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A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. Complex-valued harmonic functions that are univalent and sense-preserving in the open unit disc U can be written in the form $f = h + \bar{g}$, where h and g are analytic in U. In this paper authors introduce the class, $R_H(\beta)$, $(1 < \beta \leq 2)$ consisting of harmonic univalent functions $f = h + \bar{g}$, where h and g are of the form $h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k$ and $g(z) = \sum_{k=1}^{\infty} |b_k| z^k$ for which $\mathrm{Re}\{h'(z) + g'(z)\} < \beta$. We obtain distortion bounds extreme points and radii of convexity for functions belonging to this class and discuss a class preserving integral operator. We also show that class studied in this paper is closed under convolution and convex combinations.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. See Clunie and Sheil-Small [1].

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f'(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

Note that S_H reduces to the class of normalized analytic univalent functions if the co-analytic part of its member is zero.

A function f of the form (1.1) is harmonic starlike for $|z| = r < 1$, if

$$\frac{\partial}{\partial\theta}\Big(\arg f(re^{i\theta})\Big) = \mathrm{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \qquad |z| = r < 1. \qquad \text{See [2].}$$

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Silverman [4] proved that the coefficient conditions $\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq 1$ and $\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \leq 1$ are sufficient conditions for functions $f = h + \bar{g}$ to be harmonic starlike convex functions, respectively.

Denote by V_H the subclass of S_H consisting of functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, \qquad g(z) = \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1. \tag{1.2}$$

Recently Yalcin et al. [6] studied the class $HP(\alpha)$, $(0 \leq \alpha < 1)$ the subclass of S_H satisfying the condition

$$\operatorname{Re}\{h'(z) + g'(z)\} > \alpha. \tag{1.3}$$

Further let $V_H P(\alpha)$ be the subclass of V_H consisting of functions of the form (1.2) that satisfy condition (1.3).

Let $R_H(\beta)$, $(1 < \beta \leq 2)$, denote the subclass of V_H satisfying the condition

$$\operatorname{Re}\{h'(z) + g'(z)\} < \beta. \tag{1.4}$$

We note that the class $R_H(\beta)$ reduces to class $R(\beta)$ if co-analytic part of f is zero i.e. $g \equiv 0$ studied by Uralegaddi et al. [5]. Yalcin et al. [6] have studied the functions with negative coefficients that satisfy $\operatorname{Re}\{h'(z) + g'(z)\} > \alpha$, $(0 \leq \alpha < 1)$ for $z \in U$. we need the following Lemma due to Theorem 2.1 of [6].

Lemma 1. *Let $f = h + \bar{g} \in V_H$ be given by (1.2) and $\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq 1 - \alpha$, $(0 \leq \alpha < 1)$ then $f \in V_H P(\alpha)$.*

2. Main results

Theorem 2.1. *A function f of the form (1.2) is in $R_H(\beta)$ if and only if*

$$\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1. \tag{2.1}$$

Proof. Let $\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1$. It suffices to prove that

$$\left|\frac{h'(z) + g'(z) - 1}{h'(z) + g'(z) - (2\beta - 1)}\right| < 1, \qquad z \in U.$$

We have

$$\left|\frac{h'(z) + g'(z) - 1}{h'(z) + g'(z) - (2\beta - 1)}\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty} k|a_k| z^{k-1} + \sum_{k=1}^{\infty} k|b_k| z^{k-1}}{\sum_{k=2}^{\infty} k|a_k| z^{k-1} + \sum_{k=1}^{\infty} k|b_k| z^{k-1} - 2(\beta - 1)}\right|$$

$$\leq \left| \frac{\sum_{k=2}^{\infty} k|a_k||z|^{k-1} + \sum_{k=1}^{\infty} k|b_k||z|^{k-1}}{2(\beta-1) - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} - \sum_{k=1}^{\infty} k|b_k||z|^{k-1}} \right|$$

$$\leq \left| \frac{\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k|}{2(\beta-1) - \sum_{k=2}^{\infty} k|a_k| - \sum_{k=1}^{\infty} k|b_k|} \right|$$

which is bounded above by 1 by hypothesis and the sufficient part is proved.

Conversely, suppose that

$$\operatorname{Re}\{h'(z) + g'(z)\} < \beta, \qquad \text{i.e.}$$

$$\operatorname{Re}\left\{1 + \sum_{k=2}^{\infty} k|a_k|z^{k-1} + \sum_{k=1}^{\infty} k|b_k|z^{k-1}\right\} < \beta, \qquad z \in U.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z to be real and let $z \to 1^-$, we obtain

$$\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1,$$

which gives the necessary part. The proof of the theorem is complete.

Next we determine bounds for the class $R_H(\beta)$.

Theorem 2.2. *If $f \in R_H(\beta)$, then*

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{2}(\beta - 1 - |b_1|)r^2, \quad |z| = r < 1$$

and

$$|f(z)| \geq (1 - |b_1|)r - \frac{1}{2}(\beta - 1 - |b_1|)r^2, \quad |z| = r < 1.$$

The bounds are sharp for the functions $f(z) = z + |b_1|\bar{z} + \frac{1}{2}(\beta - 1 - |b_1|)\bar{z}^2$ and $f(z) = z + |b_1|\bar{z} + \frac{1}{2}(\beta - 1 - |b_1|)z^2$ for $|b_1| \leq \beta - 1$.

Proof. Let $f \in R_H(\beta)$. Taking the absolute value of f, we have

$$|f(z)| \leq (1 - |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k$$

$$\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$
$$\leq (1+|b_1|)r + \frac{1}{2}\sum_{k=2}^{\infty}k(|a_k|+|b_k|)r^2$$
$$\leq (1+|b_1|)r + \frac{1}{2}(\beta-1-|b_1|)r^2$$

and

$$|f(z)| \geq (1+|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k$$
$$\geq (1+|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$
$$\geq (1+|b_1|)r - \frac{1}{2}\sum_{k=2}^{\infty}k(|a_k|+|b_k|)r^2$$
$$\geq (1+|b_1|)r + \frac{1}{2}(\beta-1-|b_1|)r^2.$$

The functions $z+|b_1|\bar{z}+\frac{1}{2}(\beta-1-|b_1|)\bar{z}^2$ and $z+|b_1|\bar{z}+\frac{1}{2}(\beta-1-|b_1|)z^2$ for $|b_1|\leq\beta-1$ show that the bounds given in Theorem 2.2 are sharp.

The following result follows from the left hand inequality in Theorem 2.2.

Corollary 2.1. *If $f\in R_H(\beta)$, then*

$$\{\omega : |\omega| < \frac{1}{2}(3-\beta-|b_1|)\} \subset f(U). \tag{2.2}$$

Next we determine the extreme points of the closed convex hulls of $R_H(\beta)$, denoted by clco $R_H(\beta)$.

Theorem 2.3. *$f\in clco\, R_H(\beta)$, if and only if*

$$f(z) = \sum_{k=1}^{\infty}(\lambda_k h_k + \gamma_k g_k) \tag{2.3}$$

where $h_1(z)=z$, $h_k(z)=z+\frac{\beta-1}{k}z^k$ $(k=2,3,4,\ldots)$, $g_k(z)=z+\frac{\beta-1}{k}\bar{z}^k$ $(k=1,2,3,\ldots)$ and $\sum_{k=1}^{\infty}(\lambda_k+\gamma_k)=1$, $\lambda_k\geq 0$ and $\gamma_k\geq 0$. In particular the extreme points of $R_H(\beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (2.3) write

$$f(z) = \sum_{k=1}^{\infty}(\lambda_k h_k + \gamma_k g_k)$$

$$= z + \sum_{k=2}^{\infty} \Big(\frac{\beta - 1}{k}\Big)\lambda_k z^k + \sum_{k=1}^{\infty} \Big(\frac{\beta - 1}{k}\Big)\gamma_k \bar{z}^k.$$

Then

$$\begin{aligned}&\sum_{k=2}^{\infty} \frac{k}{\beta - 1}\Big(\frac{\beta - 1}{k}\lambda_k\Big) + \sum_{k=1}^{\infty} \frac{k}{\beta - 1}\Big(\frac{\beta - 1}{k}\gamma_k\Big)\\ &\quad= \sum_{k=2}^{\infty} \lambda_k + \sum_{k=1}^{\infty} \gamma_k\\ &\quad= 1 - \lambda_1 \leq 1,\end{aligned}$$

and so $f \in clcoR_H(\beta)$.

Conversely, suppose that $f \in clcoR_H(\beta)$. Set $\lambda_k = \frac{k}{\beta-1}|a_k|$, $(k = 2, 3, 4, \ldots)$ and $\gamma_k = \frac{k}{\beta-1}|b_k|$, $(k = 1, 2, 3, \ldots)$. Then note that by Theorem 2.1, $0 \leq \lambda_k \leq 1$, $(k = 2, 3, 4, \ldots)$ and $0 \leq \gamma_k \leq 1$, $(k = 1, 2, 3, \ldots)$. We define $\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k$ and note that by Theorem 2.1, $\lambda_1 \geq 0$. Consequently, we obtain $f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k + \gamma_k g_k)$ as required.

Theorem 2.4. *If $f \in R_H(\beta)$ then $f \in V_H P(2 - \beta)$.*

Proof. The inclusion relation is a direct consequence of Lemma 1 and Theorem 2.1.

Next we give the interrelation between the class $R_H(\beta)$ and S_H^*, where S_H^* is the class of harmonic starlike function in U.

Theorem 2.5. *$R_H(\beta) \subseteq S_H^*$, where $1 < \beta \leq 2$.*

Proof. Let $f \in R_H(\beta)$. Then by Theorem 2.1

$$\sum_{k=2}^{\infty} \frac{k}{\beta - 1}|a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1}|b_k| \leq 1. \tag{2.4}$$

Now

$$\begin{aligned}&\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k|\\ &\quad\leq \sum_{k=2}^{\infty} \frac{k}{\beta - 1}|a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1}|b_k|\\ &\quad\leq 1. \qquad \text{[using (2.4)]}\end{aligned}$$

Thus $f \in S_H^*$.

This completes the proof of Theorem 2.5.

Theorem 2.6. *Each function in the class $R_H(\beta)$ maps a disks U_r where $r < \inf_k \left\{\frac{1}{k(\beta-1-|b_1|)}\right\}^{\frac{1}{k+1}}$ onto convex domains for $\beta > 1+|b_1|$.*

Proof. Let $f \in R_H(\beta)$ and let r, be fixed is that $0 < r < 1$, then $r^{-1}f(rz) \in R_H(\beta)$ and we have

$$\begin{aligned}\sum_{k=2}^{\infty} k^2(|a_k|+|b_k|)r^{k-1} &= \sum_{k=2}^{\infty} k(|a_k|+|b_k|)(kr^{k-1}) \\ &\le \sum_{k=2}^{\infty} k(|a_k|+|b_k|) \\ &\le \beta - 1 - |b_1| \le 1,\end{aligned}$$

provided

$$\begin{aligned} kr^{k-1} &\le \frac{1}{\beta-1-|b_1|} \\ \text{or, } \; r &< \inf_k \left\{\frac{1}{k(\beta-1-|b_1|)}\right\}^{\frac{1}{k-1}}. \end{aligned}$$

The proof of Theorem 2.6 is complete.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k$$

and

$$F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \bar{z}^k$$

we define their convolution

$$(f * F)(z) = f(z) * F(z) = z + \sum_{k=2}^{\infty} |a_k A_k| z^k + \sum_{k=1}^{\infty} |b_k B_k| \bar{z}^k. \tag{2.5}$$

Using this definition, we show that the class $R_H(\beta)$ is closed under convolution.

Theorem 2.7. *For $1 < \beta \le \alpha \le 2$ let $f \in R_H(\alpha)$ and $F \in R_H(\beta)$. Then $f * F \in R_H(\beta) \subseteq R_H(\alpha)$.*

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k$ be in $R_H(\beta)$ and $F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \bar{z}^k$ be in $R_H(\alpha)$. Then the convolution $f * F$ is given by (2.5). We wish to show that the coefficients of $f * F$ satisfy the required condition given in

Theorem 2.1. For $F(z) \in R_H(\alpha)$ we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $f * F$, we have

$$\begin{aligned}&\sum_{k=2}^{\infty} \frac{k}{\beta-1}|a_k A_k| + \sum_{k=1}^{\infty} \frac{k}{\beta-1}|b_k B_k| \\ &\quad \leq \sum_{k=2}^{\infty} \frac{k}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta-1}|b_k| \\ &\quad \leq 1. \qquad \text{(Science } f \in R_H(\beta).\end{aligned}$$

Therefore $f * F \in R_H(\beta) \subseteq R_H(\alpha)$.

Next, we show that $R_H(\beta)$ is closed under convex combinations of its members.

Theorem 2.8. *The class $R_H(\beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$ let $f_i(z) \in R_H(\beta)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=2}^{\infty} |a_{k_i}| z^k + \sum_{k=1}^{\infty} |b_{k_i}| \bar{z}^k.$$

Then by Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{k}{\beta-1}|a_{k_i}| + \sum_{k=1}^{\infty} \frac{k}{\beta-1}|b_{k_i}| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \Big(\sum_{i=1}^{\infty} t_i |a_{k_i}| \Big) z^k + \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{\infty} t_i |b_{k_i}| \Big) \bar{z}^k.$$

then by Theorem 2.1, we have

$$\begin{aligned}&\sum_{k=2}^{\infty} \frac{k}{\beta-1} \Big(\sum_{i=1}^{\infty} t_i |a_{k_i}| \Big) + \sum_{k=1}^{\infty} \frac{k}{\beta-1} \Big(\sum_{i=1}^{\infty} t_i |b_{k_i}| \Big) \\ &\quad = \sum_{i=1}^{\infty} t_i \Big(\sum_{k=2}^{\infty} \frac{k}{\beta-1}|a_{k_i}| + \sum_{k=1}^{\infty} \frac{k}{\beta-1}|b_{k_i}| \Big) \\ &\quad \leq \sum_{i=1}^{\infty} t_i = 1.\end{aligned}$$

Therefore

$$\sum_{i=1}^{\infty} t_i f_i(z) \in R_H(\beta).$$

The δ-neighborhood of f is the set

$$N_\delta(f) = \Big\{F : F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \bar{z}^k \quad \text{and} \quad \sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) \le \delta\Big\}. \qquad \text{See [3].}$$

Theorem 2.9. *Let $f \in R_H(\beta)$ and $\delta \le 2-\beta$. If $F \in N_\delta(f)$, then F is harmonic starlike function.*

Proof. Let $F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \bar{z}^k$ belong to $N_\delta(f)$. We have

$$\begin{aligned}\sum_{k=2}^{\infty} k|A_k| + \sum_{k=1}^{\infty} k|B_k| &\le \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + \sum_{k=2}^{\infty} k(|a_k| + |b_k|) + |b_1 - B_1| + |b_1| \\ &\le \delta + \beta - 1 \le 1.\end{aligned}$$

Hence, $F(z)$ is harmonic starlike function.

3. A family of class preserving integral operator

let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (1.1) then $F(z)$ defined by relation

$$F(z) = \frac{c+1}{z^c} \int_0^c t^{c-1} h(t)dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t)dt}, \qquad (c > -1). \tag{3.1}$$

Theorem 3.1. *Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (1.2) and $f(z) \in R_H(\beta)$ then $F(z)$ be defined by (3.1) also belong to $R_H(\beta)$.*

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| z^k$ be in $R_H(\beta)$ then by Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k| \le 1. \tag{3.2}$$

By definition of $F(z)$, we have

$$F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k| z^k + \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| \bar{z}^k.$$

Now

$$\sum_{k=2}^{\infty} \frac{k}{\beta - 1} \Big(\frac{c+1}{c+k} |a_k|\Big) + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} \Big(\frac{c+1}{c+k} |b_k|\Big)$$

$$\leq \sum_{k=2}^{\infty} \frac{k}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta-1}|b_k|$$
$$\leq 1.$$

Thus $F(z) \in R_H(\beta)$.

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