TAMKANG JOURNAL OF MATHEMATICS
Volume 43, Number 1, 69-80, Spring 2012
doi:10.5556/j.tkjm.43.2012.69-80

Available online at http://journals.math.tku.edu.tw/

NEIGHBORHOOD CONNECTED EDGE DOMINATION IN GRAPHS

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Abstract. Let $G=(V,E)$ be a connected graph. An edge dominating set X of G is called a neighborhood connected edge dominating set (nced-set) if the edge induced subgraph $< N(X) >$ is connected. The minimum cardinality of a nced-set of G is called the neighborhood connected edge domination number of G and is denoted by $\gamma'_{nc}(G)$. In this paper, we initiate a study of this parameter.

1. Introduction

The graph $G=(V,E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3] and Haynes et.al [4].

For any $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v)$ and $N[v]=N(v)\cup\{v\}$ respectively. If $S \subseteq V$, then $N(S)=\bigcup_{v\in S} N(v)$ and $N[S]=N(S)\cup S$. If $S\subseteq V$ and $u \in S$, then the private neighbor set of u with respect to S is defined by $\mathrm{pn}[u,S]=\{v : N[v]\cap S=\{u\}\}$. For any $e\in E$, the open neighborhood and the closed neighborhood of e are denoted by $N(e)$ and $N[e]=N(e)\cup\{e\}$ respectively. If $X\subseteq E$, then $N(X)=\bigcup_{e\in X} N(e)$ and $N[X]=N(X)\cup X$. If $X\subseteq E$ and $e_1\in X$ then the private neighbor of e_1 with respect to X is defined by $\mathrm{pn}[e_1,X]=\{e_2 : N(e_2)\cap X=\{e_1\}\}$. The degree of an edge $e=uv$ of G is defined by $\deg e=\deg u+\deg v-2$. $\delta'(G)(\Delta'(G))$ is the minimum (maximum) degree among the edges of G. Let $X\subseteq E$, a graph $G-X$ is obtained from the graph G by removing the edges of X. Let H be a subgraph of G and let $e\in G$, $d(e,H)$ denotes the distance from e to H.

A subset S of V is called a dominating set of G if $N[S]=V$. The minimum(maximum) cardinality of a minimal dominating set of G is called the domination number(upper domination number) of G and is denoted by $\gamma(G)(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes et.al [4]. A survey of several advanced topics in domination is given in the book edited by Haynes et.al [5].

Corresponding author: M. P. Kulandaivel.
2010 *Mathematics Subject Classification*. 05C70.
Key words and phrases. Neighborhood connected domination, neighborhood connected edge domination.

S. Arumugam and C. Sivagnanam [1] introduced the concept of neighborhood connected domination. A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $< N(S) >$ is connected. The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$.

As an analogy to vertex domination, the concept of edge domination was introduced by Mitchell and Hedetniemi [7]. A set $X \subseteq E$ is said to be an edge dominating set if every edge in $E - X$ is adjacent to some edge in X. The edge domination number of G is the cardinality of a smallest edge dominating set of G and is denoted by $\gamma'(G)$. S. Arumugam and S. Velammal [2] introduced the concept of connected edge domination of a connected graph. An edge dominating set X of a connected graph G is called a connected edge dominating set if the edge induced subgraph $< X >$ is connected. The minimum cardinality of a connected edge dominating set of G is called connected edge domination number and is denoted by $\gamma'_c(G)$.

In this paper we study the edge analogue of neighborhood connected domination number. We need the following theorem.

Theorem 1.1. **[1]** *For any graph G,* $\gamma_{nc}(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

2. Main Results

Definition 2.1. An edge dominating set X of a connected graph G is called the neighborhood connected edge dominating set (nced-set) if the edge induced subgraph $< N(X) >$ of G is connected. The minimum cardinality of a nced-set is called the neighborhood connected edge domination number (nced-number) and is denoted by $\gamma'_{nc}(G)$.

Example. Consider the following graph G

$\gamma'(G) = 2$ $\qquad \gamma'_c(G) = 4$ $\qquad \gamma'_{nc}(G) = 3$

Remark 2.2. (i) Clearly $\gamma'_{nc}(G) \geq \gamma'(G)$. Further if X is a connected edge dominating set with $|X| > 1$ then $N(X) = E$ and hence $\gamma'_{nc}(G) \leq \gamma'_c(G)$.

(ii) For any connected graph G that is not a star $\gamma'_{nc}(G) = 1$ if and only if there exists a non cut edge e such that $\deg e = m - 1$. That is G contains two adjacent vertices u and v such that all other vertices are mutually non adjacent, adjacent to either u or v, and at least one vertex is adjacent to both u and v.

Theorem 2.3. *For any graph $G, \gamma'(G) \leq \gamma'_{nc}(G) \leq 2\gamma'(G)$. Further given two positive integers a and b with $a \leq b \leq 2a$, there exists a graph G with $\gamma'(G) = a$ and $\gamma'_{nc}(G) = b$.*

Proof. Let G be a connected graph and let X be an edge dominating set of G. Obviously pairing each $e \in X$ with a private neighbor forms a nced-set of cardinality $2\gamma'(G)$.

Now, let a and b be two positive integers with $a \leq b \leq 2a$. Let $b = a+k, 0 \leq k \leq a$. Consider the galaxy with stars $G_1, G_2, \ldots, G_a$ with $|V(G_i)| \geq 3, \quad 1 \leq i \leq a$. Join the maximum degree vertices of G_i and G_{i+1} by an edge e_i, $1 \leq i \leq a$. Let H be the graph obtained from the above graph by subdividing exactly once the edges e_i where $1 \leq i \leq a-1$. Clearly $\gamma'(H) = \gamma'_{nc}(H) = a$. Let G be the graph obtained from H by subdividing an edge of G_i exactly once where $1 \leq i \leq k$. Then $\gamma'(G) = a$ and $\gamma'_{nc}(G) = a + k = b$. □

Theorem 2.4. *For the path $P_n, n \geq 2, \gamma'_{nc}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ and let $e_i = v_i v_{i+1}$. If n is odd, then $X = \{e_j : j = 2k$ or $2k+1$ and k is odd $\}$ is a nced-set of P_n and if n is even then $X_1 = X \cup \{e_{n-1}\}$ is a nced-set of P_n. Hence $\gamma'_{nc}(P_n) \leq \left\lceil \frac{n-1}{2} \right\rceil$. Further, if X is any γ'_{nc}-set of P_n then $N(X)$ contains all the internal edges of P_n and hence $|X| \geq \left\lceil \frac{n-1}{2} \right\rceil$. Thus $\gamma'_{nc}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$. □

Corollary 2.5. *For any non-trivial path P_n,*

(i) $\gamma'_{nc}(P_n) = \gamma'(P_n)$ *if and only if* $n = 3$ *or* 5.

(ii) $\gamma'_{nc}(P_n) = \gamma'_c(P_n)$ *if and only if* $n = 2, 3, 5$ *or* 6.

Proof. Since $\gamma'(P_n) = \left\lceil \frac{n-1}{3} \right\rceil$ and $\gamma'_c(P_n) = n-3$ the corollary follows. □

Theorem 2.6. *For the cycle C_n on n vertices*

$$\gamma'_{nc}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \textit{if } n \not\equiv 3 \,(\textit{mod}\, 4) \\ \left\lfloor \frac{n}{2} \right\rfloor & \textit{if } n \equiv 3 \,(\textit{mod}\, 4) \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $n = 4k + r$ where $0 \leq r \leq 3$ and $e_i = v_i v_{i+1}$. Let $X = \big\{e_i : i = 2j, 2j+1, j$ is odd and $1 \leq j \leq 2k-1\big\}$.

$$\text{Let } X_1 = \begin{cases} X & \text{if } n \equiv 0 \pmod 4 \\ X \cup \{e_n\} & \text{if } n \equiv 1 \text{ or } 2 \pmod 4 \\ X \cup \{e_{n-1}\} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Clearly X_1 is a nced-set of C_n and hence

$$\gamma'_{nc}(C_n) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil \text{ if } n \not\equiv 3 \pmod 4 \\ \left\lfloor \frac{n}{2} \right\rfloor \text{ if } n \equiv 3 \pmod 4 \end{cases}$$

Now, let X be any γ'_{nc}-set of C_n then $< X >$ contains at most one isolated edge and

$$< N(X) >= \begin{cases} C_n - \{e\} & \text{if } n \not\equiv 0 \pmod 4 \\ C_n & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

Hence

$$|X| \geq \begin{cases} \left\lceil \frac{n}{2} \right\rceil \text{ if } n \not\equiv 3 \pmod 4 \\ \left\lfloor \frac{n}{2} \right\rfloor \text{ if } n \equiv 3 \pmod 4 \end{cases}$$

and the result follows. □

Corollary 2.7. (i) $\gamma'_{nc}(C_n) = \gamma'(C_n)$ *if and only if* $n = 3,4$ *or 7.*

(ii) $\gamma'_{nc}(C_n) = \gamma'_c(C_n)$ *if and only if* $n = 3,4$ *or 5.*

Proof. Since $\gamma'(C_n) = \left\lceil \frac{n}{3} \right\rceil$ and $\gamma'_c(C_n) = n-2$ the result follows. □

Theorem 2.8. $\gamma'_{nc}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor, n \geq 3.$

Proof. Let X be a maximum matching of K_n. Hence X is an edge dominating set. Also $< N(X) >= K_n - X$ which is connected. Hence X is a nced-set which implies $\gamma'_{nc}(K_n) \leq |X| = \left\lfloor \frac{n}{2} \right\rfloor$. Since $\gamma'(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$ the result follows. □

Theorem 2.9. $\gamma'_{nc}(K_{r,s}) = \min\{r, s\}$.

Proof. Let v be a vertex such that $\deg v = min\{r, s\}$. Let X be the set of all edges incident with v. It is clear that X is an edge dominating set. Also $< N(X) >= K_{r,s}$, if $K_{r,s}$ is not a star and $< N(X) >= K_{1,n-1}$ if $K_{r,s}$ is a star. Thus X is a nced-set. Hence $\gamma'_{nc}(K_{r,s}) \leq |X| = \deg v = \min\{r, s\}$. Since $\gamma'(K_{r,s}) = \min\{r, s\}$ the result follows. □

Theorem 2.10. *For a tree* T, $\gamma'_{nc}(T) = 1$ *if and only if* T *is a star.*

Proof. Let$\gamma'_{nc}(T) = 1$ and let $X = \{e\}$ be the γ'_{nc}-set of G. Let $e = uv$ and let $\deg u \geq 2$. If $\deg v > 1$ then $< N(X) >= T - e$ is disconnected. Hence $\deg v = 1$. Thus T is a star. The converse is obvious. □

We now proceed to obtain a characterization of minimal nced-sets.

Lemma 2.11. *A superset of a nced-set is a nced-set.*

Proof. Let X be a nced-set of a graph G and let $X_1 = X \cup \{e\}$, where $e \in E - X$. Let $e = uv$. Clearly $e \in N(X)$ and X_1 is an edge dominating set of G. Now Let $x, y \in V(< N(X_1) >)$. If $x, y \in V(< N(X) >)$ then any $x - y$ path in $< N(X) >$ is a $x - y$ path in $< N(X_1) >$. If $x \in V(< N(X) >)$ and $y \notin V(< N(X) >)$, then without loss of generality we assume $x - u$ path in $< N(X) >$, and hence $x - u$ path together with $u - y$ path gives a $x - y$ path in $< N(X_1) >$. Also if $x, y \notin V(< N(X) >)$ then (x, u, v, y) or (x, v, u, y) or (x, u, y) or (x, v, y) or (x, y) is a $x - y$ path in $< N(X_1) >$. Thus $< N(X_1) >$ is connected, so that X_1 is a nced-set of G. □

Theorem 2.12. *A nced-set X of a graph G is a minimal nced-set if and only if for every $e \in X$, one of the following holds,*

(i) $\text{pn}[e, X] \neq \emptyset$.

(ii) *There exists two vertices $x, y \in < N(X) >$ such that every $x - y$ path in $< N(X) >$ contains at least one edge of $N(X) - N(X - \{e\})$.*

Proof. Let X be a minimal nced-set of G. Let $e \in X$ and let $X_1 = X - \{e\}$. Then either X_1 is not an edge dominating set of G or $< N(X_1) >$ is disconnected. If X_1 is not an edge dominating set of G, then $\text{pn}[e, X] \neq \emptyset$. If $< N(X_1) >$ is disconnected, then there exists two vertices $x, y \in < N(X_1) >$ such that there is no $x - y$ path in $< N(X_1) >$. Since $< N(X) >$ is connected, it follows that every $x - y$ path in $< N(X_1) >$ contains atleast one edge of $N(X) - N(X - \{e\})$. Conversely, X is a nced-set of G satisfying the conditions of theorem, then X is 1-minimal and hence the result follows from Lemma 2.11. □

Theorem 2.13. *Let G be a graph with $\Delta' = m - 1$. Then $\gamma'_{nc}(G) = 1$ or 2. Further $\gamma'_{nc}(G) = 2$ if and only if G is a bistar, $B(r, s), r, s \geq 1$.*

Proof. Let $e \in E(G)$ with $\deg e = m - 1$. Then $\{e, e_1\}$, where $e_1 \in E - \{e\}$ is a nced-set of G so that $\gamma'_{nc}(G) \leq 2$. Now suppose $\gamma'_{nc}(G) = 2$. Then $< N(e) >= G - \{e\}$ is disconnected and hence e is a cut edge of G. Let $e = uv$. Since $\deg e = m - 1, N[u, v] - \{u, v\}$ is an independent set. If $\deg u$ or $\deg v$ is equal to 1 than G is a star which is a contradiction to $\gamma'_{nc}(G) = 2$. Thus $\deg u \geq 2$ and $\deg v \geq 2$. Hence G is a bistar $B(r, s), r, s \geq 1$. The converse is obvious. □

In the following theorems we obtain a bound for $\gamma'_{nc}(G)$.

Theorem 2.14. *Let G be a graph with $\Delta' < m - 1$. Then $\gamma'_{nc}(G) \leq m - \Delta'$.*

Proof. Let $e \in E(G)$ and $\deg e = \Delta'$. Since G is connected and $\Delta' < m - 1$, there exists two adjacent edges e_1 and e_2 such that $e_1 \in N(e)$ and $e_2 \notin N[e]$. Now, let $X = (N(e) - \{e_1\}) \cup \{e_2\}$. Clearly $E - X$ is a nced-set of G and hence $\gamma'_{nc}(G) \leq m - \Delta'$. □

Theorem 2.15. *Let T be a tree with $n > 2$. Then $\gamma'_{nc}(T) = m - \Delta'$ if and only if T is one of the following:*

(i) Star.

(ii) *Tree obtained from bistar $B(|X_1|, |X_2|)$ with $e = uv$ be a non-pendant edge and X_1 and X_2 are set of pendant edges which are incident with u and v respectively, by subdividing at least one edge of $X_1 \cup X_2$ and subdividing at most one edge of X_1 or X_2 once, or by subdividing exactly one edge of $X_1 \cup X_2$ twice.*

Proof. Let T be a tree with $\gamma'_{nc}(T) = m - \Delta'$. Let $e = uv \in E(T)$ and $\deg e = \Delta'$. Let $Y_1 = N(u) - \{v\} = \{v_1, v_2, \ldots, v_r\}$ and $Y_2 = N(v) - \{u\} = \{v_{r+1}, v_{r+2}, \ldots, v_{\Delta'}\}$. If $r = 0$ then T is a star graph. Let us assume $r \geq 1$ and $r < \Delta'$ and $A = V(T) - N[u, v] = \{w_1, w_2, \ldots, w_k\}$ and $T_1 = < A >$.

Case i. $E(T_1) = \emptyset$.

Suppose $\deg v_i \geq 3$ for some $v_i \in Y_1 \cup Y_2$ without loss of generality we assume $v_i \in Y_1$. Let $uv_i, v_i w_1, v_i w_2 \in E(T)$. Then $X = [E(T) - (N(e) \cup \{v_i w_1, v_i w_2\})] \cup \{uv_i\}$ is a nced-set of T and $|X| = m - \Delta' - 1$, which is a contradiction. Hence $\deg v_i \leq 2$. If $\deg v_i = 1$ for all i, $1 \leq i \leq \Delta'$ then T is a bistar which is a contradiction. Thus $\deg v_i = 2$ for some i.

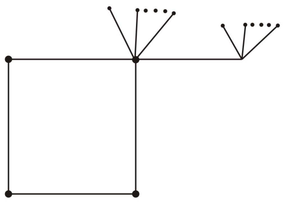
Claim. At most one vertex of Y_1 or at most one vertex of Y_2 has degree 2.

Suppose $v_1, v_2 \in Y_1$ and $v_i, v_j \in Y_2$ with $\deg v_k = 2$, for $k \in \{1, 2, i, j\}$. Let $w_k \in N(v_k) - \{u, v\}$ for $k \in \{1, 2, i, j\}$. Then $X = \left[E(T) - (N[e] \cup \{v_1 w_1, v_2 w_2, v_i w_i, v_j w_j\})\right] \cup \{uv_1, uv_2, vv_i, vv_j\}$ is a nced-set with $|X| = m - \Delta' - 1$ which is a contradiction. Hence at most one vertex of Y_1 or at most one vertex of Y_2 has degree 2.

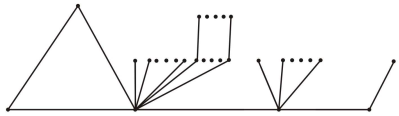
Case ii. $E(T_1) \neq \emptyset$.

Let G_1 be any non-trivial component of T_1 and we may assume without loss of generality that $v_1 \in N[V(G_1)]$. If G_1 contains more than one pendant vertex of T, then $X = [E(T) - (N(e) \cup E_1)] \cup \{uv_1\}$ where E_1 is the set of all pendant edges of T in G_1, is a nced-set of T with $|X| < m - \Delta'$ which is a contradiction. Hence G_1 is a path. Suppose $G_1 = (x_1, x_2, \ldots, x_k)$, $k \geq 3$ and let $v_1 x_1 \in E(T)$. Then $X = [E(T) - [N(e) \cup \{v_1 x_1, x_1 x_2\}]] \cup \{uv_1\}$ is a nced-set of T with $|X| = m - \Delta' - 1$ which is a contradiction. Thus $G_1 = P_2$. Now, if T has two non-trivial components $G_1 = (x_1, x_2)$ and $G_2 = (y_1, y_2)$, $x_1 \in N(v_i)$, $y_1 \in N(v_j)$ then $X = \left[E(T) - N(e) \cup \{v_i x_1, v_j y_1\}\right] \cup \{uv_i\}$ is a nced-set of T which is again a contradiction. Thus T_1 has exactly one non-trivial component. Let $X_1 = \{uv_i : 1 \leq i \leq r\}$ and $X_2 = \{vv_j : r+1 \leq j \leq \Delta'\}$ then the result follows and the converse is obvious. □

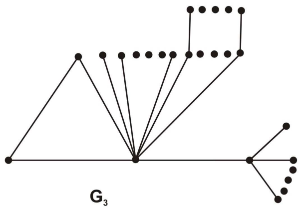
Theorem 2.16. *Let G be a unicyclic graph with cycle $C = (v_1, v_2, \ldots, v_r, v_1)$. Then $\gamma'_{nc}(G) = m - \Delta'$ if and only if G is isomorphic to C_3 or C_4 or C_5 or one of the graphs G_i, $1 \leq i \leq 23$, given in Figure* 1.



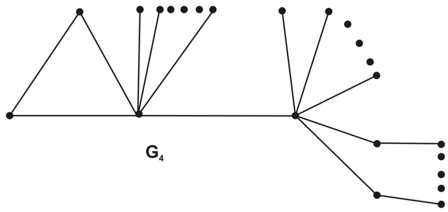
G_1



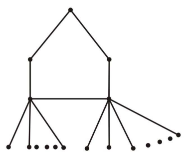
G_2



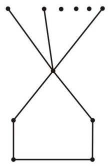
G_3



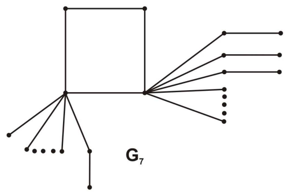
G_4



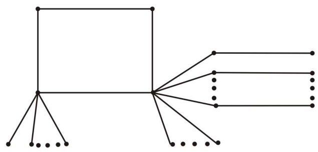
G_5



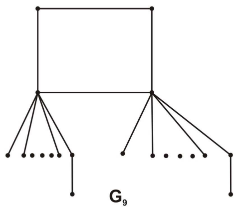
G_6



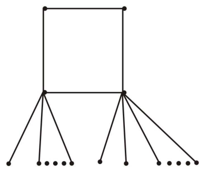
G_7



G_8



G_9



G_{10}

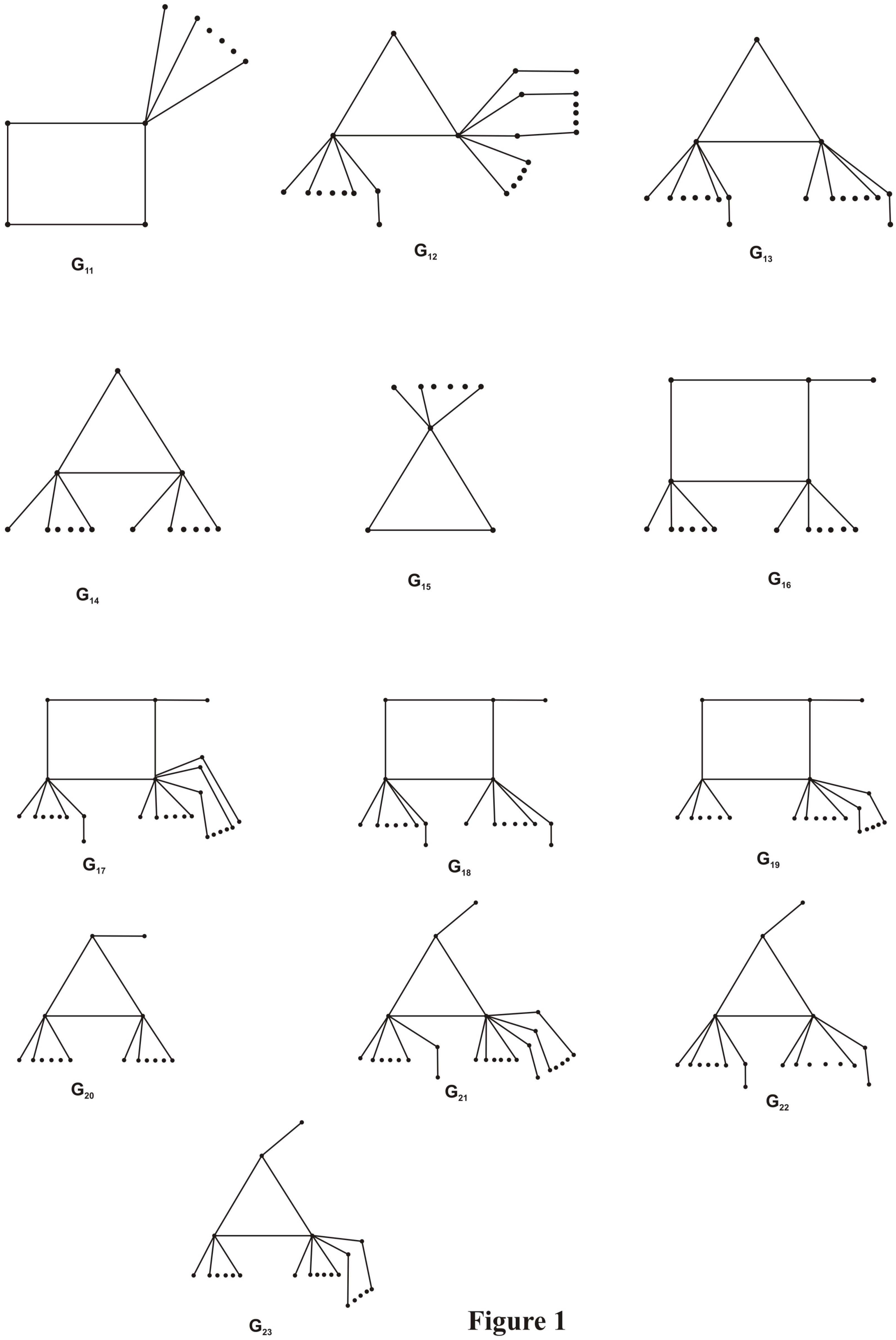


Figure 1

Proof. Let G be a unicyclic graph with cycle C and $\gamma'_{nc}(G) = m - \Delta'$. If $G = C$ then it follows from theorem 2.6 that $m \leq 5$ and hence G is isomorphic to C_3 or C_4 or C_5. Suppose $G \neq C$. Let A denote the set of all pendant edges in G and let $|A| = k$. Suppose $k \geq \Delta' + 1$. Since $E(G) - A$

is a nced-set of G we have $\gamma'_{nc}(G) \le m - \Delta' - 1$ which is a contradiction. Hence $k \le \Delta'$. Also maximum of two adjacent edges of e are in C we have $\Delta' - 2 \le k$.

Hence $\Delta' - 2 \le k \le \Delta'$. (1)

Let $e = uv$ with $\deg e = \Delta'(G)$. Suppose $d(e, C) \ge 1$, then $k = \Delta'$ or $\Delta' - 1$. Then $X = [E(G) - E(C) \cup A] \cup X_1$ where X_1 is nced-set of C, is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Hence the edge e lies on C or incident with C. Let e be incident with C and let $C = (v_1, v_2, \ldots, v_r, v_1)$. Let us assume $u = v_1$.

Claim. $r \le 4$.

Suppose $r \ge 6$. Then any γ'_{nc}-set of C does not contain at least 3 edges of C. Let X_1 be a γ'_{nc}-set of C which contains an edge adjacent to e. Then $X = [E(G) - (E(C) \cup A)] \cup X_1$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Hence $r \le 5$. Suppose $r = 5$. Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. Then $X = E(G) - [A \cup \{v_1v_2, v_2v_3, v_4v_5\}]$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Hence $r \le 4$ and it is clear that every vertex in $V(C) - \{v_1\}$ has degree 2.

Case 1.1. $r = 4$.

Let $C = (v_1, v_2, v_3, v_4, v_1)$. Suppose there exists a vertex $w \in A$ such that $d(w, e) \ge 2$. Let $d(w, u) = d(w, e)$ and let $(u, w_1, w_2 \ldots, w_k, w)$, $k \ge 1$ be the unique $u - w$ path. Then $X = [E(G) - [A \cup \{v_1v_2, v_3v_4, v_4v_1, w_1w_2\}]] \cup \{uw_1\}$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Similarly we can get a contradiction if $d(w, v) = d(w, e)$. Hence $d(w, e) = 1$ for all $w \in A$. Thus G is isomorphic to G_1.

Case 1.2. $r = 3$.

Let $C = (v_1, v_2, v_3, v_1)$ and $u = v_1$, suppose there exists a vertex $w \in A$ such that $d(w, e) \ge 3$. Let $d(w, u) = d(w, e)$ and let $(u, w_1, w_2, \ldots, w_k, w), k \ge 2$ be the unique $u - w$ path. Then $X = [E(G) - [A \cup \{v_2v_3, v_3v_1, uw_1, w_1w_2\}]] \cup \{w_kw\}$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Similarly we can get a contradiction if $d(w, v) = d(w, e)$. Hence $d(w, e) \le 2$ for all $w \in A$. Let $w_1 \in N(u) - [V(C) \cup \{v\}]$ and $\deg w_1 \ge 3$. Then $X = E(G) - [A \cup \{v_2v_3, v_3v_1\}]$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Similarly we can get a contradiction if $w_1 \in N(v) - \{u\}$. Now, let $w_1, w_2 \in N(u) - [V(C) \cup \{v\}]$ such that $\deg w_1 = \deg w_2 = 2$. Suppose there exist two vertices $w_3, w_4 \in N(v) - \{u\}$ such that $\deg w_3 = \deg w_4 = 2$. Then $X = E(G) - [A \cup \{v_2v_3, v_3v_1, e\}]$ is a nced-set of G with $|X| < m - \Delta'$. Hence at most one vertex of $N(v) - \{u\}$ is of degree 2. Then G is isomorphic to G_2 or G_3. Let $w_1, w_2 \in N(v) - \{u\}$ with $\deg w_1 = \deg w_2 = 2$. Suppose there exists a vertex $w_3 \in N(u) - [V(C) \cup \{v\}]$ such that $\deg w_3 = 2$. Then $X = E(G) - [A \cup \{v_2v_3, v_3v_1, e\}]$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Hence G is isomorphic to G_4.

Suppose e lies on C. Let $C = (v_1, v_2, \ldots, v_r, v_1)$ and $v_1v_2 = e$

Claim 1. $\deg w = 1$ or 2 for all $w \in V(G) - V(C)$.

Suppose there exist a vertex $w \in V(G) - V(C)$ with $\deg w > 2$. Then $k = \Delta' - 1$ or Δ'. If $k = \Delta' - 1$, then all the vertices of $V(C) - \{v_1, v_2\}$ have degree 2 and hence $X = E(G) - [A \cup \{v_2v_3, v_2v_1\}]$ is a nced-set of G with $|X| < m - \Delta'$. If $k = \Delta'$ then $X = E(G) - [A \cup \{v_2v_3\}]$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Hence $\deg w = 1$ or 2 for all $w \in V(G) - V(C)$.

Claim 2. Every vertex of $V(C) - \{v_1, v_2\}$ has degree 2 or 3.

It follows from (1) that $\deg v_i \le 4$ for all $i \ne 1, 2$. If there exists a vertex $v_i \in V(C)$ with $\deg v_i = 4$, then $k = \Delta'$ and $X = E(G) - [A \cup \{v_2v_3\}]$ is a nced-set of G with $|X| < m - \Delta'$. This proves claim 2.

Claim 3. $r \le 5$.

Suppose $r \ge 6$. If $k = \Delta'$ then $X = E(G) - [A \cup \{v_2v_3\}]$ is a nced-set of G with $|X| < m - \Delta'$. If $k = \Delta' - 1$ then there exists a vertex v_i such that $\deg v_i = 2$. Now $X = E(G) - [A \cup \{v_{i-1}v_i, v_iv_{i+1}\}]$ is a nced-set of G with $|X| < m - \Delta'$. If $k = \Delta' - 2$ then every vertex of $V(C) - \{v_1, v_2\}$ has degree 2 and hence $X = E(G) - [A \cup \{v_2v_3, v_{r-3}v_{r-2}, v_{r-2}v_{r-1}\}]$ is a nced-set of G with $|X| < m - \Delta'$. Thus $r \le 5$.

Claim 4. $d(w, C) \le 2$ for all $w \in A$.

Suppose there exist a pendant vertex w_1, such that $d(w_1, C) \ge 3$. Let $(w_1, w_2, \ldots, w_k, v_i), k \ge 3$ be the unique $w_1 - v_i$ path. If $k \ne \Delta - 2$ then $X = [E(G) - [A \cup \{v_2v_3, v_iw_k, w_kw_{k-1}\}]] \cup \{w_2w_1\}$ is a nced-set of G with $|X| < m - \Delta'$. If $k = \Delta - 2$, then $X = [E(G) - [A \cup \{v_2v_3, v_3v_4, v_iw_k, w_kw_{k-1}\}]] \cup \{w_2w_1\}$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Hence $d(w, C) \le 2$ for all $w \in A$.

Claim 5. If there are two P_3 attached with v_1 then at most one P_3 is attached to v_2.

Suppose not, then $X = E(G) - [A \cup \{v_1v_2, v_2v_3, v_rv_1\}]$ is a nced-set of G with $|X| < m - \Delta'$ which is a contradiction. Hence the Claim 5.

Case 2.1. $k = \Delta' - 2$.

In this case $\deg x = 1$ or 2 for all $x \in V(G) - \{v_1, v_2\}$. Now, if $r = 5$ and if there exists a vertex $w \in N(v_i) - V(C), i = 1$ or 2, such that $\deg w = 2$, then $X = E(G) - [A \cup \{v_2v_3, v_3v_4, v_5v_1\}]$ is a nced-set of G with $|X| < m - \Delta'$. Hence $\deg w = 1$ for all $w \in N(v_i) - V(C)$ and hence G is isomorphic to G_5 or G_6. If $r \le 4$ then G is isomorphic to $G_i, 7 \le i \le 15$.

Case 2.2. $k = \Delta' - 1$.

In this case $\deg v_i = 3$ for exactly one vertex $v_i \ne v_1$ and v_2 on C also $\deg x = 1$ or 2 for all $x \in V(G) - \{v_1, v_2, v_i\}$. If $r = 5$, then $X = E(G) - [A \cup B]$ where B is a set of edges in C not incident with v_i is a nced-set of G with $|X| = m - \Delta' - 1$ and hence $r = 3$ or 4. Suppose there exists a path (v_i, x_1, w_1) such that $x_1 \notin V(C)$ and $w_2 \in A$, if $r = 4$ then $X = [E(G) - [A \cup B \cup \{v_i x_1\}]] \cup \{x_1 w_1\}$ where B is $N[v_i x_1] \cap V(C)$ is a nced-set of G with $|X| < m - \Delta'$ and if $r = 3$, then $X = E(G) - [A \cup \{v_2 v_3, v_3 v_1, v_3 x_1\}] \cup \{x_1 w_1\}$ is a nced-set of G with $|X| < m - \Delta'$ and hence G is isomorphic to $G_i, 16 \leq i \leq 23$.

Case 2.3. If $k = \Delta'$.

In this case $r = 4$ or 5 and there does not exists a graph with $\gamma'_{nc}(G) = m - \Delta'$. Converse is obvious. □

Problem 2.17. Characterize the class of graphs for which $\gamma'_{nc}(G) = m - \Delta'$.

Remark 2.18. Since $\gamma'_{nc}(G) = \gamma_{nc}(L(G))$ where $L(G)$ is the line graph of G, it follows from Theorem 1.1 that $\gamma'_{nc}(G) \leq \left\lceil \frac{m}{2} \right\rceil$.

Theorem 2.19. *Let G be any graph such that both G and $\overline{G}$ are connected. Then* $\gamma'_{nc}(G) + \gamma'_{nc}(\overline{G}) \leq m + 1$.

Proof. The proof follows from Remark 2.18. □

Remark 2.20. The bounds given in Theorem 2.19 is sharp. The graph $G = C_5$, $\gamma'_{nc}(G) + \gamma'_{nc}(\overline{G}) = 6 = m + 1$.

Problem 2.21. Characterize the class of graphs for which $\gamma'_{nc}(G) + \gamma'_{nc}(\overline{G}) = m + 1$.

Theorem 2.22. *For any graph* $G, \gamma'_{nc}(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor$.

Proof. Let X be a maximum matching of the graph G. Label the edges of X by $e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_r$ such that the edges e_i and e_{i+1}, i is odd $1 \leq i \leq k-1$ are adjacent to common edge $f(e_i)$ with maximum value of k. Let $Y = \{f(e_i)/i \text{ is odd}\}$. Then $X \cup Y$ is an edge dominating set with $< N(X \cup Y) >$ is connected and hence $\gamma'_{nc}(G) \leq |X \cup Y| = \left\lfloor \frac{3n}{4} \right\rfloor$. □

Remark 2.23. The bound given in Theorem 2.22 is sharp. The graph $G = C_5$, $\gamma'_{nc}(G) = 3 = \left\lfloor \frac{3n}{4} \right\rfloor$.

Problem 2.24. Characterize the class of graphs for which $\gamma'_{nc}(G) = \left\lfloor \frac{3n}{4} \right\rfloor$.

Acknowledgement

Thanks are due to the referees for their helpful comments.

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