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WEAKLY UNIQUE FACTORIZATION MODULES

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Abstract. In this work we give the definition of weakly prime element of a module. Therefore we give a new definition of factorization in a module, which is called weakly factorization. So we call a module weakly unique factorization module if all elements have a weakly factorization which is unique. We give the relation between weakly prime elements and weakly prime submodules. Then we characterize such weakly unique factorization modules.

The study of factorization in torsion-free modules was begun in Nicolas [5]. She defined the module M to be factorial if (1) every nonzero element of M has a irreducible factorization, (2) every irreducible element of R is prime, and (3) every irreducible element of M is primitive. She showed that if M is factorial then R is a UFD. After this she showed that M is a unique factorization module (UFM) if and only if (1) every element of M has an irreducible factorization, and (2) if $x = a_1a_2\cdots a_km = b_1b_2\cdots b_tm'$ are two factorizations of $x \in M$ then $k = t$, $a_i \sim b_i$ for all $i \in \{1, 2, \ldots, k\}$ and $m \sim m'$. Later, Lu [3] gives some characterizations of UFM and relations between prime submodules and primitive elements such modules. Further she investigates polynomial modules. There is an another work about factorizations of modules, by Anderson and Valdes-Leon [1]. They generalize factorization of any modules over a ring with zero divisor, which have nonzero torsion elements. They showed that their definition and definition of Nicolas are concides if M is torsion-free module and R is an integral domain.

We give a new definition of factorization for modules, named weakly factorization, and give relations between weakly prime elements and weakly prime submodules. After this we investigate the direct sum of modules, the direct product of modules, fractions of modules and polynomial of modules.

Throughout this paper all rings, R are commutative ring with identity 1 and all modules, M are nonzero torsion-free R-module which are unitary. An R-module M is called a torsion-free module if $rm = 0$ for some nonzero $r \in R$ and $m \in M$ then $m = 0$. Let R be a ring, M a nonzero torsion free R-module and $U(R)$ the set of all unit elements of R. Let m and m' be two elements of M. We say that m divides m' in M, written $m \mid m'$, if there exists a nonzero element $r \in R$ such that $m' = rm$. If $m \mid m'$, then m is called a factor or divisor of m' in M. Similarly, an element $d \in R$ is said to divide m

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in M, written $d \mid m$, if there exists an element $m_0 \in M$ such that $m = dm_0$. If $m \mid m'$ and $m' \mid m$ then we shall say that these elements are associates in M and write $m \sim m'$. Clearly, $m \sim m'$ if and only if $m = um'$ and $vm = m'$ for some $u, v \in U(R)$. If $m \mid m'$ and m is not an associate of m', then we say that m is a proper factor of m' in M.

We will give now some definitions from [3] and [4].

Definition 1. Let M be a torsion-free R-module and m be a nonzero element of M.

(i) m is irreducible in M if $m = am'$ implies that $a \in U(R)$ for every $a \in R$ and $m' \in M$.

(ii) m is primitive in M if $m \mid am'$ implies $m \mid m'$ for all $0 \neq a \in R$ and $m' \in M$.

(iii) An irreducible element p of R is called prime to the module M if $p \mid am$ implies $p \mid a$ in R or $p \mid m$ in M.

Definition 2. Let M be an R-module. Then a submodule N of M is called pure submodule if for all $a \in R$ we have $aM \cap N = aN$.

The following Propositions are given by [3] without their proofs, we will give now their proofs.

Proposition 1. *Let M be an R-module then every w-primitive element of M is an irreducible element.*

Proof. Suppose that m is a primitive element of M and let $m = am'$ for some $a \in R$, $m' \in M$. Then $m \mid am'$ and since m is primitive we get $m \mid m'$. Since $m = am'$ implies $m' \mid m$. Therefore $m \sim m'$, hence m is a irreducible element. □

Proposition 2.[3, Proposition 1.1] *Let M be an R-module and $0 \neq m \in M$, then the following statements are equivalent:*

(i) *m is a primitive element of M,*

(ii) *the cyclic submodule Rm is a pure submodule of M and*

(iii) *for every $x \in M$ we have either $Rx \cap Rm = 0$ or $Rx \subseteq Rm$.*

Proof. $(i) \Rightarrow (ii)$ Let m be a primitive element of M and $a \in R$. Now suppose that $x \in aM \cap Rm$, then there exist elements $m' \in M$ and $r \in R$ such that $x = am' = rm$. Thus $m \mid am'$ and since m is primitive, we have $m \mid m'$. Therefore there exist $r' \in R$ such that $m' = r'm$. So $x = am' = ar'm \in aRm$. For the converse, let $y \in aRm$ then there exists an $r \in R$ such that $y = arm$. So it is clear that $y \in aM$ and $y \in Rm$.
$(ii) \Rightarrow (iii)$ Suppose that Rm is pure submodule of M and let $x \in M$ such that $Rx \cap Rm \neq 0$. Then there exists $y \in Rx \cap Rm$ such that $y = r'x$ for some $r' \in R$. Since Rm is pure submodule of M we get $r'M \cap Rm = r'Rm$. So $y = r'x \in r'M \cap Rm = r'Rm$. Thus there exist $r^* \in R$ such that $y = r'x = r'r^*m$. Hence $x = r^*m$ and this gives $Rx \subseteq Rm$.
$(iii) \Rightarrow (i)$ Suppose that for all $x \in M$ either $Rx \cap Rm = 0$ or $Rx \subseteq Rm$. Now assume that $m \mid am'$ for some $a \in R$, then there exists an $r \in R$ such that $am' = rm$. Thus $Rm' \cap Rm \neq 0$ and so by our assumption we get $Rm' \subseteq Rm$. Then there exists a $s \in R$ such that $m' = sm$ and this gives $m \mid m'$. So m is primitive.

Corollary 1.([3, Corollary]) *Let M be an R-module then two primitive elements m and m' of M are non-associates if and only if $Rm' \cap Rm = 0$.*

Definition 3. A nonzero element m of M is called weakly prime (w-prime) if for $a, b \in R$ and $m' \in M$, $m \mid abm'$ implies $m \mid am'$ or $m \mid bm'$.

Note that if m is a w-prime element of M then rm is w-prime too for all $r \in U(R)$.

Proposition 3. *Let M be an R-module, then every primitive element of M is w-prime.*

Proof. Assume that $m \mid abm'$ for some $a, b \in R$ and $m' \in M$. Then since m is primitive, we get $m \mid m'$. Hence $m \mid am'$ and $m \mid bm'$.

Example 1. *Let R be a commutative ring with an identity and $M = R[x]$, the polynomial ring over R is an R-module, then $x \in M$ is w-prime (primitive, irreducible) element.*

Example 2. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}[x]$. Then the element $2x$ is a w-prime element but is neither primitive nor irreducible.*

Theorem 1. *Let M be a cyclic module Rx over a UFD R and $m \in M$ such that $m \nsim x$. Then the following statements are equivalent;*

(i) *$m \in M$ is a w-prime element,*

(ii) *$m \sim px$ for some irreducible element p of R,*

(iii) *$Rm = pM$ for some irreducible element p of R.*

Proof. $(i) \Rightarrow (ii)$ Let m be a w-prime element of M. Here $m \in M = Rx$ and there exists an element $r \in R$ such that $m = rx$. Since R is UFD r has a factorization such that $r = p_1 \cdots p_k$ where $p_i \in R$ are all irreducible. Since m is a w-prime element of M, $m \mid p_i x$ for some $i \in \{1, 2, \ldots, k\}$. Hence $m \sim p_i x$.
$(ii) \Rightarrow (iii)$ Let $m \sim px$ for some irreducible element p of R. Then there exists a unit element $u \in R$ such that $m = upx = pux \in pRx = pM$. If $pn \in pM$ where $n \in M$ then for some $s \in R$, $pn = psx = spx = su^{-1}m \in Rm$. Thus $Rm = pM$.
$(iii) \Rightarrow (i)$ Let $Rm = pM$ for some irreducible element p of R. Assume that $m \mid abn$ for some $n \in M$ and $a, b \in R$. Then there exists an $r \in R$ such that $rm = abn$. Since $Rm = pM$ and $M = Rx$ we get for some $m' \in M$ and $r', r'' \in R$, $abn = abr'x$ and $rm = pm' = pr''x$. Thus $abr' = pr''$ and $p \mid abr'$. Therefore we obtain $p \mid a$ or $p \mid b$ or $p \mid r''$. If $p \mid a$ we get $m \mid an$ and if $p \mid b$ we get $m \mid bn$. Therefore we have $m \mid n$.

Definition 4. A submodule N of an R-module M is called weakly prime if $abk \in N$ implies $ak \in N$ or $bk \in N$ for all $k \in M$ and $a, b \in R$.

Theorem 2. *A submodule N of an R-module M is called weakly prime if and only if $abK \subseteq N$ implies $aK \subseteq N$ or $bK \subseteq N$ for some submodule K of M and $a, b \in R$.*

Proof. Suppose that N is a weakly prime submodule of M. Let $abK \subseteq N$ for some submodule K of M and $a, b \in R$. Since for all $k \in K$ we have $abk \in N$ we get $ak \in N$ or $bk \in N$. Hence $aK \subseteq N$ or $bK \subseteq N$. Indeed, if there are not $k_1, k_2 \in K$ such that $ak_1 \in N$ but $bk_1 \notin N$ and $bk_2 \in N$ but $ak_2 \notin N$, then we let the element $k_1 + k_2 \in K$ we obtain $ab(k_1 + k_2) \in N$. Thus $a(k_1 + k_2) \in N$ or $b(k_1 + k_2) \in N$. If $a(k_1 + k_2) \in N$ then $ak_2 \in N$ which is a contradiction. If $b(k_1 + k_2) \in N$ then $bk_1 \in N$ which is a contradiction. For the converse let $abk \in N$ for some $k \in M$. Then $ab(k) \subseteq N$. By our assumption $a(k) \subseteq N$ or $b(k) \subseteq N$. Thus $ak \in N$ or $bk \in N$.

Theorem 3. *Let M be an R-module. Then $m \in M$ is w-prime element if and only if Rm is w-prime submodule of M.*

Proof. Let m be a w-prime element. Assume that for any $k \in M$ and $a, b \in R$ we have $abk \in Rm$, thus $m \mid abk$. Since m is w-prime in M, $m \mid ak$ or $m \mid bk$. This implies $ak \in Rm$ or $bk \in Rm$. Now for the converse suppose that Rm is a w-prime submodule of M. Assume that $m \mid abm'$ for some $a, b \in R$ and $m' \in M$. Thus $abm' \in Rm$ and since Rm is w-prime submodule, we get $am' \in Rm$ or $bm' \in Rm$. Hence $m \mid am'$ or $m \mid bm'$.

Here first we shall note that if we let R as a module over itself then an element m is w-prime of the module R if and only if m is a prime element of the ring R. Secondly every primitive element of M is w-prime.

Definition 5. A torsion-free module M over a commutative ring with identity R is called a weakly unique factorization module (w-UFM) or w-factorial module if the following two conditions are satisfied:
(w-ufm 1) Each nonzero element $x \in M$ has a w-factorization, $x = a_1 a_2 \cdots a_k m$, where a_i's are ireducible elements in R (possibly with $k = 0$) and m is a w-prime element in M.
(w-ufm 2) if $x = a_1 a_2 \cdots a_k m = b_1 b_2 \cdots b_t m'$ are two factorizations of x, then $k = t$, $a_i \sim b_i$ and $m \sim m'$ for all $i \in \{1, 2, \ldots, k\}$.

Definition 6. Let M be an R-module and $a \in R$, $m \in M$

(1) An element $d \in R$ is called greatest common divisor (gcd) of a and m if the following two conditions hold
   (i) $d \mid a$ in R and $d \mid m$ in M, and
   (ii) if there is an element $c \in R$ such that $c \mid a$ in R and $c \mid m$ in M then c is a divisor of d.
(2) An element $m' \in M$ is called least common multiple (lcm) of ab and m if the following two conditions hold
   (i) $a \mid m'$ and $m \mid m'$ in M respectively, and
   (ii) if there is an element $n \in M$ such that $a \mid n$ and $m \mid n$ in M then m' is a factor of n.

Proposition 4.([3, Proposition 1.2]) *Let M be an R-module then*

(i) $m^* \sim \operatorname{lcm}\{a, m\}$ if and only if $aM \cap Rm = Rm^*$.

(ii) Let p be an irreducible element of R such that $\operatorname{lcm}\{p, am\}$ exists in M for any $a \in R$. If $p \nmid am$ then $pM \cap Ram = Rpam$.

Proposition 5.([3, Proposition 1.3]) *Let M be an R-module and R a GCD domain then*

(1) $\gcd\{(a, c), m\} = \gcd\{a, \gcd\{c, m\}\}$

(2) $\gcd\{ca, cm\} = c\gcd\{a, m\}$ and

(3) *if* $a \mid cm$ *and* $\gcd\{a, m\} = 1$ *then* $a \mid c$.

Theorem 4. *Let M be a module over a UFD R which satisfies (w-ufm 1). Then M is a w-UFM if and only if every weakly prime element of M is primitive.*

Proof. Let M be a w-UFM and $m \in M$ be a weakly prime element. Assume that $m \mid abm'$ for some $m' \in M$ and $a, b \in R$. Then there exist an $r \in R$ such that $rm = abm'$. Since M is a w-UFM and R is a UFD there exist irreducible elements in R, $r_1, \ldots, r_k, a_1, \ldots, a_t, b_1, \ldots, b_l, c_1, \ldots, c_n$, and weakly prime element m^* in M such that $r = r_1 \cdots r_k$, $a = a_1 \cdots a_t$, $b = b_1 \cdots b_l$ and $m' = c_1 \cdots c_n m^*$. Thus we get $r_1 \cdots r_k m = a_1 \cdots a_t b_1 \cdots b_l c_1 \cdots c_n m^*$. Since M is a w-UFM we have by the uniqueness, $k = t+l+n$ and for a suitable choice $r_i \sim a_i$, $r_j \sim b_j$, $r_s \sim c_s$ and $m \sim m^*$. Therefore there exist an $r^* \in U(R)$ such that $m^* = r^* m$. Hence $m' = c_1 \cdots c_n m^* = c_1 \cdots c_n r^* m$. Thus $m \mid m'$ and so $m \mid am'$, $m \mid bm'$. Now suppose that every weakly prime element of M is primitive and $x = a_1 \cdots a_t m = b_1 \cdots b_l m'$ are two decompositions of x. Then we have $m \mid b_1 \cdots b_l m'$, $m' \mid a_1 \cdots a_t m$. Since m and m' are w-prime elements this gives us $m \mid m'$ and $m' \mid m$. Hence $m \sim m'$, so there exist an elemnet $u \in U(R)$ such that $m = um'$. Therefore, $a_1 \cdots a_t um' = b_1 \cdots b_l m'$ implies $ua_1 \cdots a_t = b_1 \cdots b_l$. The result follows since R is a UFD. $\square$

Theorem 5. *Let M be a torsion-free R-module. Then M is a UFM if and only if M is w-UFM.*

Proof. The follows from Theorem 4 and [3, Theorem 2.1]. $\square$

With this theorem we get that in a w-UFD M, every weakly prime element of M is irreducible element of M. And this gives us that weakly factorial modules and factorial modules concides. From this note we obtain the following corollaries.

Corollary 2. *Every vector space is w-UFM.*

Theorem 6. *Let M be a module over a UFD R which satisfies (w-ufm 1). Then the following statement are equivalent:*

(i) *Every weakly prime element of M is weakly primitive*

(ii) *For any pair of elements $a, b \in R$ and $m \in M$ the* $\gcd\{ab, m\}$ *exists in R.*

(iii) *For any pair of elements $a, b \in R$ and $m \in M$ the* $\text{lcm}\{ab, m\}$ *exists in M, i.e. the submodule $aM \cap Rm$ is cyclic.*

(iv) *Every irreducible element p of R is prime to M.*

Note that if we let S be a multiplicative closed subset of a UFD R such that $0 \notin S$ then by Theorem 5 and [3, Proposition 5.1] we obtain that if M is a w-factorization R-module then the module of fractions M_s is a w-factorization R_s-module. For the converse, let S be a the multiplicative closed subset of R generated by any family P' of elements which are w-prime to M. If M_s is a w-factorization R_s-module, then M is a w-factorization R-module in the sense of Theorem 5 and [3, Proposition 5.2].

Similarly, by Theorem 5 and [3, Proposition 6.1] and [3, Theorem 6.1] we get w-factorization of $M[x]$, polynomial module over M, is a w-factorization module over the polynomial ring over R, $R[x]$.

Theorem 7. *Let M be a w-factorization module over a UFD R such that $pM \neq M$ for every non-unit element $p \in R$. Then the following statements are equivalent:*

(i) *p is prime to M,*

(ii) *pM is a weakly prime submodule of M with $(pM : M) = (p)$.*

Proof. $(i) \Rightarrow (ii)$ Let $p \in R$ be a w-prime element to M and let $abk \in pM$ for some $a, b \in R$ and $k \in M$. Since p is weakly prime to M and R is UFD $p \mid abk$ implies that $p \mid a$ or $p \mid b$ or $p \mid k$. Assume that $bk \notin pM$ then since p is prime to M and R is UFD, $p \mid a$. Thus $p \mid ak$. So pM is a weakly prime submodule of M. For the second part $(pM : M) \supseteq (p)$ is clear, now let $r \in (pM : M)$. Then for $m \in M \backslash pM$ we have $rm \in pM$. Thus $p \mid rm$. Since p is prime to M we get $p \mid r$. Hence $r \in (p)$.
$(ii) \Rightarrow (i)$ Suppose that pM is a weakly prime submodule of M with $(pM : M) = (p)$. First we show that p is an irreducible element of R. Let $p = ab$ for some $a, b \in R$, then since pM is w-prime submodule $abM \subseteq pM$ implies that $aM \subseteq pM$ or $bM \subseteq pM$. So $a \in (pM : M) = (p)$ or $b \in (pM : M) = (p)$. Thus b is a unit or a is a unit. Now secondly let $p \mid abm$ but $p \nmid ab$ for some $a, b \in R$ and $m \in M$. Thus $p \mid am$ or $p \mid bm$. If $p \mid am$ then there exists an element $m^* \in M$ such that $pm^* = am$. Since M is w-factorization and $p \nmid a$ we get $p \mid m$.

Theorem 8. *Let $\{M_i \mid i \in I\}$ be a set of modules over a UFD R. Then the following statements are equivalent:*

(i) $\prod_{i \in I} M_i$ *is a w-UFM over R,*

(ii) $\bigoplus_{i \in I} M_i$ *is a w-UFM over R,*

(iii) *each M_i is a w-UFM over R.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ it's clear.
$(iii) \Rightarrow (i)$ Now assume that each M_i is a w-UFM over R for $i \in I$. Let $M = \prod_{i\in I} M_i$ and $m = (m_i)_{i\in I} \in M$ where $m_i = a_i m_i'$ for some $a_i \in R$ and a w-prime element m_i' of M_i. First we will show that $m = (m_i)_{i\in I} \in M$ is a w-prime element in M iff $\{a_i\}_{i\in I}$ has no g.c.d. in R. Let $m \in M$ be a w-prime element. Assume that $d = \text{g.c.d.}\{a_i\}$ and set $a_i = db_i$ for $b_i \in R$. Then $m = dm'$ where $m' = (b_i m_i')_{i\in I}$. Then $m \mid m = dm'$ but $m \nmid m'$ gives us a contradiction. For the converse assume that $\{a_i\}_{i\in I}$ has no g.c.d. in R. Let $m \mid cbn$ for some $c, b \in R$ and $n = (n_i)_{i\in I} \in M$ then there exist $r \in R$ such that $rm = cbn$. So this gives us that for all $i \in I$, $rm_i = cbn_i$. Thus for all $i \in I$, $ra_i m_i' = cbn_i$. Since m_i' is w-prime and $\{a_i\}_{i\in I}$ has no g.c.d. then for all $i \in I$ we get $a_i m_i' \mid cn_i$ or $a_i m_i' \mid bn_i$. Hence $m \mid cn$ or $m \mid bn$, so m is w-prime. Now we will show that M is a w-UFM. Let $m = (m_i)_{i\in I} \in M$, since each M_i is a w-UFM over R we have a w-factorization for $m_i \in M_i$ and $i \in I$ such that $m_i = a_i m_i'$ where m_i' is w-prime in M_i. If we let $d = \text{g.c.d.}\{a_i\}$ then for $a_i = db_i$ we obtain the equition $m = dm'$ where $m' = (b_i m_i')$. Now by Theorem 7 m' is w-prime in M and since R is UFD M satisfies w-UFM1. Now, let $p \in R$ be a w-irreducible element such that $p \mid abm$ in M for some $a, b \in R$ and $m = (m_i)_{i\in I} \in M$. Then for all $i \in I$, $p \mid abm_i$ in M_i. Since M_i is w-UFM if $p \nmid ab$ then $p \mid m_i$ for all $i \in I$. Consequently $p \mid m$ and therefore M is w-UFM.

Corollary 3. *Every free module over a UFD is w-UFM.*

Corollary 4.([3, Theorem 5.1]) *Let S be a multiplication closed set of aUFD R such that $0 \notin S$. If M is a w-factorization R-module, then the module of fractions M_S is a w-factorization R_S-module.*

Corollary 5.([3, Theorem 5.2]) *Let M be a module over a UFD R which satisfies (w-ufm 1) and S be the multiplicatively closed set of R generated by any family P' of elements which are w-prime to M. If the R_S-module M_S is w-factorization so is the R-module M.*

Corollary 6. ([3, Proposition 6.1]) *Let R be an integral domain satisfying the a.c.c. for principal ideals and M an R-module. If M satisfies the a.c.c. for cyclic submodules, so do the $R[x]$-module $M[x]$.*

Corollary 7. ([3, Theorem 6.1]) *If M is a w-factorization module over a UFD R, then the $R[x]$-module $M[x]$ is also w-factorization.*

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