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$S_\imath$-OPEN SETS AND $S_\imath$-CONTINUITY IN BITOPOLOGICAL SPACES

ALIAS B. KHALAF AND ALAN M. OMER

Abstract. In this paper, we introduce and define a new class of sets, called $S_\imath$-open sets, in bitopological spaces. By using this set, we introduce and define the notion of $S_\imath$-continuity and investigate some of its properties. In particular, $S_\imath$-open sets and $S_\imath$-continuity are used to extend some known results of continuity.

1. Introduction and Preliminaries

In the literature notions of semi open, pre open, α-open, β-open and regular open sets in topological space have been introduced and studied respectively by Levine [2], Mashhour [3], Njastad [4], Monsef [1], Stone [5], and in 1968 Velicko [7], defined the concepts of δ-open and θ-open sets.

The study of bitopological spaces was first initiated by J.C.Kelly [6], and thereafter a large number of paper have been done to generalize the topological concepts to bitopological setting.

Throughout this paper, by a space X we mean a bitopological space (X, τ_1, τ_2). By $\imath Int(A)$ and $\imath Cl(A)$ we shall mean the interior and the closure of a subset A of X with respect to $\tau_\imath$, respectively, where $\imath, \jmath = 1$ or 2 and $\imath \neq \jmath$.

A subset A of X is said to be $\imath\jmath$-semi open [13] (resp., $\imath\jmath$-pre open [11], $\imath\jmath-\alpha$-open [12], $\imath\jmath$-semi-preopen [16], $\imath\jmath$-regular open [10]) if $A \subseteq \jmath Cl(\imath Int(A))$ (resp., $A \subseteq \imath Int(\jmath Cl(A))$, $A \subseteq \imath Int(\jmath Cl(\imath Int(A)))$, $A \subseteq \jmath Cl(\imath Int(\jmath Cl(A)))$, $A = \imath Int(\jmath Cl(A))$).

A point x of X is said to be $\imath\jmath-\delta$-cluster point [15] of A if $A \cap U \neq \phi$ for every $\imath\jmath$-reguler open set U containing x, the set of all $\imath\jmath-\delta$-cluster points of A is called $\imath\jmath-\delta$-closure of A, a subset A of X is said to be $\imath\jmath-\delta$-closed if the set of $\imath\jmath-\delta$-cluster points of A is a subset of A, the complement of $\imath\jmath-\delta$-closed set is $\imath\jmath-\delta$-open. A point $x \in X$ is said to be in the $\imath\jmath-\theta$-closure [14] of A, denoted by $\imath\jmath$-$Cl_\theta(A)$, if $A \cap \jmath Cl(U) \neq \phi$ for every $\imath$-open set U containing x. A subset

Corresponding author: Alias B. Khalaf.
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A of X is said to be $\imath\jmath-\theta$-closed if $A = \imath\jmath\text{-}Cl_\theta(A)$. A subset A of X is said to be $\imath\jmath-\theta$-open if $X\setminus A$ is $\imath\jmath-\theta$-closed.

The complement of $\imath\jmath$-semi open (resp., $\imath\jmath$-pre open, $\imath\jmath-\alpha$-open, $\imath\jmath$-semi-preopen, $\imath\jmath$-regular open) set is said to be $\imath\jmath$-semi closed (resp., $\imath\jmath$-pre closed, $\imath\jmath-\alpha$-closed, $\imath\jmath$-semi-preclosed, $\imath\jmath$-regular closed).

In 1991, Kheder [15] defined a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ to be $\imath\jmath$-super continuous if $f^{-1}(V)$ is $\imath\jmath-\delta$-open set in X for every $\imath$-open set V of Y, and a function f is said to be $\imath\jmath-\delta$-continuous [8] (resp., $\imath\jmath$-almost continuous [9]) if $f^{-1}(V)$ is $\imath\jmath-\delta$-open (resp., $\imath$-open)set in X for every $\imath\jmath$-regular open set V of Y.

In the present paper we introduce a new class of $\imath$-open sets called $S_\imath$-open, this class of sets lies strictly between the classes of $\imath\jmath-\delta$-open and $\imath$-open sets. We also study its fundamental properties and compare it with some other types of sets, and then we define and further topological properties such as, $S_\imath$-neighborhood, $S_\imath$-interior, $S_\imath$-closure, $S_\imath$-derived and $S_\imath$-boundary of sets. Also in this paper we introduce and investigate the concept of $S_\imath$-continuity functions and $\imath\jmath$-almost $S_\imath$-continuity. It will be shown that $S_\imath$-continuity is weaker than $\imath\jmath$-super continuity while it is stronger than both $\imath$-continuity and $\imath\jmath$-almost $S_\imath$-continuity, and $\imath\jmath$-almost $S_\imath$-continuity is weaker than $\imath\jmath-\delta$-continuity so it is stronger than $\imath\jmath$-almost continuity.

2. $S_\imath$-Open Sets

Definition 2.1. An $\imath$-open subset A of a space X is called $S_\imath$-open if for each $x\in A$, there exists an $\jmath\imath$-semi closed set F such that $x\in F\subseteq A$.The family of all $S_\imath$-open subsets of bitopological space (X, τ_1, τ_2) is denoted by $S_\imath O(X, \tau_1, \tau_2)$ or $S_\imath O(X)$.

Proposition 2.2. *A subset A of a space X is $S_\imath$-open if and only if A is $\imath$-open and it is a union of $\jmath\imath$-semi closed sets. That is, $A = \bigcup F_\alpha$ where A is $\imath$-open set and F_α is $\jmath\imath$-semi closed sets for each α.*

Proof. Obvious. □

It is clear from the definition that every $S_\imath$-open subset of a space X is $\imath$-open, but the converse is not true in general as shown by the following example.

Example 2.3. Let X=$\{a,b,c\}$, $\tau_1=\{X, \phi, \{a\}, \{a,b\}, \{a,c\}\}$ and $\tau_2=\{X, \phi, \{b\}, \{b,c\}\}$, then $\{a,b\}$ is $\imath$-open but not $S_\imath$-open.

Example 2.4. If X is an infinite set. In a bitopological space (X, τ_1, τ_2) with two cofinite topologies $\tau_1=\tau_2$ every infinite set is $S_\imath$-open, but it is not $\jmath\imath$-semi closed.

Remark 2.5. S_1-open sets means that a subset A is τ_1-open and for all $x \in A$, there exists a 21-semi closed set F such that $x \in F \subseteq A$ and S_2-open sets means that a subset A is τ_2-open and for all $x \in A$, there exists a 12-semi closed set F such that $x \in F \subseteq A$.

It can be easily seen that S_1-open sets and S_2-open sets are incomparable in general as shown by the following example.

Example 2.6. Consider X=$\{a,b,c\}$, $\tau_1 = \{X, \phi, \{c\}, \{a,b\}\}$ and $\tau_2 = \{X, \phi, \{b\}, \{b,c\}\}$, then $\{c\}$ is S_1-open but not S_2-open set and $\{b\}$ is S_2-open but not S_1-open set.

The next example shows that even if $\tau_1 \subseteq \tau_2$, then $S_1O(X) \subseteq S_2O(X)$ need not be true in general.

Example 2.7. If X=$\{a,b,c\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b,c\}\}$, then $\tau_1 \subseteq \tau_2$, $\{a\}$ is S_1-open but not S_2-open.

Proposition 2.8. *Let $\{A_\alpha, \alpha \in \Delta\}$ be a collection of S_ι-open sets in a bitopological space X. Then $\bigcup\{A_\alpha, \alpha \in \Delta\}$ is S_ι-open set.*

Proof. Since A_α is as S_ι-open set for each α, then A_α is ι-open and $\bigcup\{A_\alpha, \alpha \in \Delta\}$ is ι-open, then for all $x \in A_\alpha$, there exists $\jmath\iota$-semi closed set F such that $x \in F \subseteq A_\alpha$ this implies that for all $x \in \bigcup\{A_\alpha, \alpha \in \Delta\}$ we have $x \in F \subseteq A_\alpha \subseteq \bigcup\{A_\alpha, \alpha \in \Delta\}$, then $x \in F \subseteq \bigcup\{A_\alpha, \alpha \in \Delta\}$, $\bigcup\{A_\alpha, \alpha \in \Delta\}$ is S_ι-open set. □

Proposition 2.9. *The intersection of two S_ι-open sets is S_ι-open.*

Proof. Let A and B be two S_ι-open sets, then A and B are ι-open sets this implies that $A \cap B$ is an ι-open set, we have to prove that $A \cap B$ is S_ι-open, let $x \in A \cap B$ then $x \in A$ and $x \in B$, for all $x \in A$ there exists $\jmath\iota$-semi closed F such that $x \in F \subseteq A$ and for all $x \in B$ there exists $\jmath\iota$-semi closed E such that $x \in E \subseteq B$. Since the intersection of two $\jmath\iota$-semi closed sets is $\jmath\iota$-semi closed, then for all $x \in F \cap E \subseteq A \cap B$. This shows that $A \cap B$ is S_ι-open set. □

From propositions 2.8 and 2.9 we proved that the family of all S_ι-open subsets of a space X is a topology.

Proposition 2.10. *A subset A of a space (X, τ_1, τ_2) is S_ι-open if and only if for each $x \in A$, there exists an S_ι-open set B such that $x \in B \subseteq A$.*

Proof. Assume that A is S_ι-open set in the (X, τ_1, τ_2), then for each $x \in A$, put $B = A$ is S_ι-open set containing x such that $x \in B \subseteq A$.

conversely, suppose that for each $x \in A$, there exists an S_ι-open set B such that $x \in B_x \subseteq A$, thus $A = \cup B_x$ where $B_x \in S_\iota O(X)$ for each x, therefore A is S_ι-open. □

Proposition 2.11. *If $(X,\tau_\imath)$ is T_1 space, then $S_\jmath O(X) \equiv \tau_\jmath$.*

Proof. Let A be any subset of a space X and $A \in \tau_\jmath$, if $A = \phi$, then $A \in S_\jmath O(X)$. If $A \neq \phi$ let $x \in A$, since $(X,\tau_\imath)$ is a T_1 space, then every singlton is $\imath$-closed this implies that every singlton is $\imath\jmath$-semi closed and hence $x \in \{x\} \subseteq A$. Therefore, $A \in S_\jmath O(X)$. Hence, $\tau_\jmath \subseteq S_\jmath O(X)$, but from definition of $S_\jmath$-open sets we have $S_\jmath O(X) \subseteq \tau_\jmath$. Thus $S_\jmath O(X) \equiv \tau_\jmath$. □

Remark 2.12. Let $(X,\tau_\imath)$ is T_1 space, then the family of $\imath\jmath$-$SC(X)$ is discreate topology in X.

Proposition 2.13. *Every $\imath\jmath-\delta$-open set of a space X is $S_\imath$-open.*

Proof. Let A be $\imath\jmath-\delta$-open set in X, then for each $x \in A$, there exists an $\imath$-open set G such that $x \in G \subseteq \imath\text{-}int\jmath\text{-}clG \subseteq A$, so $A = \bigcup\{x\} \subseteq \bigcup G \subseteq \bigcup clG \subseteq A$ for each $x \in A$, this implies that $A = \bigcup G$ is an $\imath$-open set and $A = \bigcup \imath\text{-}int\jmath\text{-}clG$ is a union of $\jmath\imath$-semi closed sets, so by proposition2.2, A is $S_\imath$-open set. □

However, the converse of proposition may not be true in general as we show in the following example:

Example 2.14. Consider the space given in example2.4 is T_1, then the family of $\imath$-open, $\imath\jmath$-semi open and $S_\imath$-open sets are identical. Hence, any $\imath$-open set G is $S_\imath$-open but not $\imath\jmath-\delta$-open.

The proof of the following corollaries are clear and directly follow from their definitions.

Corollary 2.15. *Every $\imath\jmath-\theta$-open set is $S_\imath$-open.*

Corollary 2.16. *Every $\imath\jmath$-reguler open set is $S_\imath$-open.*

Proposition 2.17. *Let (X,τ_1,τ_2) be a bitopological space, and $A, B \subseteq X$. If $A \in S_\imath O(X)$ and B is $\imath$-open and $\jmath$-closed, then $A \cap B \in S_\imath O(X)$.*

Proof. Let $A \in S_\imath O(X)$ and B is $\imath$-open and $\jmath$-closed, then A is $\imath$-open set. This implies that $A \cap B \in \imath\text{-}O(X)$, now let $x \in A \cap B$, then $x \in A$ and $x \in B$, therefore there exists a $\jmath\imath$-semi closed set F such that $x \in F \subseteq A$. Since B is $\jmath$-closed, so B is $\jmath\imath$-semi closed set implying that $F \cap B$ is $\jmath\imath$-semi closed, therefore $x \in F \cap B \subseteq A \cap B$. Thus, $A \cap B$ is $S_\imath$-open set in X. □

The following diagram shows the relations among $\imath\jmath$-reguler open, $S_\imath$-open, $\imath\jmath-\delta$-open, $\imath\jmath-\theta$-open, $\tau_\imath$, $\imath\jmath-\alpha$-open, $\imath\jmath$-semi open, $\imath\jmath$-pre-open, $\imath\jmath$-semi-preopen sets in a bitopological space (X, τ_1, τ_2).

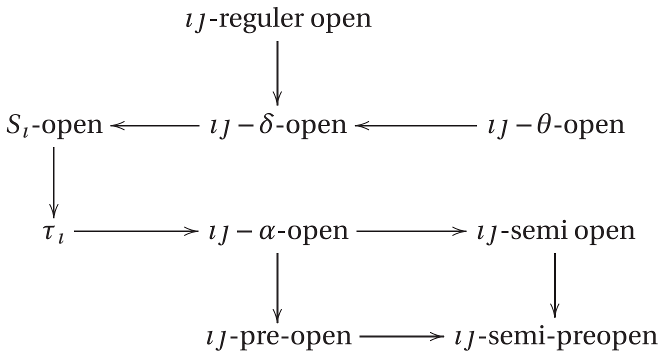

Diagram 1

The proof of the following corollaries are directtly as shown in the above diagram.

Corollary 2.18. *Every $S_\imath$-open set is $\imath\jmath$-semi open.*

Corollary 2.19. *Every $S_\imath$-open set is $\imath\jmath$-pre-open.*

This example show that the converse of the above two corollaries is not true.

Example 2.20. Considering the space given in example 2.3 the set $\{a, b\}$ is $\imath\jmath$-semi open and $\imath\jmath$-pre-open, but it is not $S_\imath$-open set.

Proposition 2.21. *For any bitopological space (X, τ_1, τ_2). We have:*

(1) *If $\tau_\jmath$ is indiscreate, then $S_\imath O(X)$ is also indiscreate.*
(2) *If $S_\imath O(X)$ is discreate, then $\tau_\jmath$ is discreate.*
(3) *$\tau_\imath$ and $\tau_\jmath$ are discreate if and only if $S_\imath O(X)$ is discreate.*

Proof. Obvious. □

Proposition 2.22. *For any subset A of a space (X, τ_1, τ_2). The following statements are equivalent:*

(1) *A is $\imath$-open and $\jmath$-closed.*
(2) *A is $S_\imath$-open and $\jmath$-closed.*
(3) *A is $\imath\jmath-\alpha$-open and $\jmath$-closed.*
(4) *A is $\imath\jmath$-pre-open and $\jmath$-closed.*

Proof. Straightforward. □

Proposition 2.23. *For any subset A of a space (X, τ_1, τ_2). The following statements are equivalent:*

(1) *A is $\imath\jmath$-regular open.*

(2) *A is $S_\imath$-open and $\jmath\imath$-semi closed.*

(3) *A is $\imath$-open and $\jmath\imath$-semi closed.*

(4) *A is $\imath\jmath - \alpha$-open and $\jmath\imath$-semi closed.*

(5) *A is $\imath\jmath$-pre-open and $\jmath\imath$-semi closed.*

Proof. Straightforward. □

Proposition 2.24. *Let (X, τ_1, τ_2) be a space and $x \in X$. If $\{x\}$ is $S_\imath$-open, then $\{x\}$ is $\jmath\imath$-semi closed.*

Proof. Obvious. □

Proposition 2.25. *Let (X, τ_1, τ_2) be a space and $x \in X$. Then $\{x\}$ is $S_\imath$-open if and only if $\{x\}$ is $\imath\jmath$-reguler open.*

Proof. Obvious. □

Proposition 2.26. *Let Y be a subset of a space (X, τ_1, τ_2). If $A \in S_\imath O(X)$ and $A \subseteq Y$, then $A \in S_\imath O(Y)$.*

Proof. Let $A \in S_\imath O(X)$, then $A \in \imath\text{-}O(X)$ and for each $x \in A$, there exists a $\jmath\imath$-semi closed set F in X such that $x \in F \subseteq A$. Since $A \in \imath\text{-}O(X)$ and $A \subseteq Y$. Then $A \in \imath\text{-}O(Y)$. Since $F \in \jmath\imath\text{-}SC(X)$ and $F \subseteq Y$. Then $F \in \jmath\imath\text{-}SC(Y)$. Hence $A \in S_\imath O(Y)$. □

Definition 2.27. Let Y be a subset of the bitopological space (X, τ_1, τ_2), we say that a subset A is $S_\imath$-open in Y if it is $S_\imath$-open in the relative bitoplogical space Y.

Proposition 2.28. *Let Y be a subset of a space (X, τ_1, τ_2). If $A \in S_\imath O(Y)$ and $Y \in \imath\jmath\text{-}RO(X)$, then $A \in S_\imath O(X)$.*

Proof. Let $A \in S_\imath O(Y)$, then $A \in \imath\text{-}O(Y)$ and for each $x \in A$, there exists a $\jmath\imath$-semi closed set F in Y such that $x \in F \subseteq A$. Since $Y \in \imath\jmath\text{-}RO(X)$, then $Y \in \imath\text{-}O(X)$ and Since $A \in \imath\text{-}O(Y)$, then $A \in \imath\text{-}O(X)$. Again since $Y \in \imath\jmath\text{-}RO(X)$, then $Y \in \jmath\imath\text{-}SC(X)$ and since $F \in \jmath\imath\text{-}SC(Y)$. Then $F \in \jmath\imath\text{-}SC(X)$. Hence, $A \in S_\imath O(X)$. □

From Proposition 2.26 and Proposition 2.28 we obtain the following result:

Corollary 2.29. *Let (X, τ_1, τ_2) be a bitopological space and A, Y subsets of X such that $A \subseteq Y \subseteq X$ and $Y \in \imath\jmath\text{-}RO(X)$. Then $A \in S_\imath O(Y)$ if and only if $A \in S_\imath O(X)$.*

Proposition 2.30. *Let Y be a subset of a space (X, τ_1, τ_2). If $A \in S_\imath O(Y)$ and $Y \in \jmath\imath\text{-}SC(X)$, then for each $x \in A$, there exists a $\jmath\imath$-semi closed set F in X such that $x \in F \subseteq A$.*

Proof. Let $A \in S_\imath O(Y)$, then $A \in \imath\text{-}O(Y)$ and for each $x \in A$, there exists a $\jmath\imath$-semi closed set F in Y such that $x \in F \subseteq A$. Since $Y \in \jmath\imath\text{-}SC(X)$, Then $F \in \jmath\imath\text{-}SC(X)$, which compleates the proof. □

The following example satisfies the conditions stated in Proposition 2.30, but $A \notin S_\imath O(X)$.

Example 2.31. Consider $X=\{a,b,c,d\}$, with two topologies $\tau_1 = \{X, \phi, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$ and $\tau_2 = \{X, \phi, \{b\}, \{c,d\}, \{b,c,d\}\}$. And let (Y, σ_1, σ_2) be a subspace of a space (X, τ_1, τ_2) such that $Y = \{a,c,d\} \in \jmath\imath\text{-}SC(X)$ with the relative topologies σ_1 and σ_2 in Y such that $\sigma_1 = \{X,\phi,\{a\},\{c\},\{a,c\}\}$ and $\sigma_2 = \{X,\phi,\{c,d\}\}$. Then the subset $\{a\}$ is S_1-open in (Y,σ_1,σ_2) but $\{a\}$ is not S_1-open in (X,τ_1,τ_2).

Proposition 2.32. *Let A and Y be any subsets of a space X. If $A \in S_\imath O(X)$ and $Y \in \imath\jmath\text{-}RO(X)$, then $A \cap Y \in S_\imath O(X)$.*

Proof. Obvious. □

Definition 2.33. A subset B of a space X is called $S_\imath$-closed if $X \setminus B$ is $S_\imath$-open. The family of all $S_\imath$-closed subsets of bitopological space (X, τ_1, τ_2) is denoted by $S_\imath C(X,\tau_1,\tau_2)$ or $S_\imath C(X)$.

Proposition 2.34. *A subset B of a space X is $S_\imath$-closed if and only if B is an $\imath$-closed set and it is an intersection of $\jmath\imath$-semi open sets.*

Proof. Obvious. □

Proposition 2.35. *Let $\{B_\alpha, \alpha \in \Delta\}$ be a collection of $S_\imath$-closed sets in a bitopological space X. Then $\bigcap\{B_\alpha, \alpha \in \Delta\}$ is $S_\imath$-closed set.*

Proof. Follows from Proposition 2.8. □

Proposition 2.36. *The union of two $S_\imath$-closed sets is $S_\imath$-closed.*

Proof. Follows from Proposition 2.9. □

All of the following results are true by using complement.

Proposition 2.37. *If $(X, \tau_\jmath)$ is T_1 space, then $S_\imath C(X) \equiv \imath - C(X)$.*

Proof. The proof is directly from Proposition 2.11. □

Remark 2.38. Let $(X, \tau_\jmath)$ is T_1 space, then the family of $\jmath\imath\text{-}SO(X)$ is discreate topology in X.

Proposition 2.39. *Let B be any subset of a space X. If $B \in \imath\jmath - \delta C(X)$, then $B \in S_\imath C(X)$.*

Proof. Similar to Proposition 2.13 taking $A = X \setminus B$. ☐

Corollary 2.40. *Each $(\imath, \jmath) - \theta$-closed set is $S_\imath$-closed.*

Corollary 2.41. *Every $\imath\jmath$-reguler closed set is $S_\imath$-closed.*

Proposition 2.42. *For any subset B of a space (X, τ_1, τ_2). The following statements are equivalent:*

(1) *B is $\imath$-closed and $\jmath$-open.*

(2) *B is $S_\imath$-closed and $\jmath$-open.*

(3) *B is $\imath\jmath - \alpha$-closed and $\jmath$-open.*

(4) *B is $\imath\jmath$-pre-closed and $\jmath$-open.*

Proof. Similar to Proposition 2.22 taking $A = X \setminus B$. ☐

Proposition 2.43. *For any subset A of a space (X, τ_1, τ_2). The following statements are equivalent:*

(1) *A is $\imath\jmath$-regular closed.*

(2) *A is $S_\imath$-closed and $\jmath\imath$-semi open.*

(3) *A is $\imath$-closed and $\jmath\imath$-semi open.*

(4) *A is $\imath\jmath - \alpha$-closed and $\jmath\imath$-semi open.*

(5) *A is $\imath\jmath$-pre-closed and $\jmath\imath$-semi open.*

Proof. Similar to Proposition 2.23 taking $A = X \setminus B$. ☐

Proposition 2.44. *Let Y be a subset of a space (X, τ_1, τ_2). If $B \in S_\imath C(X)$ and $B \subseteq Y$, then $B \in S_\imath C(Y)$.*

Proof. The proof is similar to Proposition 2.26. ☐

Proposition 2.45. *Let Y be a subset of a space (X, τ_1, τ_2). If $B \in S_\imath C(Y)$ and $Y \in \imath\jmath$-$RC(X)$, then $B \in S_\imath C(X)$.*

Proof. The proof is similar to Proposition 2.28. ☐

From Proposition 2.44 and Proposition 2.45 we obtain the following result:

Corollary 2.46. *Let (X, τ_1, τ_2) be a bitopological space and B, Y subsets of X such that $B \subseteq Y \subseteq X$ and $Y \in \imath\jmath$-$RC(X)$. Then $B \in S_\imath C(Y)$ if and only if $B \in S_\imath C(X)$.*

Proposition 2.47. *Let B and Y be any subsets of a space X. If $B \in S_\iota C(X)$ and $Y \in \iota\jmath$-$RC(X)$, then $B \cup Y \in S_\iota C(X)$.*

Proof. The proof is directly from Proposition 2.32, and using complements. □

Definition 2.48. For a subset A of a space X and $x \in X$, we introduce the following statements:

(1) A subset N of X is said to be S_ι-neighborhood of x, if there exists an S_ι-open set U in X such that $x \in U \subseteq N$.

(2) S_ι-interior of a set A (briefly, $S_\iota Int(A)$) is the union of all S_ι-open sets which are contained in A.

(3) A point $x \in X$ is said to be S_ι-limit point of A if for each S_ι-open set U containing x, $U \cap (A \setminus \{x\}) \neq \phi$. The set of all S_ι-limit points of A is called a S_ι-derived set of A and is denoted by $S_\iota D(A)$.

(4) A point $x \in X$ is said to be in S_ι-closure of A if for each S_ι-open set U containing x such that $U \cap A \neq \phi$.

(5) S_ι-closure of a set A (briefly $S_\iota Cl(A)$) is the intersection of all S_ι-closed sets containing A.

(6) S_ι-boundary of A is defined as $S_\iota Cl(A) \setminus S_\iota Int(A)$ and is denoted by $S_\iota Bd(A)$.

3. S_ι-Continuous Functions

Definition 3.1. A function $f : X \to Y$ is called S_ι-continuous at a point $x \in X$ if for each ι-open set V of Y containing $f(x)$, there exists an S_ι-open set U of X containing x such that $f(U) \subseteq V$. If f is S_ι-continuous at every point x of X, then it is called S_ι-continuous.

Definition 3.2. A function $f : X \to Y$ is called $\iota\jmath$-almost S_ι-continuous at a point $x \in X$ if for each ι-open set V of Y containing $f(x)$, there exists an S_ι-open set U of X containing x such that $f(U) \subseteq \iota Int(\jmath Cl V)$. If f is $\iota\jmath$-almost S_ι-continuous at every point x of X, then it is called $\iota\jmath$-almost S_ι-continuous.

It is obvious from the definition that S_ι-continuity implies $\iota\jmath$-almost S_ι-continuity. However, the converse is not true in general as it is shown in the following example.

Example 3.3. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{c\}, \{b, c\}\}$, $\sigma_1 = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$, $\sigma_2 = \{X, \phi, \{b, c\}\}$. We define a function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ as an identity function. Then f is $\iota\jmath$-almost S_ι-continuous but not S_ι-continuous at c, because $\{c\}$ is an ι-open set in (X, σ_1, σ_2) containing $f(c) = c$, there exists no S_ι-open set U in (X, τ_1, τ_2) containing c such that $x \in f(U) \subseteq \{c\}$.

Proposition 3.4. *Let X and Y be bitopological spaces. A function $f : X \to Y$ is $S_\imath$-continuous if and only if the inverse image under f of every $\imath$-open set in Y is an $S_\imath$-open in X.*

Proof. It is clear. □

The proof of the following corollaries follows directly from thier definitions.

Corollary 3.5. *Every $S_\imath$-continuous function is $\imath$-continuous.*

Corollary 3.6. *Every $\imath\jmath$-super continuous function is $S_\imath$-continuous.*

Corollary 3.7. *Every $\imath\jmath-\delta$-continuous function is $\imath\jmath$-almost $S_\imath$-continuous.*

Corollary 3.8. *Every $\imath\jmath$-almost $S_\imath$-continuous function is $\imath\jmath$-almost continuous.*

By Definition 3.1, Definition 3.2, Corollary 3.5, Corollary 3.6, Corollary 3.7, Corollary 3.8, we obtain the following diagram.

$$\begin{array}{ccccc}
\imath\jmath\text{-super continuous} & \longrightarrow & S_\imath\text{-continuous} & \longrightarrow & \imath\text{-continuous} \\
\downarrow & & \downarrow & & \downarrow \\
\imath\jmath-\delta\text{-continuous} & \longrightarrow & \imath\jmath\text{-almost } S_\imath\text{-continuous} & \longrightarrow & \imath\jmath\text{-almost continuous}
\end{array}$$

Diagram 2

In the sequel, we shall show that none of the implications that concerning $S_\imath$-continuity and $\imath\jmath$-almost $S_\imath$-continuity in Diagram 2 is reversible.

Example 3.9. Let $X = \{a,b,c,d\}$ with four topologies $\tau_1 = \{X, \phi, \{c\}, \{a,d\}, \{a,c,d\}\}$, $\tau_2 = \{X, \phi, \{b\}, \{a,b,d\}\}$, $\sigma_1 = \{X, \phi, \{a\}, \{b,c\}, \{a,b,c\}\}$ and $\sigma_2 = \{X, \phi, \{d\}, \{a,b,c\}\}$, then the family of $S_\imath$-open subset of X with respect to τ_1 and τ_2 is:
$S_\imath O(X) = \{X, \phi, \{c\}, \{a,c,d\}\}$. We define a function $f : (X,\tau_1,\tau_2) \to (X,\sigma_1,\sigma_2)$ as follows $f(a) = b$, $f(b) = d$, $f(c) = a$, $f(d) = c$. Then f is 1-continuous but not S_1-continuous, because $\{b,c\}$ is an $\imath$-open set in (X, σ_1, σ_2) containing $f(a) = b$, there exists no $S_\imath$-open set U in (X, τ_1, τ_2) containing a such that $b \in f(U) \subseteq \{b,c\}$.

Example 3.10. In Example 3.9. If we have $f : (X,\tau_1,\tau_2) \to (X,\sigma_1,\sigma_2)$ be a function defined as follows $f(a) = a$, $f(b) = f(c) = d$, $f(d) = b$. Then f is $\imath\jmath$-almost continuous but not $\imath\jmath$-almost $S_\imath$-continuous, because $\{a\}$ is an $\imath$-open set in (X, σ_1, σ_2) containing $f(a) = a$, there exists no $S_\imath$-open set U in (X, τ_1, τ_2) containing a such that $x \in f(U) \subseteq \imath Int(\jmath Cl\{a\})$ implies that $f(U) \subseteq \{a,b,c\}$.

Example 3.11. Consider a space X with co-finite topology. Let $f : X \to X$ be an identity function. Since X is T_1, then the family of $\imath$-open sets and $S_\imath$-open sets of X are identical. Hence f is $S_\imath$-continuous but not $\imath\jmath$-super continuous, because the $\jmath$-closure of every $\imath$-open set is the whole space X. So there exists no $\imath$-open set U containing x such that $f(\imath Int(\jmath Cl(U)) \subseteq V$ where V is an $\imath$-open set in Y.

Example 3.12. Let $X = Z$ with co-finite topology, and $Y = \{a, b\}$ with discrete topology. Let $f : X \to Y$ defined by
$f(x) = a$, if x is even
$f(x) = b$, if x is odd.
Hence f is $\imath\jmath$-almost $S_\imath$-continuous but not $\imath\jmath-\delta$-continuous, because the $\jmath$-closure of every $\imath$-open set in X is the whole space X. So there exists no $\imath$-open set U containing x such that $f(\imath Int(\jmath Cl(U)) \subseteq \imath Int(\jmath Cl(V))$ where V is an $\imath$-open set in Y.

Here, we begin with the following characterizations of $S_\imath$-continuous functions.

Proposition 3.13. *For a function $f : X \to Y$, the following statements are equivalent:*

(1) *f is $S_\imath$-continuous.*

(2) *$f^{-1}(V)$ is $S_\imath$-open set in X, for each $\imath$-open set V in Y.*

(3) *$f^{-1}(F)$ is $S_\imath$-closed set in X, for each $\imath$-closed set F in Y.*

(4) *$f(S_\imath Cl(A)) \subseteq \imath Cl(f(A))$, for each subset A of X.*

(5) *$S_\imath Cl(f^{-1}(B)) \subseteq f^{-1}(\imath Cl(B))$, for each subset B of Y.*

(6) *$f^{-1}(\imath Int(B)) \subseteq S_\imath Int(f^{-1}(B))$, for each subset B of Y.*

(7) *$\imath Int(f(A)) \subseteq f(S_\imath Int(A))$, for each subset A of X.*

Proof. Straightforward. □

Proposition 3.14. *For a function $f : X \to Y$, the following statements are equivalent:*

(1) *f is $\imath\jmath$-almost $S_\imath$-continuous.*

(2) *For each $x \in X$ and each $\imath\jmath$-regular open set V of Y containing $f(x)$, there exists a $S_\imath$-open U in X containing x such that $f(U) \subseteq V$.*

(3) *For each $x \in X$ and each $\imath\jmath-\delta$ open set V of Y containing $f(x)$, there exists a $S_\imath$-open U in X containing x such that $f(U) \subseteq V$.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and let V be any $\imath\jmath$-regular open set of Y containing $f(x)$. By (1), there exists a $S_\imath$-open set U of X containing x such that $f(U) \subseteq \imath Int(\jmath Cl(V))$. since V is $\imath\jmath$-regular open, then $\imath Int(\jmath Cl(V)) = V$. Therefore, $f(U) \subseteq V$.

(2) ⇒ (3). Let $x \in X$ and let V be any $\imath\jmath-\delta$-open set of Y containing $f(x)$. Then for each $f(x) \in V$, there exists an $\imath$-open set G containing $f(x)$ such that $G \subseteq \imath Int(\jmath Cl(G)) \subseteq V$. Since $\imath Int(\jmath Cl(G))$ is $\imath\jmath$-regular open set of Y containing $f(x)$. By (2), there exists a $S_\imath$-open set U in X containing x such that $f(U) \subseteq \imath Int(\jmath Cl(G)) \subseteq V$. This completes the proof.
(3) ⇒ (1). Let $x \in X$ and let V be any $\imath$-open set of Y containing $f(x)$. Then $\imath Int(\jmath Cl(V)$ is $\imath\jmath-\delta$-open set of Y containing $f(x)$. By (3), there exists a $S_\imath$-open set U in X containing x such that $f(U) \subseteq \imath Int(\jmath Cl(V))$. Therefore, f is $\imath\jmath$-almost $S_\imath$-continuous. □

Proposition 3.15. *For a function $f : X \to Y$, the following statements are equivalent:*

(1) *f is $\imath\jmath$-almost $S_\imath$-continuous.*
(2) *$f^{-1}(\imath Int(\jmath Cl(V)))$ is $S_\imath$-open set in X, for each $\imath$-open set V in Y.*
(3) *$f^{-1}(\imath Cl(\jmath Int(F)))$ is $S_\imath$-closed set in X, for each $\imath$-closed set F in Y.*
(4) *$f^{-1}(F)$ is $S_\imath$-closed set in X, for each $\imath\jmath$-regular closed set F of Y.*
(5) *$f^{-1}(V)$ is $S_\imath$-open set in X, for each $\imath\jmath$-regular open set V of Y.*

Proof. (1) ⇒ (2). Let V be any $\imath$-open set in Y. We have to show that $f^{-1}(\imath Int(\jmath Cl(V)))$ is $S_\imath$-open set in X. Let $x \in f^{-1}(\imath Int(\jmath Cl(V)))$. Then $f(x) \in \imath Int(\jmath Cl(V))$ and $\imath Int(\jmath Cl(V))$ is an $\imath\jmath$-regular open set in Y. Since f is $\imath\jmath$-almost $S_\imath$-continuous. Then by Proposition 3.14, there exists a $S_\imath$-open set U of X containing x such that $f(U) \subseteq \imath Int(\jmath Cl(V))$. Which implies that $x \in U \subseteq f^{-1}(\imath Int(\jmath Cl(V)))$. Therefore, $f^{-1}(\imath Int(\jmath Cl(V)))$ is $S_\imath$-open set in X.
(2) ⇒ (3). Let F be any $\imath$-closed set of Y. Then $Y \backslash F$ is an $\imath$-open set of Y. By (2), $f^{-1}(\imath Int(\jmath Cl(Y\backslash F)))$ is $S_\imath$-open set in X and $f^{-1}(\imath Int(\jmath Cl(Y\backslash F))) = f^{-1}(\imath Int(Y\backslash \jmath Int(F))) = f^{-1}(Y\backslash \imath Cl(\jmath Int(F))) = X \backslash f^{-1}(\imath Cl(\jmath Int(F)))$ is $S_\imath$-open set in X and hence $f^{-1}(\imath Cl(\jmath Int(F)))$ is $S_\imath$-closed set in X.
(3) ⇒ (4). Let F be any $\imath\jmath$-regular closed set of Y. Then F is an $\imath$-closed set of Y. By (3), $f^{-1}(\imath Cl(\jmath Int(F)))$ is $S_\imath$-closed set in X. Since F is $\imath\jmath$-regular closed set. Then $f^{-1}(\imath Cl(\jmath Int(F))) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is $S_\imath$-closed set in X.
(4) ⇒ (5). Let V be any $\imath\jmath$-regular open set of Y. Then $Y \backslash V$ is $\imath\jmath$-regular closed set of Y and by (4), we have $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is $S_\imath$-closed set in X and hence $f^{-1}(V)$ is $S_\imath$-open set in X.
(5) ⇒ (1). Let $x \in X$ and let V be any $\imath\jmath$-regular open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. By (5), we have $f^{-1}(V)$ is $S_\imath$-open set in X. Therefore, we obtain $f(f^{-1}(V)) \subseteq V$. Hence by Proposition 3.14, f is $\imath\jmath$-almost $S_\imath$-continuous. □

Proposition 3.16. *For a function $f : X \to Y$, the following statements are equivalent:*

(1) *f is $\imath\jmath$-almost $S_\imath$-continuous.*
(2) *$S_\imath Cl(f^{-1}(V)) \subseteq f^{-1}(\imath Cl(V))$, for each $\jmath\imath-\beta$-open set V of Y.*
(3) *$f^{-1}(\imath Int(F)) \subseteq S_\imath Int(f^{-1}(F))$, for each $\jmath\imath-\beta$-closed set F of Y.*

(4) $f^{-1}(\imath Int(F)) \subseteq S_\imath Int(f^{-1}(F))$, *for each* $\jmath\imath$*-semi closed set* F *of* Y.

(5) $S_\imath Cl(f^{-1}(V)) \subseteq f^{-1}(\imath Cl(V))$, *for each* $\jmath\imath$*-semi open set* V *of* Y.

Proof. (1) ⇒ (2). Let V be any $\jmath\imath-\beta$-open set of Y. Since $\imath Cl(V)$ is $\imath\jmath$-reguler closed set in Y and f is $\imath\jmath$-almost $S_\imath$-continuous. Then by Proposition 3.15, $f^{-1}(V)$ is $S_\imath$-closed set in X. Therefore, we obtain $S_\imath Cl(f^{-1}(V)) \subseteq f^{-1}(\imath Cl(V))$.
(2) ⇒ (3). Let F be any $\jmath\imath-\beta$-closed set of Y. Then $Y \setminus F$ is $\jmath\imath-\beta$-open set of Y and by (2), we have $S_\imath Cl(f^{-1}(Y \setminus F)) \subseteq f^{-1}(\imath Cl(Y \setminus F)) \Leftrightarrow S_\imath Cl(X \setminus f^{-1}(F)) \subseteq f^{-1}(Y \setminus \imath Int(F)) \Leftrightarrow X \setminus S_\imath Int(f^{-1}(F)) \subseteq X \setminus f^{-1}(\imath Int(F))$. Therefore, $f^{-1}(\imath Int(F)) \subseteq S_\imath Int(f^{-1}(F))$.
(3) ⇒ (4). This is obvious since every $\jmath\imath$-semi closed set is $\jmath\imath-\beta$-closed set.
(4) ⇒ (5). Let V be any $\jmath\imath$-semi open set of Y. Then $Y \setminus V$ is $\jmath\imath$-semi closed set and by (4), we have $f^{-1}(\imath Int(Y \setminus V)) \subseteq S_\imath Int(f^{-1}(Y \setminus V)) \Leftrightarrow f^{-1}(Y \setminus \imath Cl(V)) \subseteq S_\imath Int(X \setminus f^{-1}(V)) \Leftrightarrow X \setminus f^{-1}(\imath Cl(V)) \subseteq X \setminus S_\imath Cl(f^{-1}(V))$. Therefore, $S_\imath Cl(f^{-1}(V)) \subseteq f^{-1}(\imath Cl(V))$.
(5) ⇒ (1). Let F be any $\imath\jmath$-reguler closed set of Y. Then F is $\jmath\imath$-semi open set of Y. By (5), we have $S_\imath Cl(f^{-1}(F)) \subseteq f^{-1}(\imath Cl(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is $S_\imath$-closed set in X. Therefore, by Proposition 3.15, f is $\imath\jmath$-almost $S_\imath$-continuous. □

Proposition 3.17. *A function* $f : X \to Y$ *is* $\imath\jmath$*-almost* $S_\imath$*-continuous if and only if* $f^{-1}(V) \subseteq S_\imath Int(f^{-1}(\imath Int(\jmath Cl(V))))$ *for each* $\imath$*-open set* V *of* Y.

Proof. Necessity. Let V be any $\imath$-open set of Y. Then $V \subseteq \imath Int(\jmath Cl(V))$ and $\imath Int(\jmath Cl(V))$ is $\imath\jmath$-reguler open set in Y. Since f is $\imath\jmath$-almost $S_\imath$-continuous, by Proposition 3.15, $f^{-1}(\imath Int(\jmath Cl(V)))$ is $S_\imath$-open set in X and hence we obtain that $f^{-1}(V) \subseteq f^{-1}(\imath Int(\jmath Cl(V))) = S_\imath Int(f^{-1}(\imath Int(\jmath Cl(V))))$.
Sufficiency. Let V be any $\imath\jmath$-regular open set of Y. Then V is $\imath$-open set of Y. By hypothesis, we have $f^{-1}(V) \subseteq S_\imath Int(f^{-1}(\imath Int(\jmath Cl(V)))) = S_\imath Int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is $S_\imath$-open set in X and hence by Proposition 3.15, f is $\imath\jmath$-almost $S_\imath$-continuous. □

Corollary 3.18. *A function* $f : X \to Y$ *is* $\imath\jmath$*-almost* $S_\imath$*-continuous if and only if* $S_\imath Cl(f^{-1}(\imath Cl(\jmath Int(F)))) \subseteq f^{-1}(F)$ *for each* $\imath$*-closed set* F *of* Y.

Proposition 3.19. *Let* $f : X \to Y$ *is an* $\imath\jmath$*-almost* $S_\imath$*-continuous function and let* V *be any* $\imath$*-open subset of* Y. *If* $x \in S_\imath Cl(f^{-1}(V)) \setminus f^{-1}(V)$, *then* $f(x) \in S_\imath Cl(V)$.

Proof. Let $x \in X$ be such that $x \in S_\imath Cl(f^{-1}(V)) \setminus f^{-1}(V)$ and suppose $f(x) \notin S_\imath Cl(V)$. Then there exists an $S_\imath$-open set H containing $f(x)$ such that $H \cap V = \phi$. Then $\jmath Cl(H) \cap V = \phi$ implies $\imath Int(\jmath Cl(H)) \cap V = \phi$ and $\imath Int(\jmath Cl(H))$ is $\imath\jmath$-regular open set. Since f is $\imath\jmath$-almost $S_\imath$-continuous, by Proposition 3.15, there exists an $S_\imath$-open set U in X containing x such that $f(U) \subseteq \imath Int(\jmath Cl(H))$. Therefore, $f(U) \cap V = \phi$. However, since $x \in S_\imath Cl(f^{-1}(V))$, $U \cap f^{-1}(V) \neq \phi$ for every $S_\imath$-open set U in X containing x, so that $f(U) \cap V \neq \phi$. We have a contradiction. It follows that $f(x) \in S_\imath Cl(V)$. □

Proposition 3.20. *A function $f : X \to Y$ is S_ι-continuous if and only if f is ι-continuous and for each $x \in X$ and each ι-open set V of Y containing $f(x)$, there exists an $\jmath\iota$-semi closed set F in X containing x such that $f(F) \subseteq V$.*

Proof. Necessity. Let $x \in X$ and let V be any ι-open set of Y containing $f(x)$. Since f is S_ι-continuous, there exists an S_ι-open set U of X containing x such that $f(U) \subseteq V$. Since U is S_ι-open set. Then for each $x \in U$, there exists an $\jmath\iota$-semi closed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$. And also since f is S_ι-continuous. Then f is ι-continuous.
Sufficiency. Let V be any ι-open set of Y. We have to show that $f^{-1}(V)$ is S_ι-open set in X. Since f is ι-continuous, then $f^{-1}(V)$ is ι-open set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, there exists $\jmath\iota$-semi closed set F of X containing x such that $f(F) \subseteq V$. Which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is S_ι-open set in X. Hence by Proposition 3.13, f is S_ι-continuous. □

Proposition 3.21. *The set of all points x of X at which $f : X \to Y$ is not $\iota\jmath$-almost S_ι-continuous is identical with the union of the S_ι-boundaries of the inverse images of $\iota\jmath$-regular open subsets of Y containing $f(x)$.*

Proof. If f is not $\iota\jmath$-almost S_ι-continuousat $x \in X$, then there exists an $\iota\jmath$-regular open set V containing $f(x)$ such that for every S_ι-open set U of X containing x, $f(U) \cap (Y \setminus V) \neq \phi$. This means that for every S_ι-open set U of X containing x, we must have $U \cap (X \setminus f^{-1}(V)) \neq \phi$. Hence, it follows that $x \in S_\iota Cl(X \setminus f^{-1}(V))$. But $x \in f^{-1}(V)$ and hence $x \in S_\iota Cl(f^{-1}(V))$. This means that x belongs to the S_ι-boundary of $f^{-1}(V)$.
Conversely, suppose that x belongs to the S_ι-boundary of $f^{-1}(V_1)$ for some $\iota\jmath$-regular open subset V_1 of Y such that $f(x) \in V_1$. Suppose that f is $\iota\jmath$-almost S_ι-continuousat at x. Then by Proposition 3.14, there exists an S_ι-open set U of X containing x such that $f(U) \subseteq V_1$. Then we have $U \subseteq f^{-1}(V_1)$. This shows that $x \in S_\iota Int(f^{-1}(V_1))$. Therefore, we have $x \notin S_\iota Cl(X \setminus f^{-1}(V_1))$ and $x \notin S_\iota Bd(f^{-1}(V_1))$. But this is a contradiction. This means that f is not $\iota\jmath$-almost S_ι-continuous. □

In the next results, we find some conditions in which the restrictions of S_ι-continuous functions on subspaces are S_ι-continuous.

Proposition 3.22. *Let $f : X \to Y$ be S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous) function. If A is $\iota\jmath-\delta$-open subset of X, then $f|A : A \to Y$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous) in the subspace A.*

Proof. Let V be any ι-open (resp., $\iota\jmath$-regular-open) set of Y. Since f is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous). Then by Proposition 3.13 (resp., by Proposition 3.15), $f^{-1}(V)$ is S_ι-open set in X. Since A is $\iota\jmath-\delta$-open subset of X implies that A is S_ι-open. Then $(f|A)^{-1}(V) =$

$f^{-1}(V) \cap A$ is an S_ι-open subset of A. This shows that $f|A : A \to Y$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous). □

Corollary 3.23. *Let $f : X \to Y$ be S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous) function. If A is either $\iota\jmath$-regular open or $\iota\jmath-\theta$-open subset of X, then $f|A : A \to Y$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous) in the subspace A.*

Proof. Since every $\iota\jmath$-regular open or $\iota\jmath-\theta$-open is $\iota\jmath-\delta$-open set, This is an immediate consequence of Proposition 3.22. □

Proposition 3.24. *A function $f : X \to Y$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous). If for each $x \in X$, there exists an $\iota\jmath$-regular open set A of X containing x such that $f|A : A \to Y$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous).*

Proof. Let $x \in X$, then by hypothesis, there exists an $\iota\jmath$-regular open set A containing x such that $f|A : A \to Y$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous). Let V be any ι-open set of Y containing $f(x)$, there exists an S_ι-open set U in A containing x such that $(f|A)(U) \subseteq V$ (resp., $(f|A)(U) \subseteq \iota Int(\jmath Cl(V))$). Since A is $\iota\jmath$-reguler open set. By Proposition 2.28, U is S_ι-open set in X and hence $f(U) \subseteq V$ (resp., $f(U) \subseteq \iota Int(\jmath Cl(V))$). This shows that f is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous). □

Corollary 3.25. *Let $\{U_\alpha : \alpha \in \Delta\}$ be an $\iota\jmath$-reguler open cover of a bitopological space X. A function $f : X \to Y$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous) if and only if $f|U_\alpha : U_\alpha \to Y$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous) for each $\alpha \in \Delta$.*

Proof. This is an immediate consequence of Corollary 3.23 and Proposition 3.24. □

Proposition 3.26. *If $X = R \cup S$, where R and S are $\iota\jmath$-regular open sets and $f : X \to Y$ is a function such that both $f|R$ and $f|S$ are S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous), then f is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous).*

Proof. Let V be any ι-open (resp., $\iota\jmath$-regular open) set of Y. Then $f^{-1}(V) = (f|R)^{-1}(V) \cup (f|S)^{-1}(V)$. Since $f|R$ and $f|S$ are S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous). Then by Proposition 3.13 (resp., by Proposition 3.15), $(f|R)^{-1}(V)$ and $(f|S)^{-1}(V)$ are S_ι-open sets in R and S, respectively. Since R and S are $\iota\jmath$-regular open sets in X, then by Proposition 2.28, $(f|R)^{-1}(V)$ and $(f|S)^{-1}(V)$ are S_ι-open sets in X. Since union of two S_ι-open sets is S_ι-open. Hence $f^{-1}(V)$ is S_ι-open set in X. Therefore, by Proposition 3.13 (resp., by Proposition 3.15), f is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous). □

In general, if $X = \bigcup\{K_\alpha : \alpha \in \Delta\}$, where each K_α is an $\iota\jmath$-regular open set and $f : X \to Y$ is a function such that $f|K_\alpha$ is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous) for each α, then f is S_ι-continuous (resp., $\iota\jmath$-almost S_ι-continuous).

Proposition 3.27. *Let $X = R_1 \cup R_2$, where R_1 and R_2 are $\imath\jmath$-regular open sets in X. Let $f : R_1 \to Y$ and $g : R_2 \to Y$ be $S_\imath$-continuous (resp., $\imath\jmath$-almost $S_\imath$-continuous). If $f(x) = g(x)$ for each $x \in R_1 \cap R_2$. Then $h : R_1 \cup R_2 \to Y$ such that*

$$h(x) = \begin{cases} f(x) & if\, x \in R_1 \\ g(x) & if\, x \in R_2 \end{cases}$$

is $S_\imath$-continuous (resp., $\imath\jmath$-almost $S_\imath$-continuous).

Proof. Let O be an $\imath$-open (resp., $\imath\jmath$-regular open) set of Y. Now $h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O)$. Since f is $S_\imath$-continuous (resp., $\imath\jmath$-almost $S_\imath$-continuous), then by Proposition 3.13 (resp., by Proposition 3.15), $f^{-1}(O)$ is $S_\imath$-open set in R_1. But R_1 is $\imath\jmath$-regular open set in X. Then by Proposition 2.28, $f^{-1}(O)$ is $S_\imath$-open set in X. Similarly, $g^{-1}(O)$ is $S_\imath$-open set in R_2 and hence, $S_\imath$-open set in X. Since union of two $S_\imath$-open sets is $S_\imath$-open. Therefore, $h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O)$ is $S_\imath$-open set in X. Hence by Proposition 3.13 (resp., by Proposition 3.15), h is $S_\imath$-continuous (resp., $\imath\jmath$-almost $S_\imath$-continuous). □

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Department of Mathematics, College of Science, University of Duhok, Kurdistan-Region, Iraq.

E-mail: aliasbkhalaf@gmail.com

Department of Mathematics, College of Science, University of Duhok, Kurdistan-Region, Iraq.

E-mail: alan.m.omer@gmail.com