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ON THE IRREDUCIBILITY OF LINEAR REPRESENTATIONS OF THE PURE BRAID GROUP

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Abstract. Following up on our result in [1], we find a milder sufficient condition for the tensor product of specializations of the reduced Gassner representation of the pure braid group to be irreducible. We prove that $G_n(x_1,\dots,x_n)\otimes G_n(y_1,\dots,y_n): P_n \to GL(\mathbb{C}^{n-1}\otimes\mathbb{C}^{n-1})$ is irreducible if $x_i \neq \pm y_i$ and $x_j \neq \pm {y_j}^{-1}$ for some i and j.

1. Introduction

The pure braid group, P_n, is a normal subgroup of the braid group, B_n, on n strings. It has a lot of linear representations. One of them is the Gassner representation which comes from the embedding $P_n \to \mathrm{Aut}(F_n)$, by means of Magnus representation [3, p.119]. In our work, we consider the tensor product of complex specializations of irreducible Gassner representation of the pure braid group, namely,

$$G_n(x_1,\dots,x_n)\otimes G_n(y_1,\dots,y_n): P_n \to GL(\mathbb{C}^{n-1}\otimes\mathbb{C}^{n-1}).$$

Our previous work in [1] asserts that for $n\geq 3$ and $x_1,\dots,x_n,y_1,\dots,y_n \in \mathbb{C}-\{0,1\}$, the representation above is irreducible under the following condition: for some i and $j \in \{1,\dots,n\}$, $i<j$, $x_ix_j \neq y_iy_j$ and $x_\alpha x_j y_\alpha y_j \neq 1$, for every $\alpha \in \{1,\dots,j-1\}$ and $x_{\alpha+1}x_jy_{\alpha+1}y_j \neq 1$, for every $\alpha \in \{j,\dots,n-1\}$.

In Section 1 of our work, We define the Gassner representation of a free normal subgroup of the pure braid group of rank $n-1$ denoted by U_r where $1\leq r\leq n$. We consider $\mathbb{C}[U_r]$ to be the group algebra of U_r over $\mathbb{C}$, and let $\mathcal{A}$ be the augmentation ideal of $\mathbb{C}[U_r]$. On the other hand, if M is any P_n-module, then $\mathcal{A}M$ is a P_n-submodule of M. We first show that if $\mathbb{C}^{n-1}$ is made into a P_n-module via the specialization of the reduced Gassner representation $G_n(x_1,\dots,x_n): P_n \to GL(\mathbb{C}^{n-1})$, then $\mathcal{A}\mathbb{C}^{n-1}$ is its unique minimal nonzero P_n-submodule. Of course $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$ when $G_n(x_1,\dots,x_n)$ is irreducible.

In Section 2, we give a summary of the proofs of the important results published in [1], which will help us to prove our main theorem.

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In Section 3, we let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{C}-\{0\}$, so that $G_n(x_1, \ldots, x_n) \otimes G_n(y_1, \ldots, y_n)$ defines a diagonal action of P_n on $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$. The main technical result is Proposition 1, which gives a sufficient condition for $\mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1}$ to be the unique minimal nonzero P_n-submodule of $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$. This implies the irreducibility of the tensor product above. We then state Lemma 4 to simplify Proposition 1 and prove our main theorem that states the following: If $x_i \neq \pm y_i$ and $x_j \neq \pm {y_j}^{-1}$ for some i and $j \in \{1, \ldots, n\}$ then the tensor product above is irreducible.

Notation 1. *The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \to S_n$, defined by $\sigma_i \to (i, i+1)$, $1 \leq i \leq n-1$. It has the following generators:*

$$A_{i,r} = \sigma_{r-1}\sigma_{r-2} \ldots \sigma_{i+1}{\sigma_i}^2\sigma_{i+1}^{-1} \ldots \sigma_{r-2}^{-1}\sigma_{r-1}^{-1}, \quad 1 \leq i < r \leq n$$

We will construct for each $r = 1, \ldots, n$ a free normal subgroup of rank $n-1$, namely, U_r. Let U_r be the subgroup generated by the elements

$$A_{1,r}, A_{2,r}, \ldots, A_{r-1,r}, A_{r,r+1}, \ldots, A_{r,n},$$

where $A_{i,r}$ are those generators of P_n that become trivial after the deletion of the r-th strand. For a fixed value of r, the image of A_{ir} under the reduced Gassner representation is denoted by $\tau_{i,r}$, where $\tau_{i,r} = I - P_{i,r}Q_{i,r}$. In other words, the generators of U_r are $A_{i,r}$ where $A_{i,r} = A_{r,i}$ whenever $i > r$. It is known that U_r generates a free subgroup of P_n which is isomorphic to the subgroup U_n freely generated by $\{A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}\}$. This is intuitively clear because it is quite arbitrary how we assign indices to the braid "strings".

For simplicity, we denote $A_{i,r}$ by $\tau_{i,r}$. That is, we have

$$\tau_{1,r} = A_{1,r}, \ldots, \tau_{r-1,r} = A_{r-1,r}, \quad \tau_{r+1,r} = A_{r,r+1}, \quad \tau_{r+2,r} = A_{r,r+2}, \ldots, \tau_{n,r} = A_{r,n}$$

Definition 1. The reduced Gassner representation restricted to U_r is defined as follows: $\tau_{i,r} = I - P_{i,r}Q_{i,r}$ for $1 \leq i, r \leq n$. For $i < r$, $P_{i,r}$ is the column vector given by

$$\big(1-t_1, \ldots, 1-t_{i-1}, \underbrace{1-t_it_r}_{i}, t_r(1-t_{i+1}), \ldots, t_r(1-t_{r-1}), \underbrace{t_{r+1}-1}_{r}, t_{r+2}-1, \ldots, t_n-1\,\big)^T,$$

and for $n \geq i > r$, $P_{i,r}$ is the column vector given by

$$\big(t_r(t_1-1), \ldots,\ t_r(t_{r-1}-1), \underbrace{1-t_{r+1}, \ldots, 1-t_i}_{i-r}, 1-t_{i+1}t_r, t_r(1-t_{i+2}), \ldots, t_r(1-t_n)\,\big)^T.$$

Here T is the transpose and $Q_{i,r}$ is the row vector given by

$$Q_{i,r} = \big(0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0\,\big), \quad 1 \leq i, r \leq n.$$

The definition of the reduced Gassner representation restricted to a free normal subgroup is the same, up to equivalence, as the definition in [3]. Representations given by pseudoreflections $I - A_iB_i$ and $I - C_iD_i$ are equivalent if the inner products (B_iA_j) and (D_iC_j) are conjugate by a diagonal matrix. Here, A_i, C_i are column vectors and B_i, D_i are row vectors. For more details, see [4].

We identify $\mathbb{C}^{n-1}$ with $(n-1)\times 1$ column vectors. We let $e_1, \dots, e_{n-1}$ denote the standard basis for $\mathbb{C}^{n-1}$, and we consider matrices to act by left multiplication on column vectors.

Definition 2. If $r = a_1e_1 + \cdots + a_{n-1}e_{n-1} \in \mathbb{C}^{n-1}$, the *support* of r, denoted *supp*(r), is the set $\{e_i \mid a_i \neq 0\}$. If $s = \Sigma a_{ij}(e_i \otimes e_j) \in \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$, the *support* of s, also denoted *supp*(s), is the set $\{e_i \otimes e_j \mid a_{ij} \neq 0\}$, and a_{ij} is called the *coefficient* of $e_i \otimes e_j$ in s.

Definition 3. Given an integer $1 \leq r \leq n$ and a vector $t = (t_1, \dots, t_n)$. We define $v_{i,r}(t) = e_i - \tau_{i,r}(t)(e_i) = (I - \tau_{i,r}(t))(e_i)$. In other words, we have the following:

For $1 \leq i \leq r-1$, we have $v_{i,r}(t) =$

$$\big(1-t_1, \dots, 1-t_{i-1}, \underbrace{1-t_it_r}_{i}, t_r(1-t_{i+1}), \dots, t_r(1-t_{r-1}), \underbrace{t_{r+1}-1}_{r}, t_{r+2}-1, \dots, t_n-1\,\big)^T.$$

and for $n \geq i > r$, we have $v_{i,r}(t) =$

$$\big(t_r(t_1-1), \dots,\ t_r(t_{r-1}-1), \underbrace{1-t_{r+1}, \dots, 1-t_i}_{i-r}, 1-t_{i+1}t_r, t_r(1-t_{i+2}), \dots, t_r(1-t_n)\,\big)^T.$$

Next, we state a lemma similar to that used in [1].

Lemma 1. *For* $t = (t_1, \dots, t_n)$, *we have*

(1) $\tau_{i,r}(t)(v_{i,s}(t)) = v_{i,s} + (t_it_s - 1)v_{i,r}(t)$ *for* $1 \leq i \leq s-1$,

$\tau_{i,r}(t)(v_{i,s}(t)) = v_{i,s} + (t_{i+1}t_s - 1)v_{i,r}(t)$ *for* $1 \leq s < i$,

(2) $\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t) + (t_i - 1)v_{i,r}(t)$ *for* $i < j < s$,

$\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t) + t_s(t_i - 1)v_{i,r}(t)$ *for* $j < i < s$,

$\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t) + (1 - t_{i+1})v_{i,r}(t)$ *for* $j < s < i$,

(3) $\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t) + t_s(1 - t_i)v_{i,r}(t)$ *for* $i < s < j$,

$\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t) + (t_{i+1} - 1)v_{i,r}(t)$ *for* $s < i < j$,

$\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t) + t_s(t_{i+1} - 1)v_{i,r}(t)$ *for* $s < j < i$.

For a fixed value of r, we use Lemma 1 to determine elements in the group algbera $\mathbb{C}(P_n)$ over $\mathbb{C}$ that send the vector $v_{i,r}$ to the vector $v_{i+1,r}$ and other elements that send the vector $v_{i,r}$ to $v_{i-1,r}$.

Definition 4. Given an integer r such that $1 \leq r \leq n$. Consider the following elements of the pure braid group algebra.

$$f_{i,r} = \begin{cases} \tau_{i,r} - (t_i t_r)\tau_{i+1,r}, & 1 \leq i < r-1 \\ \tau_{i,r} - (t_i t_r)\tau_{i+2,r}, & i = r-1 \\ \tau_{i,r} - (t_{i+1} t_r)\tau_{i+1,r}, & 1 \leq r < i \leq n-1 \end{cases}$$

and

$$g_{i,r} = \begin{cases} \tau_{i,r} - (t_i t_r)\tau_{i-1,r}, & 1 \leq i \leq r-1 \\ \tau_{i,r} - (t_{i+1} t_r)\tau_{i-2,r}, & i = r+1 \leq n-1 \\ \tau_{i,r} - (t_{i+1} t_r)\tau_{i-1,r}, & r+1 < i \leq n-1. \end{cases}$$

Then we have the following lemma.

Lemma 2. *Fix an integer $1 \leq r \leq n$. For all integers $1 \leq i \leq n-1$, the action of the elements of the pure braid group algebra, namely, $f_{i,r}$ and $g_{i,r}$, on the vectors $v_{i,r}$ is given by*

$$\text{(i) } f_{i,r}(v_{i,r}) = \begin{cases} -t_i {t_r}^2 (t_{i+1} - 1) v_{i+1,r}, & 1 \leq i < r-1 \\ -t_{r-1} t_r (1 - t_{r+2}) v_{r+1,r}, & i = r-1 \leq n-3 \\ -t_{i+1} {t_r}^2 (t_{i+2} - 1) v_{i+1,r}, & 1 \leq r < i \leq n-2 \end{cases}$$

and

$$\text{(ii) } g_{i,r}(v_{i,r}) = \begin{cases} -t_i t_r (t_{i-1} - 1) v_{i-1,r}, & 1 \leq i \leq r-1 \\ -t_{r+2} {t_r}^2 (1 - t_{r-1}) v_{r-1,r}, & i = r+1 \leq n-1 \\ -t_{i+1} t_r (t_i - 1) v_{i-1,r}, & r+1 < i \leq n-1. \end{cases}$$

Notation 2. *Let $G_n(x_1, \ldots, x_n)$ denote the reduced Gassner representation of P_n under the specialization $t_i \to x_i$, where x_i is a non-zero complex number.*

Lemma 3. *Having U_r a free normal subgroup of the pure braid group, we let $G_n(x_1, \ldots, x_n) : U_r \to GL(\mathbb{C}^{n-1})$ be a specialization of the reduced Gassner representation restricted to U_r making $\mathbb{C}^{n-1}$ into a U_r-module, where $n \geq 3$. Then*

(a) *Let $\mathcal{A}$ be the kernel of the homomorphism $\mathbb{C}[U_r] \to \mathbb{C}$ induced by $\tau_{i,r} \to 1$ (the augmentation ideal). Let x be the vector $(x_1, \ldots, x_n)$. Then $\mathcal{A}\mathbb{C}^{n-1}$ is equal to the $\mathbb{C}$-vector space spanned by $v_{1,r}(x), \ldots, v_{r-1,r}(x), v_{r+1,r}, \ldots, v_{n,r}$.*

(b) *If M is a nonzero U_r-submodule of $\mathbb{C}^{n-1}$, then $\mathcal{A}\mathbb{C}^{n-1} \subseteq M$. Hence $\mathcal{A}\mathbb{C}^{n-1}$ is the unique minimal nonzero U_r-submodule of $\mathbb{C}^{n-1}$.*

(c) *If* $p(x_1,\dots,x_n) = (x_r-1)^{n-2}(x_1x_2\dots x_n-1) \neq 0$*, then* $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$*, and* $G_n(x_1,x_2,\dots,x_n)$ *restricted to* U_r *is irreducible.*

Proof. The proof is similar to that in [2]. Here, we will take the free normal subgroup, U_r, of rank $n-1$. Notice that, in the proof of (b), we need the fact that if $v_{j,r} \in M$ for some j and r then all $v_{i,r} \in M$. This is due to Lemma 1. As for (c), the determinant of the matrix, whose columns are the vectors $v_{1,r}(x),\dots,v_{n,r}(x)$, is $p(x) = (x_r-1)^{n-2}(x_1x_2\dots x_n-1)$, so if $p(x) \neq 0$ then $v_{1,r}(x),\dots v_{n,r}(x)$ is a basis for $\mathbb{C}^{n-1}$ and $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$. For more details, see [4].

Hence, $\mathcal{A}\mathbb{C}^{n-1}$ is its unique minimal nonzero U_r-submodule. Of course $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$ when $G_n(x_1,\dots,x_n)$ is irreducible. A result in [1] states that if for some i and j $(i<j)$ $x_ix_j \neq y_iy_j$ and $x_\alpha x_j y_\alpha y_j \neq 1$ for every $\alpha \in \{1,\dots,j-1\}$ and $x_{\alpha+1}x_jy_{\alpha+1}y_j \neq 1$ for every $\alpha \in \{j,\dots,n-1\}$ then $G_n(x)\otimes G_n(y) : P_n \to GL(\mathbb{C}^{n-1}\otimes\mathbb{C}^{n-1})$ is irreducible (see [1]). In our work, we improve the result by replacing the conditions above by the conditions $x_ix_j \neq y_iy_j$ for some $i \neq j$ and $x_\alpha x_k y_\alpha y_k \neq 1$ for some α and k $(\alpha \neq k)$.

2. Claims in [1] and a summary of their proofs

Before we proceed in proving Proposition 1, we state the main results obtained in our previous work [1]. The notations used here are the same as in Section 1.

Let M be a non zero P_n- submodule of $\mathbb{C}^{n-1}\otimes\mathbb{C}^{n-1}$, when $n \geq 3$. First, we observe that if $e_u \otimes e_v \in supp(m)$ for some $m \in M$ then $e_\alpha \otimes e_v \in supp(\tau_{u,r}(e_u\otimes e_v))$ for every choice of $\alpha = 1,\dots,n-1$ and $v \neq u$. This is clear because of our assumption that none of the parameters t_i's is equal to zero or one.

(1) <u>Claim 1.</u> If $x_ix_j \neq y_iy_j$ for some i and j then there exists an $s \in \{1,\dots,n-1\}$ such that $e_s\otimes e_s \in supp(m)$ for some $m \in M$ and $s \in \{1,\dots,n-1\}$.

Proof of Claim 1. Here, we may take r to be the given integer j.

Case 1: Suppose that there exists an s and $m \in M$ such that $e_s\otimes e_s \in supp(m)$, then we are done.

Case 2: Suppose that there exists (s,t) with $1 \leq s,t \leq n-1$ and $s \neq t$ such that

$$m = a(e_s\otimes e_t) + W,$$

where $a \in \mathbb{C}^*$ and $supp(W)$ does not contain $e_s\otimes e_t, e_t\otimes e_s$. We also assume that $supp(W)$ does not contain $e_\alpha \otimes e_\alpha$ for any α.

Then $\tau_{t,j}(m) = a(e_s\otimes e_t - v_{t,j}) + \tau_{t,j}(W)$, which implies that $e_s\otimes e_s \in supp(\tau_{t,j}(m))$ and so we are done .

Case 3: Suppose that for any pair (s,t) and any $m \in M$ such that $e_s\otimes e_t \in supp(m)$, we have that $e_t\otimes e_s \in supp(m)$ as well. That is, consider $m \in M$ such that

$$m = a(e_s\otimes e_t) + b(e_t\otimes e_s) + W, \quad \text{where}$$

$supp(W)$ does not contain $e_s \otimes e_t$, $e_t \otimes e_s$ and $e_\alpha \otimes e_\alpha$ for any α . In this case, W is either zero or its elements are of the form $\displaystyle\sum_{k,l}(c_{k,l}\ e_k \otimes e_l + d_{l,k}\ e_l \otimes e_k)$.

Here the constants $a,\ b,\ c_{k,l},\ d_{l,k}\ \in \mathbb{C}^*$.

Applying $\tau_{t,j}$, we observe that $e_i \in supp(\tau_{t,j}(e_t))$, where i is the integer given by the hypothesis of Proposition 1. Then

$$\tau_{t,j}(m) = a(e_s \otimes e_i) + b(e_i \otimes e_s) + W, \quad \text{where}$$

$supp(W)$ does not contain $e_s \otimes e_i$, $e_i \otimes e_s$, and both of a, b are not zeros. For simplicity, we denote $\tau_{t,j}(m)$ by m.

If $e_\alpha \otimes e_\alpha \in supp(W)$ for some α , then we are done. If not, we see that

$$aM + bN = \text{coefficient of } e_s \otimes e_s \text{ in } \tau_{i,j}(m) \text{ and}$$
$$aM(1 + y_i y_j) + bN(1 + x_i x_j) = \text{coefficient of } e_s \otimes e_s \text{ in } (\tau_{i,j})^2(m).$$

The values of M and N are not zeros and can be obtained directly from Definition 3. The determinant

$$det \begin{pmatrix} M & N \\ M(1 + y_i y_j) & N(1 + x_i x_j) \end{pmatrix} = MN(x_i x_j - y_i y_j)$$

is nonzero, since $x_i x_j - y_i y_j \neq 0$ by hypothesis. Then one of $\tau_{i,j}(m)$, $(\tau_{i,j})^2(m)$ has $e_s \otimes e_s$ in its support.

(2) Claim 2. Suppose that $e_\alpha \otimes e_\alpha \in supp(m)$ for some $m \in M$. Then $v_{\alpha,k}(x) \otimes v_{\alpha,k}(y) \in M$ if $x_\alpha x_k y_\alpha y_k \neq 1$ for $\alpha = 1, \ldots, k-1$ and $x_{\alpha+1} x_k y_{\alpha+1} y_k \neq 1$ for $\alpha = k, \ldots, n-1$.

Proof of Claim 2. A calculation shows that

$$(\tau_{\alpha,k}-1)(\tau_{\alpha,k}-\gamma_\alpha y_k)(\tau_{\alpha,k}-\beta_\alpha x_k)(e_\alpha \otimes e_\alpha) = \gamma_\alpha \beta_\alpha x_k y_k(\beta_\alpha x_k \gamma_\alpha y_k - 1)(v_{\alpha,k}(x) \otimes v_{\alpha,k}(y))$$

and

$$(\tau_{\alpha,k} - 1)(\tau_{\alpha,k} - \gamma_\alpha y_k)(\tau_{\alpha,k} - \beta_\alpha x_k)(e_u \otimes e_v) = 0 \text{ if } (u, v) \neq (\alpha, \alpha).$$

Here, we have

$$\beta_\alpha = x_\alpha, \quad \gamma_\alpha = y_\alpha \text{ for } \alpha = 1, \ldots, k-1$$

and

$$\beta_\alpha = x_{\alpha+1}, \quad \gamma_\alpha = y_{\alpha+1} \text{ for } \alpha = k, \ldots, n-1.$$

3. Main Theorem

To prove our main theorem, Theorem 1, we introduce Proposition 1 and Lemma 4.

Proposition 1. *Suppose that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$,where $x_s, y_s \in \mathbb{C} - \{0, 1\}$ for $1 \leq s \leq n$. Suppose also that $x_i x_j \neq y_i y_j$ for some $i \neq j$ and $x_\alpha x_k y_\alpha y_k \neq 1$ for some $\alpha \neq k$. Let M be a nonzero P_n-submodule of $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$ under the action of $G_n(x) \otimes G_n(y) : P_n \to GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$, where $n \geq 3$. Then M contains all $v_{p,j}(x) \otimes v_{q,j}(y)$ for $p, q \in \{1, \ldots, j-1, j+1, \ldots, n\}$. Thus M contains $\mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1}$, where the action of P_n on the first factor is induced by $G_n(x_1, \ldots, x_n)$ and the action of P_n on the second factor is induced by $G_n(y_1, \ldots, y_n)$.*

Proof. We will show that once there exists a vector $v_{a,k} \otimes v_{a,k} \in M$ for some a then all other vectors $v_{l,k} \otimes v_{l,k}$ will also be in M using the hypothesis of our proposition and Lemma 2. Here k is the integer given by the proposition and $l \in \{1, \ldots, k-1, k+1, \ldots, n\}$.

By Claim 1 of [1, p.14], we have that $e_s \otimes e_s \in supp(m)$ for some s. This is due to the fact that $x_i x_j \neq y_i y_j$ for some i and j. Having that $e_s \otimes e_s \in supp(m)$ for some $m \in M$, we write $m = \alpha_s e_s \otimes e_s + W$ and $supp(W)$ does not contain $e_s \otimes e_s$. Here $\alpha_s \in \mathbb{C}^*$. It follows that

$$\begin{aligned}\tau_{s,k}(m) &= \alpha_s \tau_{s.k}(e_s \otimes e_s) + \tau_{s,k}(W)\\ &= \alpha_s(e_s - v_{s,k} \otimes e_s - v_{s,k}) + \tau_{s,k}(W)\\ &= \alpha_s(\sum_{l=1}^{n-1} A_l e_l \otimes \sum_{l=1}^{n-1} B_l e_l) + \tau_{s,k}(W).\end{aligned}$$

This implies that $e_l \otimes e_l \in supp(\tau_{s,k}(m))$ for every $l \in \{1, \ldots, n-1\}$. In particular, let $l = \alpha$ such that $x_\alpha x_k y_\alpha y_k \neq 1$. Then we get, by Claim 2 of [1, p.14], that

$$v_{\alpha,k} \otimes v_{\alpha,k} \in M.$$

Applying Lemma 2, we have that $f_{\alpha,k}(v_{\alpha,k} \otimes v_{\alpha,k}) \in M$, which implies that $v_{\alpha+1,k} \otimes v_{\alpha+1,k} \in M$. Similarly, we also have that $g_{\alpha,k}(v_{\alpha,k} \otimes v_{\alpha,k}) \in M$, which implies that $v_{\alpha-1,k} \otimes v_{\alpha-1,k} \in M$. After a consecutive use of $f_{\alpha,k}$, $f_{\alpha+1,k}$, $\ldots$ and $g_{\alpha,k}$, $g_{\alpha-1,k}, \ldots$,, we obtain that

$$v_{l,k} \otimes v_{l,k} \in M \text{ for every } l \in \{1, \ldots, k-1, k+1, \ldots, n\}. \tag{1}$$

As in Lemma 2, for that integer k and any integer l, we find elements in the group algebra $\mathbb{C}(P_n)$ over $\mathbb{C}$ that send the vectors $v_{l,k}$ to the vectors $v_{l,k+1}$ when $1 \leq k < n$, and other elements in $\mathbb{C}(P_n)$ that send the vectors $v_{l,k}$ to the vectors $v_{l,k-1}$. For example, we can consider the element

$$\gamma_{l,k} = \tau_{l-1,k} - (t_{l-1} t_k)\tau_{l,k+1} \quad \text{when} \quad l \leq k.$$

Here, k is the integer given in proposition 1 and l is any integer in $\{1, \ldots, k-1, k+1, n\}$. We have $\gamma_{l,k}(v_{l-1,k}) = -t_{l-1} {t_k}^2 (t_l - 1) v_{l,k+1}$.
Since $v_{l-1,k} \otimes v_{l-1,k} \in M$ for every value of l, it follows that $v_{l,k+1} \otimes v_{l,k+1} \in M$ $(1 \leq k < n)$. Now apply $g_{l,k+1}$ from Lemma 2, we obtain that $v_{l-1,k+1} \otimes v_{l-1,k+1} \in M$ for every value of l.

Similarly, we find other elements in $\mathbb{C}(P_n)$ to show that given any integer $l \in \{1, \ldots, k-1, k+1, n\}$ such that $v_{l-1,k} \otimes v_{l-1,k} \in M$, we obtain that $v_{l-1,k-1} \otimes v_{l-1,k-1} \in M$. Therefore, by direct computations, and for all integers p and t, we get

$$v_{p,t} \otimes v_{p,t} \in M. \tag{2}$$

In particular, let $t = j$. Then we have that $v_{p,j} \otimes v_{p,j} \in M$. Given integers $p, q \in \{1, \ldots, j-1, j+1, \ldots, n\}$ and $p \neq q$. We will show that $v_{p,j} \otimes v_{q,j} \in M$. We consider two cases.

Case 1: Let $p = i$. (i and j are the integers given by the hypothesis of Proposition 1). By our assumption, we have that $q \neq i$. By applying $\tau_{q,j}$ on $(v_{i,j} \otimes v_{i,j})$, we obtain

$$av_{q,j} \otimes v_{i,j} + bv_{i,j} \otimes v_{q,j} \in M \quad (a \neq 0,\ b \neq 0). \tag{3}$$

Applying $\tau_{i,j}$, we obtain

$$ay_iy_jv_{q,j} \otimes v_{i,j} + bx_ix_jv_{i,j} \otimes v_{q,j} \in M. \tag{4}$$

Combining (3) and (4), we get

$$b(y_iy_j - x_ix_j)v_{i,j} \otimes v_{q,j} \in M$$

Since $x_ix_j \neq y_iy_j$, it follows that

$$v_{i,j} \otimes v_{q,j} \in M.$$

Case 2: Let $p \neq i$, $q \neq i$ and $p \neq q$. By applying $\tau_{i,j}$ on $(v_{p,j} \otimes v_{p,j})$, we obtain

$$av_{i,j} \otimes v_{p,j} + bv_{p,j} \otimes v_{i,j} \in M \quad (a \neq 0,\ b \neq 0). \tag{5}$$

Applying $\tau_{i,j}$ again, we obtain

$$ax_ix_jv_{i,j} \otimes v_{p,j} + by_iy_jv_{p,j} \otimes v_{i,j} \in M. \tag{6}$$

Combining (5) and (6), we get that

$$v_{p,j} \otimes v_{i,j} \in M.$$

Now apply $\tau_{q,j}$, we get that for $p, q \in \{1, \ldots, j-1, j+1, \ldots, n\}$

$$v_{p,j} \otimes v_{q,j} \in M.$$

In order to simplify the conditions in Proposition 1 further, we prove the next Lemma.

Lemma 4. *Given* $n \geq 3$ *and non-zero complex numbers* $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_n$. *Then the following holds.*

(i) *If* $x_v \neq \pm y_v$ *for some* v *then* $x_ax_b \neq y_ay_b$ *for some distinct* a, b.

(ii) *If* $x_u \neq \pm {y_u}^{-1}$ *for some* u *then* $x_gx_hy_gy_h \neq 1$ *for some distinct* g, h.

Proof. (i) Assume, to get contradiction, that $x_a x_b = y_a y_b$ for $a \neq b$. Then, we have that

$$x_1 x_n = y_1 y_n,$$
$$\vdots$$
$$x_{n-1} x_n = y_{n-1} y_n.$$

By solving the above equations, we get that $x_u y_v = x_v y_u$, where $1 \leq u < v \leq n-1$. Since we also have that $x_u x_v = y_u y_v$, it follows that $x_v = \pm y_v$ for every $v \in \{1, \ldots, n\}$, a contradiction.

(ii) Assume, to get contradiction, that $x_g x_h y_g y_h = 1$ for $g \neq h$. Then, we have that

$$x_1 x_n y_1 y_n = 1,$$
$$\vdots$$
$$x_{n-1} x_n y_{n-1} y_n = 1.$$

This implies that $x_u y_u = x_{u+1} y_{u+1}$, where $u \in \{1, \ldots, n-2\}$. Having that $x_u x_{u+1} y_u y_{u+1} = 1$, it follows that $x_u = \pm {y_u}^{-1}$ for every $u \in \{1, \ldots, n\}$, a contradiction.

Consider the representation $G_n(t_1, \ldots, t_n) : P_n \to GL_{n-1}(\mathbb{C}[{t_1}^{\pm 1}, \ldots, {t_n}^{\pm 1}])$, where $t_1, \ldots, t_n$ are indeterminates. Specializing $t_1, \ldots, t_n$ to nonzero complex numbers $x_1, \ldots, x_n$ defines a representation $G_n(x_1, \ldots, x_n) : P_n \to GL_{n-1}(\mathbb{C}) = GL(\mathbb{C}^{n-1})$ which is irreducible if and only if $q(x_1, \ldots, x_n) = x_1 \ldots x_n - 1 \neq 0$. (see [1] and [4]).

By combining Proposition 1 and Lemma 4, we get our main theorem.

Theorem 1. *For $n \geq 3$, consider the tensor product of irreducible representations $G_n(x_1, \ldots, x_n) \otimes G_n(y_1, \ldots, y_n) : P_n \to GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$, where $q(x_1, \ldots, x_n) \neq 0$ and $q(y_1, \ldots, y_n) \neq 0$. If for some i and j, $x_i \neq \pm y_i$ and $x_j \neq \pm {y_j}^{-1}$ then the above representation is irreducible.*

Proof. By Lemma 4 and Proposition 1, we have that $\mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1}$ is the unique minimal nonzero P_n-submodule of $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$. In particular, it is an irreducible P_n-module. The fact that $q(x_1, \ldots, x_n) \neq 0$ and $q(y_1, \ldots, y_n) \neq 0$ implies that the left factor $\mathcal{A}\mathbb{C}^{n-1}$ corresponds to the representation $G_n(x_1, \ldots, x_n)$ and the right factor $\mathcal{A}\mathbb{C}^{n-1}$ corresponds to the representation $G_n(y_1, \ldots, y_n)$.

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