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SOME CODING THEOREMS ON GENERALIZED HAVRDA-CHARVAT AND TSALLIS'S ENTROPY

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Abstract. A new measure L_α^β, called average code word length of order α and type β has been defined and its relationship with a result of generalized Havrda-Charvat and Tsallis's entropy has been discussed. Using L_α^β, some coding theorem for discrete noiseless channel has been proved.

1. Introduction

Throughout the paper $\mathbb{N}$ denotes the set of the natural numbers and for $N \in \mathbb{N}$ we set

$$\Delta_N = \left\{ \left(p_1, \dots, p_N\right) / p_i \geq 0,\ i = 1, \dots, N,\ \sum_{i=1}^{N} p_i = 1 \right\}.$$

In case there is no rise to misunderstanding we write $P \in \Delta_N$ instead of $\left(p_1, \dots, p_N\right) \in \Delta_N$.

In case $N \in \mathbb{N}$ the well-known Shannon entropy is defined by

$$H(P) = H\left(p_1, \dots, p_N\right) = -\sum_{i=1}^{N} p_i \log(p_i) \quad \left(\left(p_1, \dots, p_N\right) \in \Delta_N\right), \tag{1.1}$$

where the convention $0 \log(0) = 0$ is adapted, (see Shannon [19]).

Throughout this paper, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to the base $D\,(D > 1)$.

Let a finite set of N input symbols

$$X = \{x_1, x_2, \dots, x_N\}$$

be encoded using alphabet of D symbols, then it has been shown by Feinstien [6] that there is a uniquely decipherable code with lengths $n_1, n_2, \dots, n_N$ if and only if the Kraft inequality holds that is,

$$\sum_{i=1}^{N} D^{-n_i} \leq 1. \tag{1.2}$$

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Where D is the size of code alphabet.

Furthermore, if

$$L = \sum_{i=1}^{N} n_i p_i \tag{1.3}$$

is the average codeword length, then for a code satisfying (1.2), the inequality

$$L \geq H(P) \tag{1.4}$$

is also fulfilled and equality holds if and only if

$$n_i = -\log_D(p_i) \quad (i = 1, \ldots, N), \tag{1.5}$$

and that by suitable encoded into words of long sequences, the average length can be made arbitrarily close to $H(P)$, (see Feinstein [6]). This is Shannon's noiseless coding theorem.

By considering Renyi's entropy (see e.g. [18]), a coding theorem and analogous to the above noiseless coding theorem has been established by Campbell [3] and the authors obtained bounds for it in terms of $\mathrm{H}_\alpha(P)=\frac{1}{1-\alpha}\log_D \sum P_i^\alpha, \alpha > 0 (\neq 1)$. Kieffer [13] defined a class rules and showed $H_\alpha(P)$ is the best decision rule for deciding which of the two sources can be coded with expected cost of sequences of length n when $n \to \infty$, where the cost of encoding a sequence is assumed to be a function of length only. Further, in Jelinek [10] it is shown that coding with respect to Campbell's mean length is useful in minimizing the problem of buffer overflow which occurs when the source symbol is produced at a fixed rate and the code words are stored temporarily in a finite buffer. Concerning Campbell's mean length the reader can consult [3].

Hooda and Bhaker considered in [9] the following generalization of Campbell's mean length:

$$L^\beta(t) = \frac{1}{t}\log_D\left\{\frac{\sum p_i^\beta D^{-tn_i}}{\sum p_i^\beta}\right\}, \ \beta \geq 1$$

and proved

$$H_\alpha^\beta(P) \leq L^\beta(t) < H_\alpha^\beta(P) + 1, \quad \alpha > 0,\ \alpha \neq 1,\ \beta \geq 1$$

under the condition

$$\sum p_i^{\beta-1} D^{-n_i} \leq \sum p_i^\beta$$

where $H_\alpha^\beta(P)$ is generalized entropy of order $\alpha = \frac{1}{1+t}$ and type β studied by Aczel and Daroczy [1] and Kapur [11]. It may be seen that the mean codeword length (1.3) had been generalized parametrically and their bounds had been studied in terms of generalized measures of entropies. Here we give another generalization of (1.3) and study its bounds in terms of generalized entropy of order α and type β.

Generalized coding theorems by considering different information measure under the condition of unique decipherability were investigated by several authors, see for instance the papers [7, 9, 12, 14, 17, 20].

In this paper we study some coding theorems by considering a new information measure depending on two parameters. Our motivation is -among others- that this quantity generalizes some information measures already existing in the literature such as the Tsallis (or Havrda-Charvat) entropy, (see [8] and [21]).

2. Coding theorem

Definition. Let $N \in \mathbb{N}$ be arbitrarily fixed, $\alpha, \beta > 0,\ \alpha \neq 1$ be given real numbers. Then the information measure $H_\alpha^\beta : \Delta_N \to \mathbb{R}$ is defined by

$$H_\alpha^\beta(p_1,\dots,p_N) = \frac{1}{\alpha-1}\left(1-\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^\beta}\right) \quad \left((p_1,\dots,p_N)\in\Delta_N\right). \tag{2.1}$$

Remark (i) When $\beta = 1$, then the information measure H_α^β reduces to entropy,

$$i.e.,\quad H_\alpha(p_1,\dots,p_N) = \frac{1}{\alpha-1}\left[1-\sum_{i=1}^N p_i^\alpha\right] \quad \left((p_1,\dots,p_N)\in\Delta_N\right). \tag{2.2}$$

The measure (2.2) was characterized by Havrda-Charvat [8], Vajda [22], Tsallis [21] and Daroczy [4] by different approaches.

(ii) When $\beta = 1$ and $\alpha \to 1$, then the information measure H_α^β reduces to Shannon entropy,

$$i.e.,\quad H(P) = -\sum p_i \log p_i. \tag{2.3}$$

(iii) When $\alpha \to 1$, the information measure H_α^β is the entropy of the β-power distribution,

$$i.e.,\quad H^\beta(P) = -\frac{\sum_{i=1}^N p_i^\beta \log(p_i^\beta)}{\sum_{j=1}^N p_j^\beta}, \tag{2.4}$$

that was considered e.g. in Mathur-Mitter [16].

Definition. Let $N \in \mathbb{N},\ \alpha, \beta > 0,\ \alpha \neq 1$ be arbitrarily fixed, then the mean length L_α^β corresponding to the generalized information measure H_α^β is given by the formula

$$L_\alpha^\beta = \frac{1}{\alpha-1}\left(1-\left(\frac{\sum_{i=1}^N p_i^\beta D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^N p_j^\beta}\right)^\alpha\right), \tag{2.5}$$

where $(p_1,\dots,p_N)\in\Delta_N$ and $D,\ n_1,n_2,\dots,n_N$ are positive integers so that

$$\sum_{i=1}^{N} D^{-n_i} \le \sum_{j=1}^{N} p_j^{\beta}. \tag{2.6}$$

Remark (i) When $\beta=1$, (2.5) reduces to a mean codeword length,

$$i.e.,\quad L_\alpha=\frac{1}{\alpha-1}\left[1-\left\{\sum p_i D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}\right\}^{\alpha}\right]. \tag{2.7}$$

(ii) When $\beta=1$, $\alpha\to1$, (2.5) reduces to a mean code length $L=\sum_{i=1}^{N} n_i p_i$, defined in Shannon [19].

Also, we have used the condition (2.6) to find the bounds. It may be seen that the case $\beta=1$ inequality (2.6) reduces to (1.2).

We establish a result, that in a sense, provides a characterization of $H_\alpha^\beta(P)$ under the condition of (2.6).

Theorem 2.1. *Let $\alpha,\beta>0,\ \alpha\ne1$ be arbitrarily fixed real numbers, then for all integers $D>1$ inequality*

$$L_\alpha^\beta \ge H_\alpha^\beta(P) \tag{2.8}$$

is fulfilled. Furthermore, equality holds if and only if

$$n_i=-\log_D\left(\frac{p_i^{\alpha\beta}}{\frac{\sum_{i=1}^{N} p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^{\beta}}}\right). \tag{2.9}$$

Proof. By reverse Hölder inequality, that is, if $N\in\mathbb{N}$, $\gamma>1$ and $x_1,\dots,x_N$, $y_1,\dots,y_N$ are positive real numbers then

$$\left(\sum_{i=1}^{N} x_i^{\frac{1}{\gamma}}\right)^{\gamma}\cdot\left(\sum_{i=1}^{N} y_i^{-\frac{1}{(\gamma-1)}}\right)^{-(\gamma-1)} \le \sum_{i=1}^{N} x_i y_i. \tag{2.10}$$

Let $\gamma=\frac{\alpha}{\alpha-1}$, $\ x_i=\left(\frac{p_i^\beta}{\sum_{j=1}^{N} p_j^\beta}\right)^{\left(\frac{\alpha}{\alpha-1}\right)} D^{-n_i}$, $\ y_i=\left(\frac{p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^\beta}\right)^{\frac{1}{1-\alpha}}$ $\quad(i=1,\dots,N)$.

Putting these values into (2.10), we get

$$\left(\frac{\sum p_i^\beta D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N} p_j^\beta}\right)^{\frac{\alpha}{\alpha-1}}\left(\frac{\sum_{i=1}^{N} p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^\beta}\right)^{\frac{1}{1-\alpha}} \le \frac{\sum_{i=1}^{N} D^{-n_i}}{\sum_{j=1}^{N} p_j^\beta} \le 1,$$

where we used (2.6), too. This implies however that

$$\left(\frac{\sum_{i=1}^{N} p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^\beta}\right)^{\frac{1}{1-\alpha}} \le \left(\frac{\sum_{i=1}^{N} p_i^\beta D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N} p_j^\beta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{2.11}$$

Here two cases arise

Case 1. When $0<\alpha<1$, then raising power $(1-\alpha)$ to both sides of (2.11), we have

$$\left(\frac{\sum_{i=1}^{N} p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^{\beta}}\right) \le \left(\frac{\sum p_i^{\beta} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N} p_j^{\beta}}\right)^{\alpha}$$

we obtain the result (2.8) after simplification for $\frac{1}{\alpha-1}<0$ as $0<\alpha<1$.

i.e.,

$$L_\alpha^\beta \ge H_\alpha^\beta(P), \text{ when } 0<\alpha<1. \tag{2.12}$$

Case 2. When $\alpha>1$, then raising power $(1-\alpha)$ to both sides of (2.11), we have

$$\left(\frac{\sum_{i=1}^{N} p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^{\beta}}\right) \ge \left(\frac{\sum p_i^{\beta} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N} p_j^{\beta}}\right)^{\alpha}$$

we obtain the result (2.8) after simplification for $\frac{1}{\alpha-1}>0$ as $\alpha>1$.

i.e., $L_\alpha^\beta \ge H_\alpha^\beta(P)$, when $\alpha>1$.

Case 3. From (2.9) and after simplification, we get

$$D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)} = p_i^{\beta(\alpha-1)} \left(\frac{\sum_{i=1}^{N} p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^{\beta}}\right)^{\left(\frac{1-\alpha}{\alpha}\right)}.$$

This implies

$$\left(\frac{\sum p_i^{\beta} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N} p_j^{\beta}}\right)^{\alpha} = \left(\frac{\sum_{i=1}^{N} p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^{\beta}}\right),$$

which gives $L_\alpha^\beta = H_\alpha^\beta(P)$. □

Theorem 2.2. *Let $N\in\mathbb{N}$, $\alpha,\beta>0$, $\alpha\ne1$ be fixed. Then there exist code length $n_1,\dots,n_N$ so that*

$$H_\alpha^\beta(P) \le L_\alpha^\beta < D^{1-\alpha}\mathrm{H}_\alpha^\beta(\mathrm{P}) + \frac{1}{\alpha-1}(1-\mathrm{D}^{1-\alpha}) \tag{2.13}$$

holds. Where $H_\alpha^\beta(P)$ and L_α^β are given by (2.1) *and* (2.5) *respectively.*

Proof. Due to the previous theorem,

$$L_\alpha^\beta = H_\alpha^\beta(P)$$

holds if and only if

$$D^{-n_i} = \frac{p_i^{\alpha\beta}}{\frac{\sum_{i=1}^{N} p_i^{\alpha\beta}}{\sum_{j=1}^{N} p_j^{\beta}}},\ \alpha>0,\ \alpha\ne1,\ \beta>0,$$

i.e., $n_i = -\log_D p_i^{\alpha\beta} + \log_D \left[\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right]$.

We choose the codeword lengths n_i, $i = 1,\dots,N$ in such a way that

$$-\log_D p_i^{\alpha\beta} + \log_D \left[\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right] \le n_i < -\log_D p_i^{\alpha\beta} + \log_D \left[\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right] + 1 \tag{2.14}$$

is fulfilled for all $i = 1,\dots,N$.

From the left inequality of (2.14), we have

$$D^{-n_i} \le \frac{p_i^{\alpha\beta}}{\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}}, \tag{2.15}$$

taking sum over i, we get the generalized inequality (2.6). So there exists a generalized code with code lengths n_i, $i = 1,\dots,N$.

Case 1. Let $0 < \alpha < 1$, then (2.14) can be written as

$$p_i^{\beta(\alpha-1)} \left(\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right)^{\frac{1-\alpha}{\alpha}} \le \mathrm{D}^{-\mathrm{n_i}\left(\frac{\alpha-1}{\alpha}\right)} < p_i^{\beta(\alpha-1)} \left(\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right)^{\frac{1-\alpha}{\alpha}} \mathrm{D}^{\frac{1-\alpha}{\alpha}}. \tag{2.16}$$

Multiplying (2.16) throughout by $\frac{p_i^{\beta}}{\sum_{j=1}^N p_j^{\beta}}$ and then summing up from $i = 1$ to $i = N$, we obtain inequality (2.13) after simplification with $\frac{\alpha}{\alpha-1}$,

$$\text{i.e.,}\quad \frac{1}{\alpha-1}\left(1-\left(\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right)\right) \le \frac{1}{\alpha-1}\left(1-\frac{\sum_{i=1}^N p_i^{\beta} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^N p_j^{\beta}}\right)^{\alpha}$$
$$< D^{1-\alpha}\left\{\frac{1}{\alpha-1}\left(1-\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right)\right\} + \frac{1}{\alpha-1}\left(1-D^{1-\alpha}\right)$$

$H_\alpha^\beta(P) \le L_\alpha^\beta < D^{1-\alpha}\mathrm{H}_\alpha^\beta(\mathrm{P}) + \frac{1}{\alpha-1}(1-\mathrm{D}^{1-\alpha})$, which gives (2.13).

Case 2. Let $\alpha > 1$, then (2.14) can be written as

$$p_i^{\beta(\alpha-1)} \left(\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right)^{\frac{1-\alpha}{\alpha}} \ge \mathrm{D}^{-\mathrm{n_i}\left(\frac{\alpha-1}{\alpha}\right)} > p_i^{\beta(\alpha-1)} \left(\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right)^{\frac{1-\alpha}{\alpha}} \mathrm{D}^{\frac{1-\alpha}{\alpha}}.$$

Multiplying (2.16) throughout by $\frac{p_i^{\beta}}{\sum_{j=1}^N p_j^{\beta}}$ and then summing up from $i = 1$ to $i = N$, we obtain inequality (2.13) after simplification with $\frac{\alpha}{\alpha-1}$,

$$\text{i.e.,}\quad \frac{1}{\alpha-1}\left(1-\left(\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{j=1}^N p_j^{\beta}}\right)\right) \le \frac{1}{\alpha-1}\left(1-\frac{\sum_{i=1}^N p_i^{\beta} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^N p_j^{\beta}}\right)^{\alpha}$$

$$< D^{1-\alpha}\left\{\frac{1}{\alpha-1}\left(1-\frac{\sum_{i=1}^{N}p_i^{\alpha\beta}}{\sum_{j=1}^{N}p_j^{\beta}}\right)\right\}+\frac{1}{\alpha-1}\left(1-D^{1-\alpha}\right)$$

$H_\alpha^\beta(P) \le L_\alpha^\beta < D^{1-\alpha}\mathrm{H}_\alpha^\beta(\mathrm{P}) + \frac{1}{\alpha-1}(1-\mathrm{D}^{1-\alpha})$, which gives (2.13). □

Theorem 2.3. *For arbitrary $N \in \mathbb{N}, \alpha, \beta > 0, \alpha \neq 1$ and for every code word lengths n_i, $i = 1, \dots, N$ of Theorem 2.1, L_α^β can be made to satisfy,*

$$L_\alpha^\beta \ge H_\alpha^\beta\,(P) > H_\alpha^\beta\,(P)\,D + \frac{1}{\alpha-1}\,(1-D)\,. \tag{2.17}$$

Proof. Suppose

$$\bar{n}_i = -\log_D\left(\frac{p_i^{\alpha\beta}}{\frac{\sum_{i=1}^{N}p_i^{\alpha\beta}}{\sum_{j=1}^{N}p_j^{\beta}}}\right), \quad \alpha>0,\ \alpha\neq 1,\ \beta>0. \tag{2.18}$$

Clearly $\bar{n}_i$ and $\bar{n}_i + 1$ satisfy 'equality' in Holder's inequality (2.10). Moreover, $\bar{n}_i$ satisfies (2.6). Suppose n_i is the unique integer between $\bar{n}_i$ and $\bar{n}_i + 1$, then obviously, n_i satisfies (2.6).

Since $\alpha > 0\,(\neq 1)$, we have

$$\left(\frac{\sum_{i=1}^{N}p_i^{\beta}D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N}p_j^{\beta}}\right)^{\alpha} \le \left(\frac{\sum_{i=1}^{N}p_i^{\beta}D^{-\bar{n}_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N}p_j^{\beta}}\right)^{\alpha} < D\left(\frac{\sum_{i=1}^{N}p_i^{\beta}D^{-\bar{n}_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N}p_j^{\beta}}\right)^{\alpha}. \tag{2.19}$$

Since $\left(\frac{\sum_{i=1}^{N}p_i^{\beta}D^{-\bar{n}_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N}p_j^{\beta}}\right)^{\alpha} = \frac{\sum p_i^{\alpha\beta}}{\sum p_j^{\beta}}$.

Hence (2.19) becomes

$$\left(\frac{\sum_{i=1}^{N}p_i^{\beta}D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{j=1}^{N}p_j^{\beta}}\right)^{\alpha} \le \frac{\sum p_i^{\alpha\beta}}{\sum p_j^{\beta}} < D\left(\frac{\sum p_i^{\alpha\beta}}{\sum p_j^{\beta}}\right)$$

which gives (2.17).

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