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COHOMOLOGY OF FINITE ABELIAN p-GROUPS AND FREE ACTIONS

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Abstract. The generators of the integral cohomology ring $H^*(G,\mathbb{Z})$ of a finite abelian p-group G have been constructed and an application to free action of a finite group on a product of spheres $(S^n)^k$ has been given.

1. Introduction

In this paper Bockestein homomorphism is used to work out the generators and some basic relations of the integral cohomology ring $H^*(G,\mathbb{Z})$ of an abelian p-group $G = \mathbb{Z}_{p^{t_1}} \times \mathbb{Z}_{p^{t_2}} \times \cdots \times \mathbb{Z}_{p^{t_r}}$, aproduct of cyclic groups of orders powers of the prime p. An application is also given about the maximum possible rank for a p-group to act freely, preserving orientation, on $(S^n)^k$, $k \geq 2$, a product of k spheres of dimension n.

2. Preleminaries

Let $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_p \to 0$ be a short exact sequence, where the first map p is defined by multiplication by the prime p and second map j is by taking modulo p. The Bockestein homomorphism $\Delta : H^q(G,\mathbb{Z}_p) \to H^{q+1}(G,\mathbb{Z})$ is the connecting homomorphism . The following sequence is however exact:

$$0 \to H^q(G,\mathbb{Z}) \xrightarrow{j_*} H^q(G,\mathbb{Z}_p) \xrightarrow{\delta} H^{q+1}(G,\mathbb{Z}) \to 0.$$

j_* is monomorphism . Therefore $H^*(G,Z) = \ker\delta = \ker\Delta$. Observe that the composite $\Delta\Delta = j_*\delta j_*\delta = j_*0\delta = 0$.

3. Computations

$H^*(\mathbb{Z}_{p^t},\mathbb{Z}_p) = P[v] \otimes E[u]$, tensor product of polynomial and exterior algebras, where $\deg v = 2$, $\deg u = 1$, $p^t v = p^t u = 0$ and $\Delta u = v$. That is if $p \geq 3$. When $p = 2$,

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$H^*(\mathbb{Z}_{p^t}, \mathbb{Z}_p) = P[v]$, where $\deg v = 1$, $p^t v = 0$ and $\Delta v = v^2$[4], Let G be a finite abelian p-group. Then $G = \mathbb{Z}_{p^{t_1}} \times \mathbb{Z}_{p^{t_2}} \times \cdots \times \mathbb{Z}_{p^{t_r}}$, a product of cyclic groups of orders powers of the prime p. Without lost of generality, we can assume $t_1 \leq t_2 \leq \cdots \leq t_r$. By Kunneth formula [3] we have

$$H^*(G, \mathbb{Z}_p) \cong H^*(\mathbb{Z}_{p^{t_1}}, \mathbb{Z}_p) \otimes H^*(\mathbb{Z}_{p^{t_2}}, \mathbb{Z}_p) \otimes \cdots \otimes H^*(\mathbb{Z}_{p^{t_r}}, \mathbb{Z}_p).$$

Suppose that $p \geq 3$. Then $H^*(G, \mathbb{Z}_p) \cong P[v_1, v_2, \ldots, v_r] \otimes E[u_1, u_2, \ldots, u_r]$ where $\deg v_i = 2$, $\deg u_i = 1$, $p^{t_i} v_i = p^{t_i} u_i = 0$ and $\Delta u_i = v_i$ $(i = 1, 2, \ldots, n)$. As an operator Δ has the property $\Delta(ab) = (\Delta a)b + (-1)^{\deg a} a(\Delta b)$. The product of these generators satisfies the anticommutative rule $ab = (-1)^{\deg a \deg b} ba$. In particular $u_i u_j = -u_j u_i$ if $i \neq j$ and $u_i^2 = 0$. Multiplication by v_i is commutative. $\Delta\Delta = 0$. Therefore $\Delta v_i = 0$. The generators of $H^*(G, \mathbb{Z})$ are the elements of $\ker \Delta$. They are given as follows:

$$\sum_{k=1}^{j} (-1)^{i_k} u_{i_1} u_{i_2} \ldots \hat{u}_{i_k} \ldots u_{i_j} v_{i_k} \qquad \text{where } i, j \in \{1, 2, \ldots, r\}.$$

The number of such generators is $\binom{r}{j}$ for each j. Suppose now $p = 2$. Then $H^*(G, \mathbb{Z}_p) \cong P[v_1, v_2, \ldots, v_r]$ where $\deg v_i = 1$, $p^{r_i} v_i = 0$ and $\Delta v_i = v_i^2$ $(i = 1, 2, \ldots, n)$.

Thus the generators of $H^*(G, \mathbb{Z})$ are given as follows:

$$\sum_{k=1}^{j} v_{i_1} v_{i_2} \ldots v_{i_k}^2 \ldots v_{i_j} \qquad \text{where } i, j \in \{1, 2, \ldots, r\}.$$

They are also $\binom{r}{j}$ generators for each j. In each case there are $2^r - 1$ generators. Thus we have

Theorem. *The integeral cohomology ring $H^*(G, \mathbb{Z})$ of an abelian p-group $G = \mathbb{Z}_{p^{t_1}} \times \mathbb{Z}_{p^{t_2}} \times \cdots \times \mathbb{Z}_{p^{t_r}}$ is generated by $\alpha_k^{(d)}$; $d = 2, \ldots, r+1$; $k = 1, \ldots, \binom{r}{d-1}$ where $\deg \alpha_k^{(d)} = d$ and its additive order is p^{t_k}. Further more $(\alpha_k^{d)})^2 = 0$ if d is odd.*

Corollary. *The exponent of $H^*(G, \mathbb{Z})$ is p^{t_r}.*

Special Cases.

$H^*(\mathbb{Z}_{p^t}, \mathbb{Z})$ is generated by α where $\deg \alpha = 2$ and $p^t \alpha = 0$. $H^*(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^t}, \mathbb{Z})$ where $s \leq t$ is generated by α, β and μ where $\deg \alpha = \deg \beta = 2$ and $\deg \mu = 3$ and $p^s \alpha = p^t \beta = p^s \mu = 0$ and $\mu^2 = 0$ if $p \geq 3$. $\mu^2 = \alpha^2 \beta + \alpha \beta^2$ when $p = 2$.

Finally $H^*((\mathbb{Z}_p)^3, \mathbb{Z})$ is generated by $\alpha, \beta, \gamma, \mu, \nu, \chi, \xi$ where $\deg \alpha = \deg \beta = \deg \gamma = 2$, $\deg \mu = \deg \nu = \deg \chi = 3$, $\deg \xi = 4$. There are two different sets of relations for $p = 2$ and $p \geq 3$[4].

4. Application

W. Browder [2] proved that when n is odd and p a prime if $(\mathbb{Z}_p)^r$ acts freely, preserving orientation, on $(S^n)^k$ with trivial action on the homology, then $r \leq k$. This gives the limitation on the rank r when any finite group acts freely on $(S^n)^k$. A different but simpler proof to this result is now introduced.

By [1, Proposition 1], the following sequence is exact

$$0 \to H^m(G,\mathbb{Z}) \to H^{n+1}(G,\mathbb{Z}) \to (\mathbb{Z}_{|G|})^k \to H^{m+1}(G,\mathbb{Z}) \to H^n(G,\mathbb{Z}) \to 0$$

where $m = (k-1)n - k + 2$. m is also odd. Observe that $H^{n+1}(G,\mathbb{Z})| = |G||H^m(G,\mathbb{Z})|$ and $|G| = p^r$. From the structure of $H^*(G,\mathbb{Z})$ we have $p^r \leq |H^{n+1}(G,\mathbb{Z})|$, since $n+1$ is even and $H^m(G,\mathbb{Z})| \leq p^{k-r}$, since $m = (n-1)(k-1)+1$ is odd and $n-1$ is even. Then $p^r \leq p^r p^{k-r}$. Thus $r \leq k$.

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References

[1] K. H. Alzubaidy, *Free actions on* $(S^n)^k$, Mathematika, **32**(1985), 49–54.

[2] W. Browder, *Cohomology and group action*, Invent. Math., **71**(1983), 599-607.

[3] K. S. Brown, Cohomology and Groups, Springer-Verlag, New York, 1982.

[4] G. Lewis, *The integral cohomology rings of groups of order* p^3, Trans. Amer. Math. Soc., **132**(1968), 501-529.

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