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ON GRADED WEAK MULTIPLICATION MODULES

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Abstract. Let G be a group with identity e, and let R be a G-graded commutative ring, and let M be a graded R-module. In this paper we characterize graded weak multiplication modules.

1. Introduction

Let G be a group. A commutative ring (R,G) is called a G-graded commutative ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \bigoplus_{g\in G} R_g$ such that $1 \in R_e$ and $R_gR_h \subseteq R_{gh}$ for each g and h in G. For simplicity, we will denote the graded ring (R,G) by R. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g,h \in G$; $R_gM_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be homogeneous element of degree g. A submodule $N \subseteq M$, where M is G-graded, is called G-graded if $N = \bigoplus_{g\in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. We write $h(R) = \cup_{g\in G} R_g$ and $h(M) = \cup_{g\in G} M_g$. A graded commutative ring R is called graded integral domain, if $ab = 0$ for $a,b \in h(R)$, then $a = 0$ or $b = 0$. A graded commutative ring R is called graded field, if every homogeneous element of R is unit. A graded ideal I of R is said to be graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a,b \in h(R)$ (see, [6]). A graded ideal I of R is said to be graded maximal if $I \neq R$ and there is no graded ideal J of R such that $I \subsetneq J \subsetneq R$. If R is a graded ring and M a graded R-module, the subset $T(M)$ of M is defined by $T(M) = \{m \in M : rm = 0$ for some $0 \neq r \in h(R)\}$. If R is a graded integral domain, then $T(M)$ is a graded submodule of M (see [1]). If $T(M) = 0$, M is called graded torsion free, if $T(M) = M$, M is called graded torsion. A graded module M over a G-graded ring R is called to be graded multiplication if for each graded submodule N of M; $N = IM$ for some graded ideal I of R. One can easily show that if N is graded submodule of a graded multiplication module M, then $N = (N : M)M$ [see, 4]. Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then

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the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R=\bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g=\{r/s : r\in h(R), s\in S \text{ and } g=(degs)^{-1}(degr)\}$.

Let M be a graded module over a G-graded ring R and $S\subseteq h(R)$ be a multiplicatively closed subset of R ($0\notin S$, $1\in S$ and for any $a,b\in S$, $ab\in S$). The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M=\bigoplus_{g\in G}(S^{-1}M)_g$ where $(S^{-1}M)_g=\{m/s : m\in M, s\in S \text{ and } g=(degs)^{-1}(degm)\}$. We write $h(S^{-1}R)=\bigcup_{g\in G}(S^{-1}R)_g$ and $h(S^{-1}M)=\bigcup_{g\in G}(S^{-1}M)_g$. Let $M=\bigoplus_{g\in G}M_g$ and $N=\bigoplus_{g\in G}N_g$ are two graded modules, then a mapping $\eta : M\longrightarrow N$ is graded homomorphism if $\eta(M_g)\subseteq N_g$ for all $g\in G$. Consider the graded homomorphism $\eta : M\longrightarrow S^{-1}M$ defined by $\eta(m)=m/1$. For any graded submodule N of M, the submodule of $S^{-1}M$ generated by $\eta(N)$ is denoted by $S^{-1}N$. Similar to non graded case, one can prove that $S^{-1}N=\{\beta\in S^{-1}M : \beta=m/s \text{ for } m\in N$ and $s\in S\}$ and that $S^{-1}N\neq S^{-1}M$ if and only if $S\cap(N:M)=\phi$. Let P be any graded prime ideal of a graded ring R and consider the multiplicatively closed subset $S=h(R)-P$. We denote the graded ring of fraction $S^{-1}R$ of R by R^g_P and we call it the graded localization of R. This ring is graded local with the unique graded maximal $S^{-1}P$ which will be denoted by PR^g_P. Moreover, R^g_P-module $S^{-1}M$ is denoted by M^g_P. For graded submodules N and K of M, if $N^g_P=K^g_P$ for every graded prime (graded maximal) ideal P of R, then $N=K$.

Moreover, similar to non graded case, we have the following properties for graded submodules N and K of M:

(1) $S^{-1}(N\cap K)=S^{-1}N\cap S^{-1}K$.

(2) $S^{-1}(N:K)=(S^{-1}N:S^{-1}K)$ if K is finitely generated.

If K is a graded submodule of $S^{-1}R$-module $S^{-1}M$, then $K\cap M$ will denote the graded submodule $\eta^{-1}(K)$ of M. Moreover, similar to the non graded case one can prove that $S^{-1}(K\cap M)=K$. In this paper, we study properties of graded prime submodules and define graded weak multiplication module. Also, we characterize graded weak multiplication modules.

2. Graded prime submodules

A proper graded submodule N of a graded R-module M is defined to be graded prime if $rm\in N$ where $r\in h(R)$ and $m\in h(M)$, then $m\in N$ or $r\in(N:M)$ (see [3]).

The set of all graded prime submodules of a graded R-module M is denoted by $GSpec_R(M)$.

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.1. *Let M be a graded module over a graded ring R. Then the following hold:*

(i) *If I and J are graded ideals of R, then $I+J$ and $I\bigcap J$ are graded ideals.*

(ii) *If N is a graded submodule, $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of M.*

(iii) *If N and K are graded submodules of M, then $N+K$ and $N\bigcap K$ are also graded submodules of M and $(N :_R M)$ is a graded ideal of R.*

(iv) *Let N_λ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.*

Theorem 2.2. *Let R be a graded ring and $M \neq 0$ a graded R-module, then R is a graded field if and only if every proper graded submodule of M is a graded prime submodule of M and $T(M) \neq M$.*

Proof. ($\Rightarrow$)Let N be a proper graded submodule of M. Suppose that $r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$. If $r_g = 0$, then $r_g = 0 \in (N : M)$, so let $r_g \neq 0$. Hence $m_h = r_g^{-1}(r_g m_h) \in N$ since R is graded field. So N is a graded prime submodule.

Now we show that $T(M) \neq M$. Let $T(M) = M$, and $0 \neq m \in M$. Then $r_g m = 0$ for some $0 \neq r_g \in h(R)$. So $m = 0$ since R is graded field, a contradiction.

($\Leftarrow$)Let $m_g \in M - T(M)$, so $Ann(m_g) = 0$. It is easy to see that every proper graded submodule of Rm_g is a graded prime submodule, and $Rm_g \cong R/Ann(m_g) = R$ as graded R-modules. Therefore, every proper graded ideal of R is a graded prime ideal, hence R is a graded field (see, [2]). □

Lemma 2.3. *Let M be a graded R-module, $S \subseteq h(R)$ a multiplicatively closed set and N a graded prime submodule of M such that $S \cap (N : M) = \emptyset$. Then $S^{-1}(N : M) = (S^{-1}N : S^{-1}M)$.*

Proof. Let $r/s \in (S^{-1}N : S^{-1}M)$. Let $m \in M$, so $m = \sum_{g\in G} m_g$. We show that for any $g \in G$, $rm_g \in N$. $r/s.m_g/1 \in S^{-1}N$. Hence $rm_g/s = n/t$ for some $n \in N$ and $t \in S$. So $s'trm_g = ss'n \in N$ for some $s' \in S$ and since $s't \notin (N : M)$, so N graded prime submodule gives $rm_g \in N$. Therefore, $rm \in N$, so $(S^{-1}N : S^{-1}M) \subseteq S^{-1}(N : M)$. It is clear that $S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$. Thus $S^{-1}(N : M) = (S^{-1}N : S^{-1}M)$. □

Theorem 2.4. *Let P be a graded prime ideal of R, $S \subseteq h(R)$ a multiplicatively closed set such that $S = h(R) - P$ and M a graded R-module. Then there exists a one-to-one correspondence between the graded P-prime submodules of M and the PR_p^g-prime R_p^g-submodules of M_p^g.*

Proof. Let N be a graded P-prime submodule of M. suppose $r/s.m/t \in N_P^g$ and $m/t \notin N_P^g$ where $r/s \in h(R_P^g)$ and $m/t \in h(M_P^g)$. Hence $rm \in N$ and $m \notin N$ where $r \in h(R)$ and $m \in h(M)$, so $r \in (N : M)$ since N is graded prime. Hence $r/s \in (N :_R M)_P^g = (N_P^g :_{R_P^g} M_P^g) = PR_P^g$ by Lemma 2.3. Hence N_P^g is a graded PR_P^g-prime R_P^g-submodule of M_P^g. Let N_P^g be a graded prime submodule of M_P^g. Let $rm \in N$ where $r \in h(R)$ and $h(M)$. So $r/1.m/1 \in N_P^g$ where

$r/1 \in h(R_P^g)$ and $m/1 \in h(M_P^g)$. Thus $m/1 \in N_P^g$ or $r/1 \in (N_P^g : M_P^g) = (N :_R M)_P^g = PR_P^g$ since N_P^g is a graded prime. Hence $r \in (N : M) = P$ or $m \in N$. Therefore, N is a graded P-prime submodule of M. □

3. Graded weak multiplication modules

Definition 3.1. Let M be a graded R-module. M is called a graded weak multiplication module if $GSpec_R M = \emptyset$ or for every graded prime submodule N of M, we have $N = IM$ where I is a graded ideal of R.

One can easily show that if M is a graded weak multiplication module, then $N = (N : M)M$ for every graded prime submodule N of M.

Theorem 3.2. *A graded R-module M is a graded weak multiplication module if and only if the graded R_P-module M_P is a graded weak multiplication module for every graded prime (or graded maximal) ideal P of R.*

Proof. Let M be a graded weak multiplication R-module and K a graded prime submodule of M_P where P is a graded prime ideal of R. So $K = N_P^g$ for some graded prime submodule N of M. So $N = IM$ for some graded ideal I of R. Hence $K = N_P^g = (IM)_P^g = I_P^g M_P^g$ because if $x/s \in (IM)_P^g$, then $x/s = am/t$ for some $a \in I$, $m \in M$ and $t \in S = R - P$, then $x/s = a/t.m/t \in I_P^g M_P^g$, so $(IM)_P^g \subseteq I_P^g M_P^g$, and clearly, $I_P^g M_P^g \subseteq (IM)_P^g$, so $(IM)_P^g = I_P^g M_P^g$. Thus M_P^g is a graded weak multiplication R_P^g-module.

Conversely, let N be a graded prime submodule of M. We show that $(N/(N : M)M)_P^g = 0$ for every graded maximal ideal P of R. If $(N : M) \subseteq P$, then by Theorem 2.4, N_P^g is a graded prime submodule of M_P^g. So $N_P^g = (N_P^g : M_P^g)M_P^g$, and by Lemma 2.3, $(N_P^g : M_P^g) = (N : M)_P^g$. Hence $(N/(N : M)M)_P^g = N_P^g/(N :_R M)_P^g M_P^g = N_P^g/(N_P^g :_{R_P^g} M_P^g)M_P^g = 0$. If $(N : M) \not\subseteq P$, then clearly $N_P^g = M_P^g$ and $(N : M)_P^g = R_P^g$. So $(N/(N : M)M)_P^g = N_P^g/(N : M)_P^g M_P^g = M_P^g/M_P^g = 0$. Hence $(N/(N : M)M)_P^g = 0$ for every graded maximal ideal P of R. So $N/(N : M)M = 0$, as needed. □

Remark 3.3. Let R be a graded integral domain. The quotient field Q of R is defined by $Q = S^{-1}R$ where $S = h(R) - \{0\}$. Indeed $Q = \bigoplus_{g \in G} Q_g$ where $Q_g = \{r/s : r, s \in h(R),\ s \neq 0 \text{ and } g = (degs)^{-1}(degr)\}$ that is a graded field which is called the graded quotient field.

Definition 3.4. Let R be a graded domain with the graded quotient field Q, the $Grank(M)$ is defined to be the maximal number of homogeneous elements of M linearly independent over M. In fact, $Grank(M)$ is the dimension of the graded vector space QM over graded field Q, that is $Grank(M) = Grank_Q(QM)$.

Theorem 3.5. *Let M be a graded weak multiplication module over a graded domain, then*

(i) *If M is a non-zero graded torsion free module, then $Grank(M) = 1$.*

(ii) *If M is a graded torsion module, then $Grank(M) = 0$.*

(iii) *M is either graded torsion or graded torsion free.*

Proof. (i) First, Let R is a graded field, so $0 \neq M$ is a graded vector space which is a graded weak multiplication module. If $Grank(M) > 1$, then let $0 \neq W \subset M$. According to Theorem 2.2, W is a graded prime submodule of M, so $W = IM$ for some graded ideal I of the graded field R. So $I = 0$ or $I = R$, a contradiction. Hence $Grank(M) \leq 1$, and since $M \neq 0$, then $Grank(M) = 1$. Now let M be a non-zero graded torsion free R-module, then $QM \neq 0$, which Q is a graded quotient field of R. By Theorem 3.2, QM is a graded weak multiplication Q-module, and since we have proved in above $Grank_Q(QM) = 1$, so $Grank(M) = Grank_Q(QM) = 1$.

(ii) Suppose that M is a graded torsion module, then $QM = 0$ and therefore $Grank(M) = Grank_Q(QM) = 0$.

(iii) Let $T(M) \neq M$. Then $T(M)$ is a graded prime submodule and $(T(M) : M) = 0$ by [1, Proposition 2.5]. Therefore since M is graded weak multiplication, we have $T(M) = (T(M) : M)M = 0$, so the proof is complete. □

A graded module M over a G-graded ring R is called locally graded cyclic if M_P^g is graded cyclic R_p^g-module for all graded maximal ideal P of R.

A graded module M over a G-graded ring R is called to be graded finitely generated if $M = \sum_{i=1}^{n} Rx_{g_i}$ where $x_{g_i} \in h(M)$. A graded R-module M is called graded cyclic if $M = Rx_g$ where $x_g \in h(M)$.

Theorem 3.6. *A graded finitely generated module is a graded multiplication module if and only if it is locally graded cyclic.*

Proof. See [4, Proposition 2.4]. □

Theorem 3.7. *Every graded finitely generated graded weak multiplication module is graded multiplication module.*

Proof. Suppose that M is a graded finitely generated graded weak multiplication R-module. We show that M is locally graded cyclic, and by Theorem 3.5, M is a graded multiplication module. By localization and Theorem 3.2, we can assume that M is a graded finitely generated graded weak multiplication R-module where R is a graded local ring. Let m be the only graded maximal ideal of R. Obviously M/mM is a graded finitely generated graded weak multiplication R/m-module. If $M = mM$, then by Nakayama Lemma for graded case (see [4]), we have $M = 0$, so it is graded cyclic.

If $mM \neq M$, then $Grank_{R/m}(M/mM) = 1$, by Theorem 3.5 (i), so M is graded cyclic. □

The graded dimension of a G-graded R is the supremum of the lengths of all chains of graded prime ideals in R and denoted by $Gdim(R)$.

Theorem 3.8. *Let R is a graded domain. Then the following are equivalent.*

(i) *Gdim(R)=0.*

(ii) *For every graded weak multiplication R-module M, if $T(M) = 0$, then M is graded cyclic.*

(iii) *For every graded weak multiplication R-module M, if $T(M) = 0$, then M is graded multiplication.*

Proof. (i)⇒(ii) First let R be a graded field. Let M a graded torsion-free graded weak multiplication R-module. If $M = 0$, then M is graded cyclic. So let $M \neq 0$. Hence M is a non-zero graded weak multiplication vector space over the graded field R. By Theorem 3.5, $Grank(M) = 1$. That is $M \cong R$, so M is graded cyclic.

Now we prove the general case. Let $0 \neq M$. By [1, Proposition 2.5], $T(M) = 0$ is a graded prime submodule of M. Hence $(T(M) : M)$ is a graded prime ideal of R and since $Gdim(R) = 0$, $R/(T(M) : M)$ is a graded field. Since $T(M) = 0$, so $M \cong M/0 = M/T(M)$ is a graded torsion-free graded weak multiplication $R/(T(M) : M)$-module. So M is a graded torsion-free graded weak multiplication module over a graded field $R/(T(M) : M)$. Therefore as we have proved in above M is a graded cyclic $R/(T(M) : M)$-module and by the fact $(r + (T(M) : M))m = rm$ for all $r \in R$ and $m \in M$, then M is a graded cyclic R-module.

(ii)⇒(iii) By [4, Proposition 2.4].

(iii)⇒(i) Let P be a graded prime ideal of R. It is enough to prove that R/P is a graded field. If Q is the graded quotient field of R, then 0 is the only graded prime ideal of R/P-module Q. Hence Q is a graded torsion-free graded weak multiplication R/P-module. Therefore by assumption Q is a graded multiplication module. Since R/P is a graded submodule of Q, so $R/P = IQ$ for some graded ideal I of R/P. Clearly, $IQ = Q$, so $Q = R/P$, hence P is a graded maximal ideal of R. Therefore, $Gdim(R) = 0$. □

Corollary 3.9. *Let R is a graded integral domain. Then the following are equivalent.*

(i) *R is a graded field.*

(ii) *Every graded weak multiplication R-module is graded cyclic.*

(iii) *Every graded weak multiplication R-module is graded multiplication.*

Proof. If R is graded field, and since every graded weak multiplication R-module is a graded vector space, it is a graded torsion-free graded weak multiplication R-module, so the proof follow by Theorem 3.8. □

Corollary 3.10. *If m is a graded maximal ideal of the graded ring R which is a minimal graded prime ideal and $m \neq m^2$, then the following are equivalent.*

(i) *m is a graded weak multiplication R-module.*

(ii) *There is no graded ideal between m^2 and m.*

(iii) $GSpec_R m = \{m^2\}$.

Proof. By localization and Theorem 3.2 we can assume that R is a graded local ring with the only graded prime ideal m.

(i)⇒(ii) Let m be a graded weak multiplication R-module. If $m^2 \subseteq I \subseteq m$ where I is graded ideal of R, we show that I is a graded prime submodule of m. Let $ra \in I$ where $r \in h(R)$ and $a \in h(m)$. Suppose that $a \notin I$, then r is not unit, hence $r \in m$. Therefore $rm \subseteq m^2 \subset I$, that is I is a graded prime submodule of m.

Since m is graded weak multiplication module, and I a graded prime submodule, then $I = mm_1$ for some graded ideal m_1 of R. If $m_1 = R$, then $I = mR = m$, which is a contradiction. So $m_1 \subseteq m$, that is $m^2 \subseteq I = mm_1 \subseteq m^2$, thus there is no graded ideal between m^2 and m.

(ii)⇒ (iii) Suppose that there is no graded ideal between m^2 and m. If I is a graded prime submodule of the graded R-module m, then $(I : m)$ is a graded prime ideal by [3, Proposition 2.5]. Since m is the only graded prime ideal of R, then $(I : m) = m$. Therefore, $m^2 \subseteq I \subseteq m$, and by assumption $I = m^2$, hence $GSpec_R m = \{m^2\}$. □

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