)\right\|_4 \|\nabla V^D(t)\|_4\|\Delta V^D(t)\|_2 \\
\le\ & c\left(\|\overline{\mathbf{S}^D}(t)\|_2^2 + \|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2 + \|\overline{\mathbf{S}_\Delta^D}(t)\|_2^2\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \Delta R(D)\cdot\Delta\overline{\mathbf{S}^D}\,\mathrm{d}x &\le c(\|\widetilde{w^D}(t)\|_2 + \|\nabla\widetilde{w^D}(t)\|_2 + \|\Delta\widetilde{w^D}(t)\|_2)\|\overline{\mathbf{S}_\Delta^D}(t)\|_2 \\
&\le c\left(\|\overline{\mathbf{S}_\Delta^D}(t)\|_2^2 + \|\Delta w^D(t)\|_2^2\right),
\end{aligned}$$

respectively. Hence from (4.19), we obtain

$$\frac{1}{2}\frac{d\|\overline{\mathbf{S}_\Delta^D}(t)\|_2^2}{dt} \le c\left(2\|\overline{\mathbf{S}_\Delta^D}(t)\|_2^2 + \|\Delta w_0\|_2^2\,\mathrm{e}^{-\frac{D\lambda}{4}t} + \frac{c_{11}}{D} + \frac{c_{12}(t) + c_{13}(T)}{D^2}\right).$$

Noting $\|\overline{\mathbf{S}_\Delta^D}(0)\|_2^2 = 0$, Gronwall's lemma leads to

$$\|\overline{\mathbf{S}_\nabla^D}(t)\|_2^2 \le \left(\frac{6c\,\|\Delta w_0\|_2^2}{12c + D\lambda} + \frac{c_{11}}{2D}\right)\mathrm{e}^{4ct} + \frac{2ct(c_{12}(t) + c_{13}(t))}{D^2}.$$

Thus putting $c_{14}(t) = \left(\frac{6c}{\lambda}\|\Delta w_0\|_2^2 + \frac{c_{11}}{2}\right)\mathrm{e}^{4ct} + \frac{2ct(c_{12}(t)+c_{13}(t))}{D}$, (4.10) can be derived. □

Let $T>0$ be arbitrarily fixed constant. Consequently, using (4.8)-(4.10) in the right hand sides of (4.5) and (4.6), we have

$$\|\overline{\mathbf{S}^D}(t)\|_\infty \leq \begin{cases} c\sqrt{\dfrac{C_{12}(T)+C_{13}(T)}{D}} & \text{if } n=1, \\ c\sqrt{\dfrac{C_{12}(T)+C_{14}(T)}{D}} & \text{if } n=2,3 \end{cases}$$

for $t \in [0,T]$. Then defining $T(D)$ by

$$T(D) = \begin{cases} \max\left\{0, \dfrac{4\log\left(\|\nabla w_0\|_2^2 D\right)}{D\lambda}\right\} & \text{if } n=1, \\ \max\left\{0, \dfrac{4\log\left(\|\Delta w_0\|_2^2 D\right)}{D\lambda}\right\} & \text{if } n=2,3 \end{cases}$$

and using (1.12) in (4.4), we obtain

$$\|W^D(t)\|_\infty \leq \begin{cases} c\sqrt{\dfrac{C_{12}(T)+C_{13}(T)}{D}} + c_5\sqrt{\dfrac{1}{D}} & \text{if } n=1, \\ c\sqrt{\dfrac{C_{12}(T)+C_{14}(T)}{D}} + c_5\sqrt{\dfrac{1}{D}} & \text{if } n=2,3 \end{cases}$$

for $t \in [T(D),T]$. Thus, the proof of Theorem 4 is complete.

5. Concluding remarks

To study the occurrence of harmful algal blooms observed in lakes and rivers, we discussed a three-component reaction-diffusion system and its shadow system which was derived as the diffusion rate of the predator D tended to infinity. Under the zero-flux boundary conditions, we showed that a solution of the full system for arbitrarily given initial data (1.4) for (1.2) and (1.8) for (1.6) is well approximated by the one of the shadow system if D is very large. Precisely speaking, for any fixed $T>0$, any solution tends to that of the shadow system for $0<t<T$, as D tends to infinity. Of course, this is not a satisfactory result, because numerical simulation suggests that this result extends to the case for $0<t<\infty$. Unfortunately, the method used in this paper is unable to answer to this problem. We think that our approach is required to combine with the theory of global attractors for the full system and its shadow system which are discussed in [13] and [14]. However, the situation is rather difficult, because of the following reason: If $d(\mu)$ is $\frac{1}{1+(\mu/\delta)}$ with some constant $\delta>0$ (for instance $\delta=0.01$), the equilibrium E_4 undergoes Hopf bifurcations and the system exhibits oscillatory behaviour when μ increases. This is significantly different from the situation when $\delta=1$. The extension of our result to the case for $0<t<\infty$ will be part of our future work.

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