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PRE-GRÜSS TYPE INEQUALITIES IN 2-INNER PRODUCT SPACES

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Abstract. Some pre-Grüss type inequalities in 2-inner product space and applications for determinantal inequalities are given.

1. Introduction

Let f, g be two functions defined and integrable on $[a,b]$. Assume that

$$\varphi \leq f(x) \leq \Phi \quad \text{and} \quad \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a,b]$, where φ, Φ, γ, Γ are given real constant. Then the following inequality is well known in the literature as the Grüss inequality ([9, p.296])

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \leq \frac{1}{4}|\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

In this inequality, G. Grüss has proved that, the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one, and is achieved for

$$f(x) = g(x) = sgn\Big(x - \frac{a+b}{2}\Big).$$

In [3], S. S. Dragomir has proved the Grüss type inequality in real or complex inner product spaces. Further, S. S. Dragomir et al. have given some pre-Grüss type inequalities in real or complex inner product spaces [7].

In [8], the authors have proved the Grüss type inequality in 2-inner product spaces. Recently, in [4-6, 11], the authors have further given some refinements, generalizations, extensions and alternative proofs of Grüss type inequality in 2-inner product spaces.

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The purpose of this paper, we will establish the corresponding versions of pre-Grüss inequality for both real and complex 2-inner product spaces. Also, some determinantal inequalities are point out.

2. Preliminaries and Lemmas

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot \mid \cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2\mathrm{I}_1$) $(x, x \mid z) \geq 0$ and $(x, x \mid z) = 0$ if and only if x and z are linearly dependent,
($2\mathrm{I}_2$) $(x, x \mid z) = (z, z \mid x)$,
($2\mathrm{I}_3$) $(y, x \mid z) = \overline{(x, y \mid z)}$,
($2\mathrm{I}_4$) $(\alpha x, y \mid z) = \alpha(x, y \mid z)$ for any scalar $\alpha \in \mathbb{K}$,
($2\mathrm{I}_5$) $(x + x', y \mid z) = (x, y \mid z) + (x', y \mid z)$.

$(\cdot, \cdot \mid \cdot)$ is called a 2-*inner product* on X and $(X, (\cdot, \cdot \mid \cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product spaces can be immediately obtained as follows [2]:

(1) If $\mathbb{K} = \mathbb{R}$, then ($2\mathrm{I}_3$) reduces to

$$(y, x \mid z) = (x, y \mid z).$$

(2) From ($2\mathrm{I}_3$) and ($2\mathrm{I}_4$), we have

$$(0, y \mid z) = 0, \quad (x, 0 \mid z) = 0$$

and also

$$(x, \alpha y \mid z) = \overline{\alpha}(x, y \mid z). \tag{2.1}$$

(3) Using ($2\mathrm{I}_2$)-($2\mathrm{I}_5$), we have

$$(z, z \mid x \pm y) = (x \pm y, x \pm y \mid z) = (x, x \mid z) + (y, y \mid z) \pm 2\mathrm{Re}(x, y \mid z)$$

and

$$\mathrm{Re}(x, y \mid z) = \frac{1}{4}[(z, z \mid x + y) - (z, z \mid x - y)]. \tag{2.2}$$

In the real case $\mathbb{K} = \mathbb{R}$, (2.2) reduces to

$$(x, y \mid z) = \frac{1}{4}[(z, z \mid x + y) - (z, z \mid x - y)] \tag{2.3}$$

and, using this formula, it is easy to see that, for any $\alpha \in \mathbb{R}$,

$$(x, y \mid \alpha z) = \alpha^2 (x, y \mid z). \tag{2.4}$$

In the complex case, using (2.1) and (2.2), we have

$$\operatorname{Im}(x, y \mid z) = \operatorname{Re}[-i(x, y \mid z)] = \frac{1}{4}[(z, z \mid x + iy) - (z, z \mid x - iy)],$$

which, in combination with (2.2), yields

$$(x, y \mid z) = \frac{1}{4}[(z, z \mid x + y) - (z, z \mid x - y)] + \frac{i}{4}[(z, z \mid x + iy) - (z, z \mid x - iy)]. \tag{2.5}$$

Using the above formula and (2.1), we have, for any $\alpha \in \mathbb{C}$,

$$(x, y \mid \alpha z) = |\alpha|^2 (x, y \mid z). \tag{2.6}$$

However, for $\alpha \in \mathbb{R}$, (2.6) reduces to (2.4).

Also, from (2.6) it follows that

$$(x, y \mid 0) = 0.$$

(4) For any three given vectors x, y, $z \in X$, consider the vector $u = (y, y \mid z)x - (x, y \mid z)y$. By ($2I_1$), we know that $(u, u \mid z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u \mid z) \geq 0$ can be rewritten as,

$$(y, y \mid z)[(x, x \mid z)(y, y \mid z) - |(x, y \mid z)|^2] \geq 0. \tag{2.7}$$

For $x = z$, (2.7) becomes

$$-(y, y \mid z) \mid (z, y \mid z)|^2 \geq 0,$$

which implies that

$$(z, y \mid z) = (y, z \mid z) = 0 \tag{2.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (2.8) holds too. Thus (2.8) is true for any two vectors y, $z \in X$. Now, if y and z are linearly independent, then $(y, y \mid z) > 0$ and, from (2.7), it follows that

$$|(x, y \mid z)|^2 \leq (x, x \mid z)(y, y \mid z). \tag{2.9}$$

Using (2.8), it is easy to check that (2.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (2.9) holds for any three vectors x, y, $z \in X$ and it is strict unless the vectors $u = (y, y \mid z)x - (x, y \mid z)y$ and z are linearly dependent. In fact, we have the equality in (2.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot \mid \cdot))$, we can define a function $\|\cdot \mid \cdot\|$ on $X \times X$ by

$$\|x \mid z\| = \sqrt{(x, x \mid z)} \tag{2.10}$$

for all x, $z \in X$.

It is easy to see that this function satisfies the following conditions:

($2N_1$) $\|x \mid z\| \geq 0$ and $\|x \mid z\| = 0$ if and only if x and z are linearly dependent,
($2N_2$) $\|x \mid z\| = \|x \mid z\|$,
($2N_3$) $\|\alpha x \mid z\| = |\alpha| \, \|x \mid z\|$ for any scalar $\alpha \in \mathbb{K}$,
($2N_4$) $\|x + x' \mid z\| \leq \|x \mid z\| + \|x' \mid z\|$.

Any function $\|\cdot \mid \cdot\|$ defined on $X \times X$ and satisfying the conditions ($2N_1$)-($2N_4$) is called a 2-*norm* on X and $(X, \|\cdot \mid \cdot\|)$ is called a *linear* 2-*normed space* [10]. Whenever a 2-inner product space $(X, (\cdot, \cdot \mid \cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot \mid \cdot\|)$ with the 2-norm defined by (2.10).

Let $(X; (\cdot, \cdot \mid \cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1 \leq i \leq n}$ are linearly independent vectors in the 2-inner product space X, and, for a given $z \in X$, $(f_i, f_j \mid z) = \delta_{ij}$ for all i, $j \in \{1, \ldots, n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(f_i)_{1 \leq i \leq n}$ is z-*orthonormal*), then the following inequality is the corresponding *Bessel's inequality* (see for example [2]) for the z-*orthonormal family* $(f_i)_{1 \leq i \leq n}$ in the 2-inner product space $(X; (\cdot, \cdot \mid \cdot))$:

$$\sum_{i=1}^{n} |(x, f_i \mid z)|^2 \leq \|x \mid z\|^2 \tag{2.11}$$

for any $x \in X$. For more details on this inequality, see the recent paper [2] and the references therein.

The following result can be found in [4, Corollary 1]:

Let x, z, $e \in X$ with $\|e \mid z\| = 1$ and φ, $\Phi \in K$ with $\varphi \neq \Phi$. Then

$$\mathrm{Re}(\Phi e - x, e - \varphi e \mid z) \geq 0$$

if and only if

$$\left\| x - \frac{\varphi + \Phi}{2} \cdot e \mid z \right\| \leq \frac{1}{2} |\Phi - \varphi|.$$

We shall use the following lemma:

Lemma 1.([4]) *Let x, z, $e \in X$ with $\|e \mid z\| = 1$. Then one has the following representation*

$$0 \leq \|x \mid z\|^2 - |(x, e \mid z)|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e \mid z\|^2.$$

In [6]*, the following result and lemma hold.*

Let $\{e_i\}_{i \in I}$ be a family of z-orthornormal vectors in X, F a finite part of I and φ_i, Φ_i $(i \in F)$, real or complex numbers. The following statements are equivalent for $x \in X$.

(i) $\mathrm{Re}\left(\sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \varphi_i e_i \mid z \right) \geq 0,$

(ii) $\left\| x - \sum_{i \in F} \frac{\varphi_i + \Phi_i}{2} \cdot e_i \mid z \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{1/2}.$

Lemma 2. *If* (i) *or* (ii) *hold, then we have the inequality*

$$\begin{aligned}0 &\le \|x \mid z\|^2 - \sum_{i\in F} |(x, e_i \mid z)|^2 \\ &\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \varphi_i|^2 - \operatorname{Re}\left(\sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \varphi_i e_i \mid z\right) \\ &\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \varphi_i|^2.\end{aligned}$$

We also need the following lemma.

Lemma 3.([5]) *Let* $\{e_i\}_{i\in I}$, F, φ_i, Φ_i, $i \in F$ *and* x, $z \in X$ *so that either* (i) *or* (ii) *hold. Then we have the inequality*

$$\begin{aligned}0 &\le \|x \mid z\|^2 - \sum_{i\in F} |(x, e_i \mid z)|^2 \\ &\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \varphi_i|^2 - \sum_{i\in F} \left|\frac{\varphi_i + \Phi_i}{2} - (x, e_i \mid z)\right|^2 \\ &\quad \left(\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \varphi_i|^2\right).\end{aligned}$$

3. Pre-Grüss Inequalities in 2-Inner Product Spaces

We start with the following result.

Theorem 1. *Let* $(X, (\cdot, \cdot \mid \cdot))$ *be an* 2*-inner product space over* $\mathbb{K}\,(\mathbb{K} = R, C)$*, and* e, $z \in X$, $\|e \mid z\| = 1$*. If* φ, Φ *are real or complex numbers and* x, y *are vectors in* X *such that the condition*

$$Re(\Phi e - x, x - \varphi e \mid z) \ge 0, \tag{3.1}$$

holds or, equivalently, the following assumption

$$\left\|x - \frac{\varphi + \Phi}{2} \cdot e \mid z\right\| \le \frac{1}{2}|\Phi - \varphi|, \tag{3.2}$$

is valid, then one has the inequality

$$|(x, y \mid z) - (x, e \mid z)(e, y \mid z)| \le \frac{1}{2}|\Phi - \varphi| \cdot \sqrt{\|y \mid z\|^2 - |(y, e \mid z)|^2} \tag{3.3}$$

and

$$\begin{aligned}&|(x, y \mid z) - (x, e \mid z)(e, y \mid z)| \\ &\le \frac{1}{2}|\Phi - \varphi| \cdot \|y \mid z\| - (\operatorname{Re}(\Phi e - x, x - \varphi e \mid z))^{1/2} \cdot |(y, e \mid z)|.\end{aligned} \tag{3.4}$$

Proof. If we apply Schwarz's inequality in 2-inner product space for the vectors $x - (x, e \mid z)e$, $y - (y, e \mid z)e$, then it can be easily shown that

$$|(x, y \mid z) - (x, e \mid z)(e, y \mid z)| \leq [\|x \mid z\|^2 - |(x, e \mid z)|^2]^{\frac{1}{2}}[\|y \mid z\|^2 - |(y, e \mid z)|^2]^{\frac{1}{2}}, \quad (3.5)$$

for any x, y, $z \in X$ and $e \in X$, $\|e \mid z\| = 1$.

Using Lemma 1 and condition (3.2) we have

$$[\|x \mid z\|^2 - |(x, e \mid z)|^2]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e \mid z\| \leq \left\| x - \frac{\varphi + \Phi}{2} \cdot e \mid z \right\| \leq \frac{1}{2}|\Phi - \varphi|,$$

and so, by (3.5), the desired inequality (3.3) is obtained.

By simple computation, we also observe that the following identities are valid.

$$\begin{aligned} 0 &\leq \|x \mid z\|^2 - |(x, e \mid z)|^2 \\ &= Re[(\Phi - (x, e \mid z))(\overline{(x, e \mid z)} - \overline{\varphi})] - Re(\Phi e - x, x - \varphi e \mid z). \end{aligned} \quad (3.6)$$

Using the elementary inequality for complex numbers.

$$4Re(a\overline{b}) \leq |a + b|^2, \qquad a, b \in \mathbb{K}(\mathbb{K} = R, C), \quad (3.7)$$

we have

$$Re((\Phi - (x, e, \mid z))(\overline{(x, e \mid z)} - \overline{\varphi})) \leq \frac{1}{4}|\Phi - \varphi|^2.$$

Consequently, by (3.1), (3.5), (3.6) and (3.7), we have

$$\begin{aligned} &|(x, y \mid z) - (x, e \mid z)(e, y \mid z)|^2 \\ &\leq \left[(\frac{1}{2}|\Phi - \varphi|)^2 - ([Re(\Phi e - x, x - \varphi e \mid z)]^{\frac{1}{2}})^2 \right] \cdot [\|y \mid z\|^2 - |(y, e \mid z)|^2]. \end{aligned} \quad (3.8)$$

Finally, using the elementary inequality for positive real numbers

$$(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2, \quad (3.9)$$

we have

$$\begin{aligned} &\left[(\frac{1}{2}|\Phi - \varphi|)^2 - ([Re(\Phi e - x, x - \varphi e \mid z)]^{\frac{1}{2}})^2 \right] [\|y \mid z\|^2 - |(y, e \mid z)|^2] \\ &\leq \left[\frac{1}{2}|\Phi - \varphi| \, \|y \mid z\| - [Re(\Phi e - x, x - \varphi e \mid z)]^{\frac{1}{2}} \cdot |(y, e \mid z)| \right]^2. \end{aligned} \quad (3.10)$$

The desired inequality (3.4) follows immediately from (3.8) and (3.10).

4. Pre-Grüss Inequalities Associated to Orthonormal Families in 2-Inner Product Spaces

Theorem 2. *Let $\{e_i\}_{i\in I}$ be family of z-orthonormal vectors in X, F a finite part of I, φ_i, $\Phi_i \in K$, $i \in F$ and x, y are vectors in X such that either the condition*

$$Re\Big(\sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\varphi_i e_i \mid z\Big) \geq 0, \tag{4.1}$$

or equivalently,

$$\|x - \sum_{i\in F}\frac{\Phi_i+\varphi_i}{2}e_i \mid z\| \leq \frac{1}{2}\Big(\sum_{i\in F}|\Phi_i-\varphi_i|^2\Big)^{\frac{1}{2}} \tag{4.2}$$

holds. Then we have the following inequalities:

$$\begin{aligned} &|(x,y\mid z) - \sum_{i\in F}(x,e_i\mid z)(e_i,y\mid z)| \\ &\leq \frac{1}{2}\Big(\sum_{i\in F}|\Phi_i-\varphi_i|^2\Big)^{\frac{1}{2}}\sqrt{\|y\mid z\|^2 - \sum_{i\in F}|(y,e_i\mid z)|^2}; \end{aligned} \tag{4.3}$$

$$\begin{aligned} &|(x,y\mid z) - \sum_{i\in F}(x,e_i\mid z)(e_i,y\mid z)| \\ &\leq \frac{1}{2}\Big(\sum_{i\in F}|\Phi_i-\varphi_i|^2\Big)^{\frac{1}{2}}\|y\mid z\| \\ &\quad -\Big(Re(\sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\varphi_i e_i\mid z)\Big)^{\frac{1}{2}}\Big(\sum_{i\in F}|(y,e_i\mid z)|^2\Big)^{\frac{1}{2}}; \end{aligned} \tag{4.4}$$

$$\begin{aligned} &|(x,y\mid z) - \sum_{i\in F}(x,e_i\mid z)(e_i,y\mid z)| \\ &\leq \frac{1}{2}\Big(\sum_{i\in F}|\Phi_i-\varphi_i|^2\Big)^{\frac{1}{2}}\|y\mid z\| \\ &\quad -\Big(\sum_{i\in F}\left|\frac{\Phi_i+\varphi_i}{2} - (x,e_i\mid z)\right|^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in F}|(y,e_i\mid z)|^2\Big)^{\frac{1}{2}}; \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &|(x,y\mid z) - \sum_{i\in F}(x,e_i\mid z)(e_i,y\mid z)| \\ &\leq \frac{1}{2}\Big(\sum_{i\in F}|\Phi_i-\varphi_i|^2\Big)^{\frac{1}{2}}\|y\mid z\| - \sum_{i\in F}\left|\frac{\Phi_i+\varphi_i}{2} - (x,e_i\mid z)\right|\cdot|(y,e_i\mid z)|\,. \end{aligned} \tag{4.6}$$

Proof. It is obvious that

$$\begin{aligned}(x, y \mid z) &- \sum_{i\in F}(x, e_i \mid z)(e_i, y \mid z)\\ &= \Big(x - \sum_{i\in F}(x, e_i \mid z)e_i, y - \sum_{i\in F}(e_i, y \mid z)e_i \mid z\Big).\\ &= \Big(x - \sum_{i\in F}(x, e_i \mid z)e_i, y - \sum_{i\in F}(y, e_i \mid z)e_i \mid z\Big). \qquad (4.7)\end{aligned}$$

Using (2.9), we have

$$\begin{aligned}&\left|\Big(x - \sum_{i\in F}(x, e_i \mid z)e_i, y - \sum_{i\in F}(y, e_i \mid z)e_i \mid z\Big)\right|^2\\ &\le \left\|x - \sum_{i\in F}(x, e_i \mid z)e_i \mid z\right\|^2 \cdot \left\|y - \sum_{i\in F}(y, e_i \mid z)e_i \mid z\right\|^2\\ &= \left(\|x \mid z\|^2 - \sum_{i\in F}|(x, e_i \mid z)|^2\right)\left(\|y \mid z\|^2 - \sum_{i\in F}|(y, e_i \mid z)|^2\right). \qquad (4.8)\end{aligned}$$

Using the third inequality of Lemma 2, we have

$$\|x \mid z\|^2 - \sum_{i\in F}|(x, e_i \mid z)|^2 \le \frac{1}{4}\sum_{i\in F}|\Phi_i - \varphi_i|^2 \qquad (4.9)$$

the inequality (4.3) follows from (4.7), (4.8) and (4.9).

Using the second inequality of Lemma 2, we also have

$$\begin{aligned}&\left|\Big(x - \sum_{i\in F}(x, e_i \mid z)e_i, y - \sum_{i\in F}(y, e_i \mid z)e_i \mid z\Big)\right|^2\\ &\le \left(\Big[\frac{1}{2}\Big(\sum_{i\in F}|\Phi_i - \varphi_i|^2\Big)^{\frac{1}{2}}\Big]^2 - \Big(\Big[Re\Big(\sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\varphi_i e_i \mid z\Big)\Big]^{\frac{1}{2}}\Big)^2\right)\\ &\quad\times\left(\|y \mid z\|^2 - \Big[\Big(\sum_{i\in F}|(y, e_i \mid z)|^2\Big)^{\frac{1}{2}}\Big]^2\right). \qquad (4.10)\end{aligned}$$

By the elementary inequality (3.9) and (4.10), we have

$$\begin{aligned}&\left|\left(x-\sum_{i\in F}(x,e_i\mid z)e_i, y-\sum_{i\in F}(y,e_i\mid z)e_i\mid z\right)\right|^2\\ &\quad\le\left[\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\varphi_i|^2\right)^{\frac{1}{2}}\cdot\|y|z\|\right.\\ &\quad\left.-\left(Re\Big(\sum_{i\in F}\Phi_ie_i-x,x-\sum_{i\in F}\varphi_ie_i\mid z\Big)\right)^{\frac{1}{2}}\Big(\sum_{i\in F}|(y,e_i\mid z)|^2\Big)^{\frac{1}{2}}\right]^2\end{aligned}$$

which gives the desired result (4.4).

Similarly, applying Lemma 3 we have

$$\begin{aligned}&\left|\left(x-\sum_{i\in F}(x,e_i\mid z)e_i, y-\sum_{i\in F}(y,e_i\mid z)e_i\mid z\right)\right|^2\\ &\quad\le\left(\left[\frac{1}{2}\Big(\sum_{i\in F}|\Phi_i-\varphi_i|^2\Big)^{\frac{1}{2}}\right]^2-\left[\Big(\sum_{i\in F}\Big|\frac{\Phi_i+\varphi_i}{2}-(x,e_i\mid z)\Big|^2\Big)^{\frac{1}{2}}\right]^2\right)\\ &\quad\times\left(\|y|z\|^2-\left[\left(\sum_{i\in F}\Big|(y,e_i\mid z)\Big|^2\right)^{\frac{1}{2}}\right]^2\right)\end{aligned}\tag{4.11}$$

By the elementary inequality (3.9) and (4.11), we have

$$\begin{aligned}&\left|\left(x-\sum_{i\in F}(x,e_i\mid z)e_i, y-\sum_{i\in F}(y,e_i\mid z)e_i\mid z\right)\right|^2\\ &\quad\le\left[\frac{1}{2}\Big(\sum_{i\in F}|\Phi_i-\varphi_i|^2\Big)^{\frac{1}{2}}\cdot\|y\mid z\|\right.\\ &\quad\left.-\left(\sum_{i\in F}\Big|\frac{\Phi_i+\varphi_i}{2}-(x,e_i\mid z)\Big|^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}\Big|(y,e_i\mid z)\Big|^2\right)^{\frac{1}{2}}\right]^2.\end{aligned}$$

which gives the desired result (4.5).

Further, on utilizing (4.11) and the Aczél's inequality [9, p.117]

$$(a_1^2-a_2^2-\cdots-a_n^2)(b_1^2-b_2^2-\cdots-b_n^2)\le(a_1b_1-a_2b_2-\cdots a_nb_n)^2,$$

provided $a_1^2 - a_2^2 - \cdots - a_n^2 > 0$ or $b_1^2 - b_2^2 - \cdots - b_n^2$, we have

$$\begin{aligned}&\left|\Big(x - \sum_{i\in F}(x, e_i \mid z)e_i, y - \sum_{i\in F}(y, e_i \mid z)e_i \mid z\Big)\right|^2\\ &\quad\le \left(\frac{1}{2}\Big(\sum_{i\in F}|\Phi_i - \varphi_i|^2\Big)^{\frac{1}{2}} \cdot \|y \mid z\| - \sum_{i\in F}\Big|\frac{\Phi_i + \varphi_i}{2} - (x, e_i \mid z)\Big| \cdot |(y, e_i \mid z)|\right)^2\end{aligned}$$

which gives the desired result (4.6). This completes the proof.

Remark 3. Taking $F = \{1\}$ in Theorem 2, we note that (4.3) and (4.4) reduce to (3.3) and (3.4), respectively. Also, both (4.5) and (4.6) reduce to

$$\begin{aligned}&|(x, y \mid z) - (x, e_1 \mid z)(e_1, y \mid z)|\\ &\le \frac{1}{2}|\Phi_1 - \varphi_1| \cdot \|y \mid z\| - \left|\frac{\Phi_1 + \varphi_1}{2} - (x, e_1 \mid z)\right| \cdot |(y, e_1 \mid z)|\end{aligned}$$

which is a new pre-Grüss type inequality in 2-inner product spaces.

5. Determinantal Integral Inequalities

Let $(\Omega, \sum, \mu)$ be a measure space consisting of a set Ω, $\sum$ be a σ-algebra of subsets of Ω and be μ a countably additive and positive measure on $\sum$ with value in $\mathbb{R} \cup \{\infty\}$.

Denote by $L_\rho^2(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are 2-ρ-integrable on Ω, i.e., $\int_\Omega \rho(s)|f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L_\rho^2(\Omega)$ by formula

$$(f, g \mid h)_\rho := \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t)\\ h(s) & h(t)\end{vmatrix} \begin{vmatrix} g(s) & g(t)\\ h(s) & h(t)\end{vmatrix} d\mu(s)d\mu(t), \tag{5.1}$$

where by

$$\begin{vmatrix} f(s) & f(t)\\ h(s) & h(t)\end{vmatrix}$$

we denote the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t)\\ h(s) & h(t)\end{bmatrix}.$$

Define the 2-norm on $L_\rho^2(\Omega)$ expressed by

$$\|f \mid h\|_\rho := \left(\frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t)\\ h(s) & h(t)\end{vmatrix}^2 d\mu(s)d\mu(t)\right)^{\frac{1}{2}} \tag{5.2}$$

A simple calculation with integrals reveals that

$$(f,g \mid h)_\rho = \begin{vmatrix} \int_\Omega \rho fg d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \tag{5.3}$$

and

$$\|f \mid h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho fh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{\frac{1}{2}}, \tag{5.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s)f(s)g(s)d\mu(s)$, we have written $\int_\Omega \rho fg d\mu$.

We recall that the pair of functions $(q,p) \in L^2_\rho(\Omega) \times L^2_\rho(\Omega)$ is called *synchronous* if

$$(q(x) - q(y))(p(x) - p(y)) \geq 0$$

for a.e. $x, y \in \Omega$.

We note that, if $\Omega = [a,b]$, then a sufficient condition for synchronicity is that the functions are both monotonic increasing or decreasing. This condition is not necessary.

Now, suppose that $h \in L^2_\rho(\Omega)$ is such that $h(x) \neq 0$ for a.e. $x \in \Omega$. Then, by the definition of 2-inner product $(f,g \mid h)_\rho$, we have

$$(f,g \mid h)_\rho = \frac{1}{2}\int_\Omega \int_\Omega \rho(s)\rho(t)h^2(s)h^2(t)\Big(\frac{f(s)}{h(s)} - \frac{f(t)}{h(t)}\Big)\Big(\frac{g(s)}{h(s)} - \frac{g(t)}{h(t)}\Big)d\mu(s)d\mu(t) \tag{5.5}$$

and thus a *sufficient condition* for the inequality

$$(f,g \mid h)_\rho \geq 0 \tag{5.6}$$

to hold, is that, the pair of functions $(\frac{f}{h}, \frac{g}{h})$ are synchronous. It is obvious that, this condition is not necessary.

Using the representations (5.3), (5.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, one may state some interesting determinantal integral inequalities as follows.

Proposition 4. *Let $f, g, h, u \in L^2_\rho(\Omega)$ with $h \neq 0$ a.e. and*

$$\int_\Omega \rho u^2 d\mu \int_\Omega \rho h^2 d\mu - \Big(\int_\Omega \rho uh d\mu\Big)^2 = 1.$$

If M and m are real numbers with the property that

$$\Big(M \cdot \frac{u}{h} - \frac{f}{h}, \frac{f}{h} - m \cdot \frac{u}{h}\Big)$$

is synchronous on Ω, then we have the following determinantal integral Pre-Grüss type inequality

$$|G_\rho(f,g)| \leq \frac{|M-m|}{2}\left(\det\begin{bmatrix} \int_\Omega \rho g^2 d\mu & \int_\Omega \rho gh d\mu \\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix} - \left|\det\begin{bmatrix} \int_\Omega \rho gu d\mu & \int_\Omega \rho gh d\mu \\ \int_\Omega \rho uh d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right|^2\right)^{\frac{1}{2}}$$

and

$$|G_\rho(f,g)| \le \frac{|M-m|}{2}\left(\det\begin{bmatrix}\int_\Omega \rho g^2 d\mu & \int_\Omega \rho g h d\mu\\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right)^{\frac{1}{2}}$$
$$-\left(\det\begin{bmatrix}\int_\Omega \rho(Mu-f)(f-mu)d\mu & \int_\Omega \rho(Mu-f)hd\mu\\ \int_\Omega \rho(f-mu)hd\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right)^{\frac{1}{2}}$$
$$\times\left|\det\begin{bmatrix}\int_\Omega \rho g u d\mu & \int_\Omega \rho g h d\mu\\ \int_\Omega \rho u h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|$$

where

$$G_\rho(f,g) = \det\begin{bmatrix}\int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu\\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}$$
$$-\det\begin{bmatrix}\int_\Omega \rho f u d\mu & \int_\Omega \rho f h d\mu\\ \int_\Omega \rho u h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\cdot\det\begin{bmatrix}\int_\Omega \rho g u d\mu & \int_\Omega \rho g h d\mu\\ \int_\Omega \rho u h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}.$$

The proof follows by applying for the 2-inner product $(\cdot,\cdot \mid \cdot)_\rho$ defined in (5.1) and Theorem 1.

If one applies Theorem 2 for the same 2-inner product, then one can state the following interesting determinantal integral inequalities.

Proposition 5. *Let $f,g,h \in L^2_\rho(\Omega)$ with $h(x) \neq 0$ a.e. $x \in \Omega$ and $(f_i)_{i\in I}$ a family of functions in $L^2_\rho(\Omega)$ with the property that*

$$\begin{vmatrix}\int_\Omega \rho f_i f_j d\mu & \int_\Omega \rho f_i h d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{vmatrix} = \delta_{i,j}$$

for any $i,j \in I$, where $\delta_{i,j}$ is the Kronecker delta.

If we assume that there exist real numbers M_i, m_i, $i \in F$, where F is a given finite part of I, such that the functions

$$\Big(\sum_{i\in F} M_i \cdot \frac{f_i}{h} - \frac{f}{h}, \frac{f}{h} - \sum_{i\in F} m_i \cdot \frac{f_i}{h}\Big)$$

are synchronous on Ω and define

$$F_\rho(f,g) = \det\begin{bmatrix}\int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu\\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}$$
$$-\sum_{i\in F}\det\begin{bmatrix}\int_\Omega \rho f f_i d\mu & \int_\Omega \rho f h d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\cdot\det\begin{bmatrix}\int_\Omega \rho g f_i d\mu & \int_\Omega \rho g h d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}.$$

Then we have the inequalities

$$|F_\rho(f,g)| \le \frac{1}{2}\Big(\sum_{i\in F}|M_i-m_i|^2\Big)^{\frac{1}{2}} \times \left(\det\begin{bmatrix}\int_\Omega \rho g^2 d\mu & \int_\Omega \rho gh d\mu\\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix} - \sum_{i\in F}\left|\det\begin{bmatrix}\int_\Omega \rho g f_i d\mu & \int_\Omega \rho gh d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|^2\right)^{\frac{1}{2}},$$

$$|F_\rho(f,g)| \le \frac{1}{2}\Big(\sum_{i\in F}|M_i-m_i|^2\Big)^{\frac{1}{2}}\cdot\left(\det\begin{bmatrix}\int_\Omega \rho fg d\mu & \int_\Omega \rho fh d\mu\\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right)^{\frac{1}{2}} - \left(\det\begin{bmatrix}\int_\Omega\sum_{i\in F}\rho(M_i f_i - f)(f-m_i f_i)d\mu & \int_\Omega\sum_{i\in F}\rho(M_i f_i-f)h d\mu\\ \int_\Omega \rho(f-m_i f_i)h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right)^{\frac{1}{2}} \times \left(\sum_{i\in F}\left|\det\begin{bmatrix}\int_\Omega \rho g f_i d\mu & \int_\Omega \rho gh d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|^2\right)^{\frac{1}{2}},$$

$$|F_\rho(f,g)| \le \frac{1}{2}\Big(\sum_{i\in F}|M_i-m_i|^2\Big)^{\frac{1}{2}}\cdot\left(\det\begin{bmatrix}\int_\Omega \rho fg^2 d\mu & \int_\Omega \rho gh d\mu\\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right)^{\frac{1}{2}} - \left(\sum_{i\in F}\left|\frac{M_i+m_i}{2}-\det\begin{bmatrix}\int_\Omega \rho f f_i d\mu & \int_\Omega \rho fh d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|^2\right)^{\frac{1}{2}} \times \left(\sum_{i\in F}\left|\det\begin{bmatrix}\int_\Omega \rho g f_i d\mu & \int_\Omega \rho gh d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|^2\right)^{\frac{1}{2}}$$

and

$$|F_\rho(f,g)| \le \frac{1}{2}\Big(\sum_{i\in F}|M_i-m_i|^2\Big)^{\frac{1}{2}}\cdot\left(\det\begin{bmatrix}\int_\Omega \rho g^2 d\mu & \int_\Omega \rho gh d\mu\\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right)^{\frac{1}{2}} - \sum_{i\in F}\left|\frac{M_i+m_i}{2}-\det\begin{bmatrix}\int_\Omega \rho f f_i d\mu & \int_\Omega \rho fh d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|\cdot\left|\begin{bmatrix}\int_\Omega \rho g f_i d\mu & \int_\Omega \rho gh d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|.$$

References

[1] Y. J. Cho, C. S. Lin, S. S. Kim and A. Misiak, Theory of 2-Inner Product Spaces, Nova Science Publishers, Inc., New York, 2001.

[2] Y. J. Cho, M. Matić and J. E. Pečarić, *On Gram's determinant in 2-inner product spaces*, J. Korean Math. Soc. **38**(2001), 1125–1156.

[3] S. S. Dragomir, *A generalization of Grüss inequality in inner product spaces and applications*, J. Math. Anal. Appl., **237**(1999), 74–82.

[4] S. S. Dragomir, Y. J. Cho, S. M. Kang, S. S. Kim and J. S. Jung, *Some Grüss type inequalities in* 2*-inner product Spaces and applications for determinantal integral inequalities*, Panamer. math. J. **15**(2005), 79–94.

[5] S. S. Dragomir, Y. J. Cho, S. S. Kim and Y.-H. Kim, *On Bessel's and Grüss' inequalities for Orthornormal families in* 2*-inner product spaces and applications*, RGMIA Research Report Collection, **6**(4)(2003), Article 8. [ON LINE http://rgmia.vu.edu.au/v6n4.html]

[6] S. S. Dragomir, Y. J. Cho, S. S. Kim and J. Roumeliotis, *A reverse of Bessel's inequality in* 2*-inner product spaces and some Grüss type related results with applications*, Appl. Math. Comput. **20**(2006), 279–292.

[7] S. S. Dragomir, J. E. Pečarić and B. Tepeš, *Pre-Grüss type inequalities in inner product spaces*, Nonlinear Funct. Anal. Appl. **9**(2004), 627–639.

[8] S. S. Kim, S. S. Dragomir, A. White and Y. J. Cho, *On the Grüss type inequality in* 2*-inner product spaces and applications*, PanAmerican Math. J., **11**(2001), 89–97.

[9] D. S. Mitrinovic, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993.

[10] R. W. Freese and Y. J. Cho, Geometry of Linear 2-Normed Spaces, Nova Science Publishers, Inc., New York, 2001.

[11] R. W. Freese, S. S. Dragomir, Y. J. Cho and S. S. Kim, Some companions of Grüss inequality in 2-inner product spaces and applications for determinantal integral inequalities, Commun. Korean Math. Soc. **20**(2005), 487–503.

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