ALMOST $p_s$-CONTINUOUS FUNCTIONS

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Abstract. The purpose of this paper is to introduce a new class of functions called almost $p_s$-continuous function by using $p_s$-open sets in topological spaces. Some properties and characterizations of this function are given.

1. Introduction

Throughout this paper, a space $X$ mean a topological space with out any separation axiom. We recall the following definitions, notations and terminology. The closure (resp. interior) of a subset $A$ of $X$ is denoted by $ClA$ (resp. $IntA$). A subset $A$ of $X$ is said to be preopen [19] (resp. semi-open [17], $\alpha$-open [22], $\beta$-open [1], regular open [31] and regular semi-open [5]) if $A \subseteq IntClA$ (resp. $A \subseteq ClIntA$, $A \subseteq IntClIntA$, $A \subseteq ClIntClA$, $A = IntClA$ and $A = sIntsClA$). The complement of a preopen (resp. semi-open, $\alpha$-open, $\beta$-open, regular open and regular semi-open) set is said to be preclosed (resp. semi-closed, $\alpha$-closed, $\beta$-closed, regular closed and regular semi-open). The family of all preopen (resp. semi-open, $\alpha$-open, regular open, regular semi-open and regular closed) subsets of a topological space $X$ is denoted by $PO(X)$ (resp. $SO(X)$, $\alpha O(X)$, $RO(X)$, $R SO(X)$ and $RC(X)$). A function $f : X \rightarrow Y$ is said to be precontinuous [19] (resp. super continuous [20]) if the inverse image of each open subset of $Y$ is preopen (resp. $\delta$-open) in $X$. A function $f : X \rightarrow Y$ is said to be almost precontinuous [11] (resp. almost continuous in the sense of Singal and Singal [30], almost $\alpha$-continuous [23], R-map [6], almost strongly $\theta$-continuous [27], almost s-continuous [14], weakly $\theta$-irresolute [10] and $\theta$-irresolute [16]) if the inverse image of each regular open subset of $Y$ is preopen (resp., open, $\alpha$-open, regular open, $\theta$-open, closed, semi-closed and intersection of regular open sets) in $X$. A function $f : X \rightarrow Y$ is said to be $\delta$-continuous [24] (resp., almost strongly $\theta$-continuous [27]) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists an open set $U$ of $X$ containing $x$ such that $f(IntClU) \subseteq IntClV$ (resp., $f(ClU) \subseteq sClV$). A function $f : X \rightarrow Y$ is said to be irresolute [7] if the inverse image of each semi-open subset of $Y$ is semi-open in $X$. A function $f : X \rightarrow Y$ is said to be weakly quasi-continuous [25] (resp. S-continuous [33]) if for every $F \in RC(Y)$, $f^{-1}(F) \in SO(X)$ (resp. $f^{-1}(F)$

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is the union of regular closed sets of \( X \). A function \( f : X \to Y \) is said to be preopen \([19]\) (resp., semi-open \([26]\)) if the image of each open set of \( X \) is preopen (resp., semi-open) in \( Y \).

**Definition 1.1** ([15]). A preopen subset \( A \) of a space \( X \) is called \( p_s \)-open if for each \( x \in A \), there exists a semi-closed set \( F \) such that \( x \in F \subseteq A \).

The family of all \( p_s \)-open subsets of a topological space \( X \) is denoted by \( P_s O(X) \).

**Definition 1.2** ([32]). A subset \( A \) of a space \( X \) is called \( \delta \)-open (resp., \( \theta \)-open) if for each \( x \in A \), there exists an open set \( G \) such that \( x \in G \subseteq \text{IntCl} G \subseteq A \) (resp., \( x \in G \subseteq \text{Cl} G \subseteq A \)).

The intersection of all \( p_s \)-closed (resp. preclosed, semi-closed, \( \alpha \)-closed and \( \delta \)-closed) sets of \( X \) containing \( A \) is called the \( p_s \)-closure (resp. preclosure, semi-closure, \( \alpha \)-closure and \( \delta \)-closure) of \( A \) and is denoted by \( P_s \text{Cl} A \) (resp. \( p \text{Cl} A, s \text{Cl} A, \alpha \text{Cl} A \) and \( \text{Cl}_\delta A \)). The union of all \( p_s \)-open (resp. preopen, semi-open, \( \alpha \)-open and \( \delta \)-open) sets of \( X \) contained in \( A \) is called the \( p_s \)-interior (resp. preinterior, semi-interior, \( \alpha \)-interior and \( \delta \)-interior) of \( A \) and is denoted by \( P_s \text{Int} A \) (resp. \( p \text{Int} A, s \text{Int} A, \alpha \text{Int} A \) and \( \text{Int}_\delta A \)).

**Proposition 1.3** ([15]). A subset \( A \) of a space \( X \) is \( p_s \)-open if and only if \( A \) is preopen and it is a union of semi-closed sets.

**Definition 1.4** ([13]). A subset \( A \) of a space \( X \) is called \( \theta \)-semi-open if for each \( x \in A \), there exists a semi-open set \( G \) such that \( x \in G \subseteq \text{Cl} G \subseteq A \). The family of all \( \theta \)-semi-open subsets of a topological space \( X \) is denoted by \( \theta \text{SO}(X) \).

**Definition 1.5.** A space \( X \) is \( s \)-regular \([3]\) (resp., semi-regular \([28]\)) if for each \( x \in X \) and each open set \( G \) containing \( x \), there exists a semi-open (resp., regular open) set \( H \) such that \( x \in H \subseteq s \text{Cl} H \subseteq G \) (resp., \( x \in H \subseteq \text{Cl} H \subseteq G \)).

**Definition 1.6.** A space \( X \) is said to be:

1. extremally disconnected \([8]\) if \( \text{Cl} U \) is open for each open set \( U \).
2. hyperconnected \([9]\) if every nonempty open subset of \( X \) is dense in \( X \).
3. locally indiscrete \([9]\) if every open subset of \( X \) is closed.
4. semi-\( T_1 \) \([18]\) if to each pair of distinct points \( x, y \) of \( X \), there exists a pair of semi-open sets, one containing \( x \) but not \( y \) and the other containing \( y \) but not \( x \).

**Proposition 1.7.** The following statements are true:

1. A space \( X \) is semi-\( T_1 \) if and only if for any point \( x \in X \), the singleton set \( \{x\} \) is semi-closed. \([18]\).
2. A space \( X \) is extremally disconnected if and only if \( \text{RO}(X) = \text{RC}(X) \). \([11]\).
(3) If a space $X$ is semi-$T_1$, then $P_3O(X) = PO(X)$. [15].

(4) If a topological space $(X, \tau)$ is locally indiscrete, then $P_3O(X) = \tau$. [15].

(5) If a topological space $(X, \tau)$ is $s$-regular, then $\tau \subseteq P_3O(X)$. [15].

**Proposition 1.8** ([15]). For any subset $A$ of a space $X$. The following are equivalent:

1. $A$ is clopen.
2. $A$ is $p_3$-open and closed.
3. $A$ is $\alpha$-open and closed.
4. $A$ is preopen and closed.

**Proposition 1.9** ([15]). For any subset $A$ of a space $X$. The following are equivalent:

1. $A$ is regular open.
2. $A$ is $p_3$-open and semi-closed.
3. $A$ is open and semi-closed.
4. $A$ is $\alpha$-open and semi-closed.
5. $A$ is preopen and semi-closed.

**Lemma 1.10** ([15]). The following properties are true:

1. For any subset $A$ of a space $X$. If $A \in \theta SO(X)$ and $A \in PO(X)$, then $A \in P_3O(X)$.
2. If $(X, \tau)$ is extremally disconnected space and if $A \in \theta SO(X)$, then $A \in P_3O(X)$.
3. If $(Y, \tau_Y)$ is a subspace of a space $(X, \tau)$, if $A \in P_3O(Y, \tau_Y)$ and $Y \in RO(X, \tau)$, then $A \in P_3O(X)$.
4. If either $B \in RSO(X)$ or $B$ is an open subspace of a space $X$ and $A \in P_3O(X)$, then $A \cap B \in P_3O(B)$.

**Lemma 1.11.** The following statements are true:

1. If $R \in RO(X)$ and $P \in PO(X)$, then $R \cap P \in RO(P)$. [9].
2. Let $A$ be a subset of a space $(X, \tau)$. Then $A \in PO(X, \tau)$ if and only if $sClA = IntClA$. [12].
3. Let $Y$ be a dense subspace of $X$. If $O$ is regular open in $Y$, then $O = Y \cap IntClO$. [29].
4. A subset $A$ of a space $(X, \tau)$ is $\beta$-open if and only if $ClA$ is regular closed. [4].

**Lemma 1.12.** Let $A$ be a subset of a topological space $(X, \tau)$, then the following statement are true:

1. If $A \in SO(X)$, then $Cl_5A = ClA = P_3ClA = pClA = \alpha ClA$. [15].
2. If $A \in \beta O(X)$, then $\alpha ClA = ClA$. [2].
**Definition 1.13** ([15]). A function \( f : X \rightarrow Y \) is called \( p_s \)-continuous at a point \( x \in X \) if for each open set \( V \) of \( Y \) containing \( f(x) \), there exists a \( p_s \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \). Equivalently, a function \( f : X \rightarrow Y \) is \( p_s \)-continuous if and only if \( f^{-1}(V) \) is \( p_s \)-open set in \( X \) for each open set \( V \) in \( Y \).

**Proposition 1.14** ([21]). A function \( f : X \rightarrow Y \) is almost precontinuous if and only if \( f^{-1}(V) \) is \( p_s \)-open set in \( X \), for every \( \delta \)-open set \( V \) in \( Y \).

**Lemma 1.15.** The following results can be proved easily:

1. If \( f : X \rightarrow Y \) is almost precontinuous and \( Y \) is semi-regular, then \( f \) is precontinuous.
2. If \( f : X \rightarrow Y \) is almost continuous and \( Y \) is semi-regular, then \( f \) is continuous.

**Theorem 1.16** ([15]). If \( f : X \rightarrow Y \) is a continuous and open function and \( V \) is a \( p_s \)-open set of \( Y \), then \( f^{-1}(V) \) is a \( p_s \)-open set of \( X \).

**Theorem 1.17** ([12]). A function \( f : X \rightarrow Y \) is preopen if and only if \( f^{-1}(Cl(V)) \subseteq Cl(f^{-1}(V)) \), for each semi-open set \( V \) of \( Y \).

2. Almost \( p_s \)-Continuous Functions

In this section, we introduce the concept of almost \( p_s \)-continuous functions by using \( p_s \)-open sets. Some properties and characterizations are given.

**Definition 2.1.** A function \( f : X \rightarrow Y \) is called almost \( p_s \)-continuous at a point \( x \in X \) if for each open set \( V \) of \( Y \) containing \( f(x) \), there exists a \( p_s \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq Int(Cl(V)) \). If \( f \) is almost \( p_s \)-continuous at every point of \( X \), then it is called almost \( p_s \)-continuous.

**Lemma 2.2.** The following results follows directly from their definitions:

1. Every \( p_s \)-continuous function is almost \( p_s \)-continuous.
2. Every almost \( p_s \)-continuous function is almost precontinuous.

**Proposition 2.3.** If a function \( f : X \rightarrow Y \) is \( \delta \)-continuous, then \( f \) is almost \( p_s \)-continuous.

**Proof.** Let \( x \in X \) and \( V \) be any open set of \( Y \) containing \( f(x) \). Since \( f \) is \( \delta \)-continuous, there exists an open set \( U \) of \( X \) containing \( x \) such that \( f(Int(Cl(U))) \subseteq Int(Cl(V)) \). Since \( Int(Cl(U)) \) is a regular open set, hence it is \( p_s \)-open set of \( X \) containing \( x \). Therefore, \( f \) is almost \( p_s \)-continuous.

From Lemma 2.2, Proposition 2.3 and Diagram 3.1 in [15], we obtain the following diagram:
super continuous $\rightarrow$ $\delta$-continuous $\rightarrow$ $p_s$-continuous $\rightarrow$ almost $p_s$-continuous $\rightarrow$ precontinuous $\rightarrow$ almost precontinuous

Diagram 2.1

In the sequel, we shall show that none of the implications that concerning almost $p_s$-continuity in Diagram 2.1 is reversible.

**Example 2.4.** Let $X = \{a, b, c, d\}$ with the two topologies $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\phi, X, \{c\}, \{a, d\}, \{a, c, d\}\}$; then the family of $p_s$-open subsets of $X$ with respect to $\tau$ is: $P_sO(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. Then $f$ is almost $p_s$-continuous, but it is not $p_s$-continuous, because $\{a, d\}$ is an open set in $(X, \tau)$ containing $f(d) = d$, there exist no $p_s$-open set $U$ in $(X, \tau)$ containing $d$ such that $d \in f(U) \subseteq \{a, d\}$.

**Example 2.5.** Let $X = \{a, b, c, d\}$ with the two topologies $\tau = \{\phi, X, \{c\}, \{a, d\}, \{a, c, d\}\}$ and $\sigma = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$; then the family of $p_s$-open subsets of $X$ with respect to $\tau$ is: $P_sO(X) = \{\phi, X, \{c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be a function defined as follows: $f(a) = f(b) = f(c) = a$ and $f(d) = b$. Then $f$ is almost precontinuous (see Example 4.5 [21]). However $f$ is not almost $p_s$-continuous since $\{b, c\}$ is an open set in $(X, \tau)$ containing $f(d) = b$, there exist no $p_s$-open set $U$ in $(X, \tau)$ containing $d$ such that $f(\{d\}) = b \in f(U) \subseteq \text{IntCl}\{b, c\} = \{b, c\}$.

**Example 2.6.** Let $X = \{a, b, c, d\}$ with the two topologies $\tau = \{\phi, X, \{c\}, \{a, d\}, \{a, c, d\}\}$, $\{a, c, d\}$ and let $Y = \{x, y, z\}$ with the topology $\sigma = \{\phi, Y, \{x\}, \{y\}, \{x, y\}\}$; then the family of $p_s$-open subset of $X$ with respect to $\tau$ is: $P_sO(X) = \{\phi, X, \{c\}, \{a, d\}, \{a, c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be a function defined as follows: $f(a) = z$ and $f(b) = f(c) = f(d) = y$. Then $f$ is almost $p_s$-continuous. But $f$ is not almost continuous [21] Example 4.2 and hence it is not $\delta$-continuous.

**Theorem 2.7.** For a function $f : X \rightarrow Y$, the following statements are equivalent:

1. $f$ is almost $p_s$-continuous.
2. For each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a $p_s$-open set $U$ in $X$ containing $x$ such that $f(U) \subseteq sClV$. 
(3) For each \( x \in X \) and each regular open set \( V \) of \( Y \) containing \( f(x) \), there exists a \( p_s \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \).

(4) For each \( x \in X \) and each \( \delta \)-open set \( V \) of \( Y \) containing \( f(x) \), there exists a \( p_s \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( x \in X \) and let \( V \) be any open set of \( Y \) containing \( f(x) \). By (1), there exists a \( p_s \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq IntCIV \). Since \( V \) is open, hence \( V \) is preopen set. Therefore, by Lemma 1.11(2), \( f(U) \subseteq sCIV \).

(2) \( \Rightarrow \) (3). Let \( x \in X \) and let \( V \) be any regular open set of \( Y \) containing \( f(x) \). Then \( V \) is an open set of \( Y \) containing \( f(x) \). By (2), there exists a \( p_s \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq sCIV \). Since \( V \) is regular open and hence is preopen set. Therefore, by Lemma 1.11(2), \( f(U) \subseteq IntCIV \). Since \( V \) is regular open, then \( f(U) \subseteq V \).

(3) \( \Rightarrow \) (4). Let \( x \in X \) and let \( V \) be any \( \delta \)-open set of \( Y \) containing \( f(x) \). Then for each \( f(x) \in V \), there exists an open set \( G \) containing \( f(x) \) such that \( G \subseteq IntCIG \subseteq V \). Since \( IntCIG \) is a regular open set of \( Y \) containing \( f(x) \), by (3), there exists a \( p_s \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq IntCIV \). This completes the proof.

(4) \( \Rightarrow \) (1). Let \( x \in X \) and let \( V \) be any open set of \( Y \) containing \( f(x) \). Then \( IntCIV \) is \( \delta \)-open set of \( Y \) containing \( f(x) \). By (4), there exists a \( p_s \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq IntCIV \). Therefore, \( f \) is almost \( p_s \)-continuous. \( \Box \)

**Theorem 2.8.** For a function \( f : X \rightarrow Y \), the following statements are equivalent:

(1) \( f \) is almost \( p_s \)-continuous.

(2) \( f^{-1}(IntCIV) \) is \( p_s \)-open set in \( X \), for each open set \( V \) in \( Y \).

(3) \( f^{-1}(CICIntF) \) is \( p_s \)-closed set in \( X \), for each closed set \( F \) in \( Y \).

(4) \( f^{-1}(F) \) is \( p_s \)-closed set in \( X \), for each regular closed set \( F \) of \( Y \).

(5) \( f^{-1}(V) \) is \( p_s \)-open set in \( X \), for each regular open set \( V \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( V \) be any open set in \( Y \). We have to show that \( f^{-1}(IntCIV) \) is \( p_s \)-open set in \( X \). Let \( x \in f^{-1}(IntCIV) \). Then \( f(x) \in IntCIV \) and \( IntCIV \) is a regular open set in \( Y \). Since \( f \) is almost \( p_s \)-continuous, by Theorem 2.7, there exists a \( p_s \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq IntCIV \). Which implies that \( x \in U \subseteq f^{-1}(IntCIV) \). Therefore, \( f^{-1}(IntCIV) \) is \( p_s \)-open set in \( X \).

(2) \( \Rightarrow \) (3). Let \( F \) be any closed set of \( Y \). Then \( Y \setminus F \) is an open set of \( Y \). By (2), \( f^{-1}(IntCl(Y \setminus F)) \) is \( p_s \)-open set in \( X \) and \( f^{-1}(IntCl(Y \setminus F)) = f^{-1}(Int(Y \setminus IntF)) = f^{-1}(Y \setminus ClIntF) = X \setminus f^{-1}(ClIntF) \) is \( p_s \)-open set in \( X \) and hence \( f^{-1}(ClIntF) \) is \( p_s \)-closed set in \( X \).

(3) \( \Rightarrow \) (4). Let \( F \) be any regular closed set of \( Y \). Then \( F \) is a closed set of \( Y \). By (3), \( f^{-1}(ClIntF) \) is \( p_s \)-closed set in \( X \). Since \( F \) is regular closed set, then \( f^{-1}(ClIntF) = f^{-1}(F) \). Therefore,
$f^{-1}(F)$ is $p_s$-closed set in $X$.

(4) $\Rightarrow$ (5). Let $V$ be any regular open set of $Y$. Then $Y \setminus V$ is regular closed set of $Y$ and by (4), we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is $p_s$-closed set in $X$ and hence $f^{-1}(V)$ is $p_s$-open set in $X$.

(5) $\Rightarrow$ (1). Let $x \in X$ and let $V$ be any regular open set of $Y$ containing $f(x)$. Then $x \in f^{-1}(V)$. By (5), we have $f^{-1}(V)$ is $p_s$-open set in $X$. Therefore, we obtain $f(f^{-1}(V)) \subseteq V$. Hence by Theorem 2.7, $f$ is almost $p_s$-continuous.

The following result can be proved easily from the above theorem.

**Proposition 2.9.** Let $f : X \to Y$ be a function. Let $B$ be any basis for $\tau_s$ in $Y$. Then $f$ is almost $p_s$-continuous if and only if for each $B \in B$, $f^{-1}(B)$ is a $p_s$-open subset of $X$.

**Theorem 2.10.** For a function $f : X \to Y$, the following statements are equivalent:

1. $f$ is almost $p_s$-continuous.
2. $f(P_sClA) \subseteq Cl_\delta f(A)$, for each $A \subseteq X$.
3. $P_sCl f^{-1}(B) \subseteq f^{-1}Cl_\delta(B)$, for each $B \subseteq Y$.
4. $f^{-1}(F)$ is $p_s$-closed set in $X$, for each $\delta$-closed set $F$ of $Y$.
5. $f^{-1}(V)$ is $p_s$-open set in $X$, for each $\delta$-open set $V$ of $Y$.
6. $f^{-1}(Int_\delta B) \subseteq P_sInt f^{-1}(B)$, for each $B \subseteq Y$.

**Proof.** (1) $\Rightarrow$ (2). Let $A$ be a subset of $X$. Since $Cl_\delta f(A)$ is $\delta$-closed set in $Y$, so $Cl_\delta f(A) = \cap \{F_\alpha : F_\alpha \in RC(Y), \alpha \in \Lambda\}$, where $\Lambda$ is an index set. Then $A \subseteq f^{-1}(Cl_\delta f(A)) = f^{-1}(\cap \{F_\alpha : \alpha \in \Lambda\}) = \cap \{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$. By (1) and Theorem 2.8, $f^{-1}(Cl_\delta f(A))$ is $p_s$-closed set of $X$. Hence $P_sCl A \subseteq f^{-1}(Cl_\delta f(A))$. Therefore, we obtain that $f(P_sCl A) \subseteq Cl_\delta f(A)$.

(2) $\Rightarrow$ (3). Let $B$ be any subset of $Y$. Then $f^{-1}(B)$ is a subset of $X$. By (2), we have $f(P_sCl f^{-1}(B)) \subseteq Cl_\delta f(f^{-1}(B)) = Cl_\delta B$. Hence $P_sCl f^{-1}(B) \subseteq f^{-1}(Cl_\delta B)$.

(3) $\Rightarrow$ (4). Let $F$ be any $\delta$-closed set of $Y$. By (3), we have $P_sCl f^{-1}(F) \subseteq f^{-1}(Cl_\delta F) = f^{-1}(F)$ and hence $f^{-1}(F)$ is $p_s$-closed set in $X$.

(4) $\Rightarrow$ (5). Let $V$ be any $\delta$-open set of $Y$. Then $Y \setminus V$ is $\delta$-closed set of $Y$ and by (4), we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is $p_s$-closed set in $X$. Hence $f^{-1}(V)$ is $p_s$-open set in $X$.

(5) $\Rightarrow$ (6). For each subset $B$ of $Y$. We have $Int_\delta B \subseteq B$. Then $f^{-1}(Int_\delta B) \subseteq f^{-1}(B)$. By (5), $f^{-1}(Int_\delta B)$ is $p_s$-open set in $X$. Then $f^{-1}(Int_\delta B) \subseteq P_sInt f^{-1}(B)$.

(6) $\Rightarrow$ (1). Let $x \in X$ and $V$ be any regular open set of $Y$ containing $f(x)$. Since $V$ is a regular open set, hence it is $\delta$-open and by (6), $f^{-1}(Int_\delta V) \subseteq P_sInt f^{-1}(V)$. Therefore, $f^{-1}(V) \subseteq P_sInt f^{-1}(V)$, so $f^{-1}(V)$ is a $p_s$-open set in $X$ which contains $x$ and clearly $f(f^{-1}(V)) \subseteq V$. Hence, by Theorem 2.7, $f$ is almost $p_s$-continuous.

**Theorem 2.11.** For a function $f : X \to Y$, the following statements are equivalent:
(1) $f$ is almost $p_s$-continuous.

(2) $P_sCl f^{-1}(V) \subseteq f^{-1}(CIV)$, for each $\beta$-open set $V$ of $Y$.

(3) $f^{-1}(Int F) \subseteq P_sInt f^{-1}(F)$, for each $\beta$-closed set $F$ of $Y$.

(4) $f^{-1}(Int F) \subseteq P_sInt f^{-1}(F)$, for each semi-closed set $F$ of $Y$.

(5) $P_sCl f^{-1}(V) \subseteq f^{-1}(CIV)$, for each semi-open set $V$ of $Y$.

**Proof.** (1) $\Rightarrow$ (2). Let $V$ be any $\beta$-open set of $Y$. It follows from Lemma 1.11(4) that $CIV$ is regular closed set in $Y$. Since $f$ is almost $p_s$-continuous, by Theorem 2.8, $f^{-1}(CIV)$ is $p_s$-closed set in $X$. Therefore, we obtain $P_sCl f^{-1}(V) \subseteq f^{-1}(CIV)$.

(2) $\Rightarrow$ (3). Let $F$ be any $\beta$-closed set of $Y$. Then $Y \setminus F$ is $\beta$-open set of $Y$ and by (2), we have $P_sCl f^{-1}(Y \setminus F) \subseteq f^{-1}(Cl(Y \setminus F))$ and $P_sCl(X \setminus f^{-1}(F)) \subseteq f^{-1}(Y \setminus Int F)$ and hence, $X \setminus P_sInt f^{-1}(F) \subseteq X \setminus f^{-1}(Int F)$. Therefore, $f^{-1}(Int F) \subseteq P_sInt f^{-1}(F)$.

(3) $\Rightarrow$ (4). Obvious since every semi-closed set is $\beta$-closed.

(4) $\Rightarrow$ (5). Let $V$ be any semi-open set of $Y$. Then $Y \setminus V$ is semi-closed set in $Y$ and by (4), we have $f^{-1}(Int(Y \setminus V)) \subseteq P_sInt f^{-1}(Y \setminus V)$ and $f^{-1}(Y \setminus CIV) \subseteq P_sInt(X \setminus f^{-1}(V))$ and hence, $X \setminus f^{-1}(CIV) \subseteq X \setminus P_sCl f^{-1}(V)$. Therefore, $P_sCl f^{-1}(V) \subseteq f^{-1}(CIV)$.

(5) $\Rightarrow$ (1). Let $F$ be any regular closed set of $Y$. Then $F$ is a semi-open set of $Y$. By (5), we have $P_sCl f^{-1}(F) \subseteq f^{-1}(CIV) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is a $p_s$-closed set in $X$. Therefore, by Theorem 2.8, $f$ is almost $p_s$-continuous. $\Box$

**Corollary 2.12.** For a function $f : X \rightarrow Y$, the following statements are equivalent:

(1) $f$ is almost $p_s$-continuous.

(2) $P_sCl f^{-1}(V) \subseteq f^{-1}(aCIV)$, for each $\beta$-open set $V$ of $Y$.

(3) $P_sCl f^{-1}(V) \subseteq f^{-1}(Cl V)$, for each $\beta$-open set $V$ of $Y$.

(4) $P_sCl f^{-1}(V) \subseteq f^{-1}(P_sCIV)$, for each semi-open set $V$ of $Y$.

(5) $P_sCl f^{-1}(V) \subseteq f^{-1}(pCIV)$, for each semi-open set $V$ of $Y$.

**Proof.** (1) $\Rightarrow$ (2). Follows from Theorem 2.11 and Lemma 1.12(2).

(2) $\Rightarrow$ (3). Follows from the fact that $aCIV \subseteq Cl V$.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5). Follows from Theorem 2.11 and Lemma 1.12(1).

(5) $\Rightarrow$ (1). Follows from Theorem 2.11 and Lemma 1.12(1).

$\Box$

The following result also can be concluded directly.

**Corollary 2.13.** For a function $f : X \rightarrow Y$, the following statements are equivalent:

(1) $f$ is almost $p_s$-continuous.
(2) $f^{-1}(\alpha \operatorname{Int} F) \subseteq P_s \operatorname{Int} f^{-1}(F)$, for each $\beta$-closed set $F$ of $Y$.
(3) $f^{-1}(\operatorname{Int}_s F) \subseteq P_s \operatorname{Int} f^{-1}(F)$, for each $\beta$-closed set $F$ of $Y$.
(4) $f^{-1}(P_s \operatorname{Int} F) \subseteq P_s \operatorname{Int} f^{-1}(F)$, for each semi-closed set $F$ of $Y$.
(5) $f^{-1}(p \operatorname{Int} F) \subseteq P_s \operatorname{Int} f^{-1}(F)$, for each semi-closed set $F$ of $Y$.

**Theorem 2.14.** A function $f : X \to Y$ is almost $p_s$-continuous if and only if $f^{-1}(V) \subseteq P_s \operatorname{Int} f^{-1}((\operatorname{Int} C l V))$ for each preopen set $V$ of $Y$.

**Proof.** Necessity. Let $V$ be any preopen set of $Y$. Then $V \subseteq \operatorname{Int} C l V$ and $\operatorname{Int} C l V$ is a regular open set in $Y$. Since $f$ is almost $p_s$-continuous, by Theorem 2.8, $f^{-1}(\operatorname{Int} C l V)$ is $p_s$-open set in $X$ and hence we obtain that $f^{-1}(V) \subseteq f^{-1}(\operatorname{Int} C l V) = p_s \operatorname{Int} f^{-1}(\operatorname{Int} C l V)$.

Sufficiency. Let $V$ be any regular open set of $Y$. Then $V$ is a preopen set of $Y$. By hypothesis, we have $f^{-1}(V) \subseteq P_s \operatorname{Int} f^{-1}(\operatorname{Int} C l V) = P_s \operatorname{Int} f^{-1}(V)$. Therefore, $f^{-1}(V)$ is $p_s$-open set in $X$ and hence by Theorem 2.8, $f$ is almost $p_s$-continuous. □

We obtain the following corollary.

**Corollary 2.15.** The following statements are equivalent for a function $f : X \to Y$ :

(1) $f$ is almost $p_s$-continuous.
(2) $f^{-1}(V) \subseteq P_s \operatorname{Int} f^{-1}(s \operatorname{Cl} V)$ for each preopen set $V$ of $Y$.
(3) $P_s \operatorname{Cl} f^{-1}(C l \operatorname{Int} F) \subseteq f^{-1}(F)$ for each preclosed set $F$ of $Y$.
(4) $P_s \operatorname{Cl} f^{-1}(s \operatorname{Int} F) \subseteq f^{-1}(F)$ for each preclosed set $F$ of $Y$.

**Corollary 2.16.** For a function $f : X \to Y$, the following statements are equivalent:

(1) $f$ is almost $p_s$-continuous.
(2) For each neighborhood $V$ of $f(x), x \in P_s \operatorname{Int} f^{-1}(s \operatorname{Cl} V)$.
(3) For each neighborhood $V$ of $f(x), x \in P_s \operatorname{Int} f^{-1}(\operatorname{Int} C l V)$.

**Proof.** Follows from Theorem 2.14 and Corollary 2.15. □

**Theorem 2.17.** Let $f : X \to Y$ be an almost $p_s$-continuous function and let $V$ be any open subset of $Y$. If $x \in P_s \operatorname{Cl} f^{-1}(V) \setminus f^{-1}(V)$, then $f(x) \notin P_s \operatorname{Cl} V$.

**Proof.** Let $x \in X$ be such that $x \in P_s \operatorname{Cl} f^{-1}(V) \setminus f^{-1}(V)$ and suppose $f(x) \notin P_s \operatorname{Cl} V$. Then there exists a $p_s$-open set $H$ containing $f(x)$ such that $H \cap V = \emptyset$. Then $\operatorname{Int} C l H \cap V = \emptyset$ implies $\operatorname{Int} C l H \cap V = \emptyset$ and $\operatorname{Int} C l H$ is a regular open set. Since $f$ is almost $p_s$-continuous, by Theorem 2.7, there exists a $p_s$-open set $U$ in $X$ containing $x$ such that $f(U) \subseteq \operatorname{Int} C l H$. Therefore, $f(U) \cap V = \emptyset$. However, since $x \in P_s \operatorname{Cl} f^{-1}(V), U \cap f^{-1}(V) \neq \emptyset$ for every $p_s$-open set $U$ in $X$ containing $x$, so that $f(U) \cap V \neq \emptyset$. We have a contradiction. It follows that $f(x) \in P_s \operatorname{Cl} V$. □
Theorem 2.18. If a function \( f : X \rightarrow Y \) is almost precontinuous. Then the following statements are equivalent:

1. \( f \) is almost \( p_s \)-continuous.
2. For each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists a semi-closed set \( F \) in \( X \) containing \( x \) such that \( f(F) \subseteq \text{IntCl}V \).
3. For each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists a semi-closed set \( F \) in \( X \) containing \( x \) such that \( f(F) \subseteq sClV \).
4. For each \( x \in X \) and each regular open set \( V \) of \( Y \) containing \( f(x) \), there exists a semi-closed set \( F \) in \( X \) containing \( x \) such that \( f(F) \subseteq V \).
5. For each \( x \in X \) and each \( \delta \)-open set \( V \) of \( Y \) containing \( f(x) \), there exists a semi-closed set \( F \) in \( X \) containing \( x \) such that \( f(F) \subseteq V \).

Proof. (1) \( \Rightarrow \) (2). Let \( x \in X \) and let \( V \) be any open set of \( Y \) containing \( f(x) \). By (1), there exists a \( p_s \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq \text{IntCl}V \). Since \( U \) is \( p_s \)-open set, so for each \( x \in U \) there exists a semi-closed set \( F \) in \( X \) such that \( x \in F \subseteq U \). Therefore, we have \( f(F) \subseteq \text{IntCl}V \).

(2) \( \Rightarrow \) (3). Obvious.

(3) \( \Rightarrow \) (4). Let \( x \in X \) and let \( V \) be any regular open set of \( Y \) containing \( f(x) \). Then \( V \) is an open set of \( Y \) containing \( f(x) \). By (3), there exists a semi-closed set \( F \) in \( X \) containing \( x \) such that \( f(F) \subseteq sClV \). Since \( V \) is regular open and hence is preopen. Therefore, by Lemma 1.11(2), \( f(F) \subseteq \text{IntCl}V \). Since \( V \) is regular open, then \( f(F) \subseteq V \).

(4) \( \Rightarrow \) (5). Let \( x \in X \) and let \( V \) be any \( \delta \)-open set of \( Y \) containing \( f(x) \). Then for each \( f(x) \in V \), there exists an open set \( G \) containing \( f(x) \) such that \( G \subseteq \text{IntClG} \subseteq V \). Since \( \text{IntClG} \) is a regular open set of \( Y \) containing \( f(x) \), by (4), there exists a semi-closed set \( F \) in \( X \) containing \( x \) such that \( f(F) \subseteq \text{IntClG} \subseteq V \). This completes the proof.

(5) \( \Rightarrow \) (1). Let \( V \) be any \( \delta \)-open set of \( Y \). We have to show that \( f^{-1}(V) \) is \( p_s \)-open set in \( X \). Since \( f \) is almost precontinuous, by Proposition 1.14, \( f^{-1}(V) \) is preopen set in \( X \). Let \( x \in f^{-1}(V) \), then \( f(x) \in V \). By hypothesis, there exists a semi-closed set \( F \) of \( X \) containing \( x \) such that \( f(F) \subseteq V \). Which implies that \( x \in F \subseteq f^{-1}(V) \). Therefore, \( f^{-1}(V) \) is \( p_s \)-open set in \( X \). Hence by Theorem 2.10, \( f \) is almost \( p_s \)-continuous.

\( \square \)

Theorem 2.19. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is almost \( p_s \)-continuous if and only if \( f : (X, \tau) \rightarrow (Y, \sigma_s) \) is \( p_s \)-continuous.

Proof. Necessity. Let \( H \in \sigma_s \), then \( H \) is a regular open set in \( (Y, \sigma) \). Since \( f : (X, \tau) \rightarrow (Y, \sigma) \) is almost \( p_s \)-continuous, by Theorem 2.8, \( f^{-1}(H) \) is \( p_s \)-open set in \( X \). Therefore, \( f : (X, \tau) \rightarrow (Y, \sigma_s) \) is \( p_s \)-continuous.
Sufficiency. Let \( G \) be any regular open set in \((Y, \sigma)\). Then \( G \in \sigma_S\). Since \( f : (X, \tau) \to (Y, \sigma_S) \) is \( p_s \)-continuous, by Definition 1.13, \( f^{-1}(G) \) is \( p_s \)-open set in \( X \). Therefore, by Theorem 2.8, \( f : (X, \tau) \to (Y, \sigma) \) is almost \( p_s \)-continuous.

\begin{proof}
\begin{enumerate}
\item Necessity. Let \( H \in \sigma_S\), then \( H \) is a regular open set in \((Y, \sigma)\). Since \( f : (X, \tau) \to (Y, \sigma) \) is almost \( p_s \)-continuous, by Theorem 2.8, \( f^{-1}(H) \) is \( p_s \)-open set in \( X \). Since \( X \) is locally indiscrete space, by Proposition 1.7(4), \( f^{-1}(H) \) is open set in \( X \). Therefore, \( f : (X, \tau) \to (Y, \sigma_S) \) is continuous.
\item Sufficiency. Let \( G \) be any regular open set in \((Y, \sigma)\). Then \( G \in \sigma_S\). Since \( f : (X, \tau) \to (Y, \sigma_S) \) is continuous, so \( f^{-1}(G) \) is open set in \( X \). Since \( X \) is locally indiscrete space, by Proposition 1.7(4), \( f^{-1}(G) \) is \( p_s \)-open set in \( X \). Therefore, by Theorem 2.8, \( f : (X, \tau) \to (Y, \sigma) \) is almost \( p_s \)-continuous.
\end{enumerate}
\end{proof}

3. Properties and Comparisons

In this section, we give some properties of almost \( p_s \)-continuous functions and compare it with other types of continuous functions.

\begin{proposition}
Let \( f : X \to Y \) be an almost \( p_s \)-continuous function. If \( A \) is either open or regular semi-open subset of \( X \), then \( f|A : A \to Y \) is almost \( p_s \)-continuous in the subspace \( A \).
\end{proposition}

\begin{proof}
Let \( V \) be any regular open set of \( Y \). Since \( f \) is almost \( p_s \)-continuous, by Theorem 2.8, \( f^{-1}(V) \) is \( p_s \)-open set in \( X \). Since \( A \) is either open or regular semi-open subset of \( X \). By Lemma 1.10(4), \( (f|A)^{-1}(V) = f^{-1}(V) \cap A \) is a \( p_s \)-open subset of \( A \). This shows that \( f|A : A \to Y \) is almost \( p_s \)-continuous.
\end{proof}

\begin{corollary}
Let \( f : X \to Y \) be almost \( p_s \)-continuous function. If \( A \) is regular open subset of \( X \), then \( f|A : A \to Y \) is almost \( p_s \)-continuous in the subspace \( A \).
\end{corollary}

\begin{proof}
Follows from Proposition 3.1
\end{proof}

\begin{theorem}
a function \( f : X \to Y \) is almost \( p_s \)-continuous. If for each \( x \in X \), there exists a regular open set \( A \) of \( X \) containing \( x \) such that \( f|A : A \to Y \) is almost \( p_s \)-continuous.
\end{theorem}

\begin{proof}
Let \( x \in X \), then by hypothesis, there exists a regular open set \( A \) containing \( x \) such that \( f|A : A \to Y \) is almost \( p_s \)-continuous. Let \( V \) be any open set of \( Y \) containing \( f(x) \), there exists a \( p_s \)-open set \( U \) in \( A \) containing \( x \) such that \( (f|A)(U) \subseteq IntCIV \). Since \( A \) is regular open, by Lemma 1.10(3), \( U \) is \( p_s \)-open set in \( X \) and hence \( f(U) \subseteq IntCIV \). This shows that \( f \) is almost \( p_s \)-continuous.
\end{proof}
Corollary 3.4. Let \( \{U_\gamma : \gamma \in \Delta\} \) be a regular open cover of a topological space \( X \). A function \( f : X \to Y \) is almost \( p_s \)-continuous if and only if \( f|_{U_\gamma} : U_\gamma \to Y \) is almost \( p_s \)-continuous for each \( \gamma \in \Delta \).

Proof. This is an immediate consequence of Corollary 3.2 and Theorem 3.3. \( \square \)

Theorem 3.5. If \( X = R \cup S \), where \( R \) and \( S \) are regular open sets and \( f : X \to Y \) is a function such that both \( f|R \) and \( f|S \) are almost \( p_s \)-continuous, then \( f \) is almost \( p_s \)-continuous.

Proof. Let \( V \) be any regular open set of \( Y \). Then \( f^{-1}(V) = (f|R)^{-1}(V) \cup (f|S)^{-1}(V) \). Since \( f|R \) and \( f|S \) are almost \( p_s \)-continuous, by Theorem 2.8, \( (f|R)^{-1}(V) \) and \( (f|S)^{-1}(V) \) are \( p_s \)-open sets in \( R \) and \( S \), respectively. Since \( R \) and \( S \) are regular open sets in \( X \), then by Lemma 1.10(3), \( (f|R)^{-1}(V) \) and \( (f|S)^{-1}(V) \) are \( p_s \)-open sets in \( X \). Since a union of two \( p_s \)-open sets is \( p_s \)-open, hence \( f^{-1}(V) \) is \( p_s \)-open set in \( X \). Therefore, by Theorem 2.8, \( f \) is almost \( p_s \)-continuous. \( \square \)

In general, if \( X = \bigcup \{K_\gamma : \gamma \in \Delta\} \), where each \( K_\gamma \) is a regular open set and \( f : X \to Y \) is a function such that \( f|K_\gamma \) is almost \( p_s \)-continuous for each \( \gamma \), then \( f \) is almost \( p_s \)-continuous.

Theorem 3.6. Let \( f : X \to Y \) be almost \( p_s \)-continuous and let \( A \) be either open or regular semi-open subset of \( X \) such that \( f(A) \) is dense in \( Y \). Then \( f|A : A \to f(A) \) is almost \( p_s \)-continuous.

Proof. Let \( O \) be a regular open set of \( f(A) \). Then by Lemma 1.11(3), \( O = f(A) \cap \text{Int} \text{Cl} O \). Thus, \( (f|A)^{-1}(O) = (f|A)^{-1}(f(A) \cap \text{Int} \text{Cl} O) = (f|A)^{-1}(f(A)) \cap (f|A)^{-1}(\text{Int} \text{Cl} O) = A \cap f^{-1}(\text{Int} \text{Cl} O) = A \cap f^{-1}(O) \). Since \( f \) is almost \( p_s \)-continuous, by Theorem 2.8, \( f^{-1}(O) = f^{-1}(\text{Int} \text{Cl} O) \) is \( p_s \)-open in \( X \). Since \( A \) is either open or regular semi-open subset of \( X \). Then by Lemma 1.10(4), \( (f|A)^{-1}(O) \) is \( p_s \)-open set in the subspace \( A \). Therefore, by Theorem 2.8, \( f|A : A \to f(A) \) is almost \( p_s \)-continuous. \( \square \)

Theorem 3.7. Let \( X = R_1 \cup R_2 \), where \( R_1 \) and \( R_2 \) are regular open sets in \( X \). Let \( f : R_1 \to Y \) and \( g : R_2 \to Y \) be almost \( p_s \)-continuous. If \( f(x) = g(x) \) for each \( x \in R_1 \cap R_2 \). Then \( h : R_1 \cup R_2 \to Y \) such that \( h(x) = f(x) \) for \( x \in R_1 \) and \( h(x) = g(x) \) for \( x \in R_2 \) is almost \( p_s \)-continuous.

Proof. Let \( O \) be a regular open set of \( Y \). Now \( h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O) \). Since \( f \) and \( g \) are almost \( p_s \)-continuous, by Theorem 2.8, \( f^{-1}(O) \) and \( g^{-1}(O) \) are \( p_s \)-open set in \( R_1 \) and \( R_2 \) respectively. But \( R_1 \) and \( R_2 \) are both regular open sets in \( X \). Then by Lemma 1.10(3), \( f^{-1}(O) \) and \( g^{-1}(O) \) are \( p_s \)-open sets in \( X \). Since union of two \( p_s \)-open sets is \( p_s \)-open, so \( h^{-1}(O) \) is a \( p_s \)-open set in \( X \). Hence by Theorem 2.8, \( h \) is almost \( p_s \)-continuous. \( \square \)

Theorem 3.8. Let \( f : X \to Y \) be almost \( p_s \)-continuous surjection and \( A \) be an open subset of \( X \). If \( f \) is an open function, then the function \( g : A \to f(A) \), defined by \( g(x) = f(x) \) for each \( x \in A \), is almost \( p_s \)-continuous.
Proof. Suppose that $H = f(A)$. Let $x \in A$ and $V$ be any open set in $H$ containing $g(x)$. Since $H$ is open in $Y$ and $V$ is open in $H$, so $V$ is open in $Y$. Since $f$ is almost $p_s$-continuous, hence there exists a $p_s$-open set $U$ in $X$ containing $x$ such that $f(U) \subseteq Int Cl V$. Taking $W = U \cap A$, since $A$ is either open or regular semi-open subset of $X$, by Lemma 1.11(3), $W$ is a $p_s$-open set in $A$ containing $x$ and $g(W) \subseteq Int Cl Y \cap H = Int Cl H V$. Then $g(W) \subseteq Int Cl H V$. This shows that $g$ is almost $p_s$-continuous. □

Theorem 3.9. Let $f : X \to Y$ be almost $p_s$-continuous. If $Y$ is a preopen subset of $Z$, then $f : X \to Z$ is almost $p_s$-continuous.

Proof. Let $V$ be any regular open set of $Z$. Since $Y$ is preopen, by Lemma 1.11(1), $V \cap Y$ is a regular open set in $Y$. Since $f : X \to Y$ is almost $p_s$-continuous, by Theorem 2.8, $f^{-1}(V \cap Y)$ is a $p_s$-open set in $X$. But $f(x) \in Y$ for each $x \in X$. Thus $f^{-1}(V) = f^{-1}(V \cap Y)$ is a $p_s$-open set of $X$. Therefore, by Theorem 2.8, $f : X \to Z$ is almost $p_s$-continuous. □

Theorem 3.10. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition function $gof : X \to Z$ is almost $p_s$-continuous if $f$ and $g$ satisfy one of the following conditions:

1. $f$ is $p_s$-continuous and $g$ is almost continuous.
2. $f$ is almost $p_s$-continuous and $g$ is $\delta$-continuous.
3. $f$ is continuous and open and $g$ is almost $p_s$-continuous.
4. $f$ is almost $p_s$-continuous and $g$ is continuous and open.

Proof. (1). Let $W$ be any regular open subset of $Z$. Since $g$ is almost continuous, so $g^{-1}(W)$ is open subset of $Y$. Since $f$ is $p_s$-continuous, by Definition 1.13, $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is $p_s$-open subset in $X$. Therefore, by Theorem 2.8, $gof$ is almost $p_s$-continuous.

(2). Let $W$ be any $\delta$-open subset of $Z$. Since $g$ is $\delta$-continuous, so $g^{-1}(W)$ is $\delta$-open subset of $Y$. Since $f$ is almost $p_s$-continuous, by Theorem 2.10, $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is $p_s$-open subset in $X$. Therefore, by Theorem 2.10, $gof$ is almost $p_s$-continuous.

(3). Let $W$ be any regular open subset of $Z$. Since $g$ is almost $p_s$-continuous, by Theorem 2.8, $g^{-1}(W)$ is $p_s$-open subset of $Y$. Since $f$ is continuous and open, by Theorem 1.16, $f^{-1}(g^{-1}(W)) = (gof)^{-1}(W)$ is a $p_s$-open set in $X$. Hence by Theorem 2.8, $gof$ is almost $p_s$-continuous.

(4). Let $x \in X$ and $W$ be an open set of $Z$ containing $g(f(x))$. Since $g$ is continuous, then $g^{-1}(W)$ is an open set of $Y$ containing $f(x)$. Since $f$ is almost $p_s$-continuous, there exists a $p_s$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq Int Cl g^{-1}(W)$. Also, since $g$ is continuous, then we obtain $(gof)(U) \subseteq g(Int g^{-1}(Cl W))$. Since $g$ is open, we obtain $(gof)(U) \subseteq Int Cl W$. Therefore, $gof$ is almost $p_s$-continuous. □
Theorem 3.11. If \( f : X \to Y \) is almost \( p_s \)-continuous and \( g : Y \to Z \) is super continuous functions. Then the composition function \( g \circ f : X \to Z \) is \( p_s \)-continuous.

Proof. Let \( W \) be any open subset of \( Z \). Since \( g \) is super continuous, then \( g^{-1}(W) \) is \( \delta \)-open subset of \( Y \). Since \( f \) is almost \( p_s \)-continuous, by Theorem 2.10, \( (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \) is \( p_s \)-open subset in \( X \). Therefore, by Definition 1.13, \( g \circ f \) is \( p_s \)-continuous. \( \square \)

Theorem 3.12. If \( f : X \to Y \) is an almost \( p_s \)-continuous function and \( Y \) is semi-regular. Then \( f \) is \( p_s \)-continuous.

Proof. Let \( x \in X \) and let \( V \) be any open set of \( Y \) containing \( f(x) \). By the semi-regularity of \( Y \), there exists a regular open set \( G \) of \( Y \) such that \( f(x) \in G \subseteq V \). Since \( f \) is almost \( p_s \)-continuous, by Theorem 2.7, there exists a \( p_s \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq G \subseteq V \). Therefore, \( f \) is \( p_s \)-continuous. \( \square \)

Corollary 3.13. Let \( f : X \to Y \) be a function and \( X \) is locally indiscrete space. Then \( f \) is almost \( p_s \)-continuous if and only if \( f \) is almost continuous.

Proof. Follows from the definition and Proposition 1.7(4). \( \square \)

Corollary 3.14. If \( X \) is a locally indiscrete space and \( Y \) is semi-regular space, then the following statements are equivalent for a function \( f : X \to Y \):

1. \( f \) is \( p_s \)-continuous.
2. \( f \) is almost \( p_s \)-continuous.
3. \( f \) is almost continuous.
4. \( f \) is continuous.

Proof. Follows from Theorem 3.12, Corollary 3.13 and Proposition 1.7(4). \( \square \)

Corollary 3.15. Let \( f : X \to Y \) be a function and \( X \) is s-regular space. If \( f \) is almost continuous, then \( f \) is almost \( p_s \)-continuous.

Proof. Proof. Follows from Proposition 1.7(5). \( \square \)

Corollary 3.16. Let \( f : X \to Y \) be a function and \( X \) is semi-T\(_1\) space. Then \( f \) is almost \( p_s \)-continuous if and only if \( f \) is almost precontinuous.

Proof. Proof. Follows from Proposition 1.7(1). \( \square \)

Corollary 3.17. If \( X \) is a semi-T\(_1\) space and \( Y \) is semi-regular space, then the following statements are equivalent for a function \( f : X \to Y \):
(1) \( f \) is \( p_s \)-continuous.

(2) \( f \) is almost \( p_s \)-continuous.

(3) \( f \) is almost precontinuous.

(4) \( f \) is precontinuous.

**Proof.** Follows from Theorem 3.12, Corollary 3.16 and Proposition 1.7(1).

**Theorem 3.18.** If a function \( f : X \rightarrow Y \) is almost strongly \( \theta \)-continuous, then \( f \) is almost \( p_s \)-continuous.

**Proof.** Let \( V \) be any regular open set of \( Y \). Since \( f \) is almost strongly \( \theta \)-continuous, so \( f^{-1}(V) \) is \( \theta \)-open set and hence it is \( p_s \)-open set. Therefore, by Theorem 2.8, \( f \) is almost \( p_s \)-continuous.

**Theorem 3.19.** The following statements are equivalent for a function \( f : X \rightarrow Y \):

(1) \( f \) is R-map.

(2) \( f \) is almost \( p_s \)-continuous and either weakly \( \theta \)-irresolute or weakly quasi-continuous.

(3) \( f \) is almost continuous and either weakly \( \theta \)-irresolute or weakly quasi-continuous.

(4) \( f \) is almost \( \alpha \)-continuous and either weakly \( \theta \)-irresolute or weakly quasi-continuous.

(5) \( f \) is almost precontinuous and either weakly \( \theta \)-irresolute or weakly quasi-continuous.

**Proof.** Follows from their definitions and Proposition 1.9.

**Theorem 3.20.** If \( f : X \rightarrow Y \) is almost \( \theta s \)-continuous and almost precontinuous, then \( f \) is almost \( p_s \)-continuous.

**Proof.** Let \( V \) be a regular open set in \( Y \). Since \( f \) is almost \( \theta s \)-continuous and almost precontinuous function, then \( f^{-1}(V) \) is both \( \theta \)-semi-open and pre-open set in \( X \). Therefore, \( f^{-1}(V) \) is \( p_s \)-open set in \( X \). Hence by Theorem 2.8, \( f \) is almost \( p_s \)-continuous.

**Theorem 3.21.** Let \( f : X \rightarrow Y \) be a function and \( X \) be extremally disconnected space. If \( f \) is almost \( \theta s \)-continuous, then \( f \) is almost \( p_s \)-continuous.

**Proof.** Let \( V \) be a regular open set in \( Y \). Since \( f \) is almost \( \theta s \)-continuous, then \( f^{-1}(V) \) is \( \theta \)-semi-open set in \( X \). Therefore, by Lemma 1.10, \( f^{-1}(V) \) is \( p_s \)-open set in \( X \). Hence by Theorem 2.8, \( f \) is almost \( p_s \)-continuous.

**Theorem 3.22.** Let \( Y \) be an extremally disconnected space. If \( f : X \rightarrow Y \) is an almost precontinuous and either \( S \)-continuous or \( \theta \)-irresolute function, then \( f \) is almost \( p_s \)-continuous.
Proof. Let $V$ be a regular open set in $Y$. Since $f$ is almost precontinuous, then $f^{-1}(V)$ is preopen set of $X$. Since $Y$ is extremally disconnected space, by Proposition 1.7(2), $V$ is regular closed set of $Y$. Since $f$ is either $S$-continuous or $\theta$- irresolute, then $f^{-1}(V)$ is the union of regular closed sets of $X$ and hence is the union of semi-closed sets of $X$. By Proposition 1.3, $f^{-1}(V)$ is $p_s$-open set of $X$. Therefore, by Theorem 2.8, $f$ is almost $p_s$-continuous. □

Corollary 3.23. Let $f : X \to Y$ be either $S$-continuous or $\theta$- irresolute function and $Y$ be an extremally disconnected space. Then $f$ is almost $p_s$-continuous if and only if $f$ is almost pre-continuous.

Theorem 3.24. If $Y$ is a hyperconnected space, then every function $f : X \to Y$ is almost $p_s$-continuous.

Proof. Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Since $Y$ is a hyperconnected space, then $ClV = Y$ and hence $IntClV = Y$. Therefore, we have $f(U) \subseteq IntClV$, for any $p_s$-open set $U$ in $X$. This shows that $f$ is almost $p_s$-continuous. □

Theorem 3.25. If a function $f : X \to Y$ is almost $p_s$-continuous and semi-open, then $f(P_sCl(V)) \subseteq P_sClf(V)$ for each open set $V$ of $X$.

Proof. Let $V$ be any open set of $X$. Since $f$ is semi-open, then $f(V)$ is semi-open set in $Y$. Since $f$ is almost $p_s$-continuous, then by Corollary 2.12, we obtain that $P_sClf^{-1}(f(V)) \subseteq f^{-1}(P_sClf(V))$ which implies that $f(P_sClV) \subseteq P_sClf(V)$. □

Corollary 3.26. If a function $f : X \to Y$ is almost $p_s$-continuous and semi-open, then $P_sIntf(F) \subseteq f(P_sIntF)$ for each closed set $F$ of $X$.

Theorem 3.27. If a function $f : X \to Y$ is irresolute and preopen. Then $f$ is almost $p_s$-continuous if and only if $P_sClf^{-1}(V) = f^{-1}(P_sClV)$ for each semi-open set $V$ of $Y$.

Proof. Necessity. Let $V$ be any semi-open set of $Y$. Since $f$ is almost $p_s$-continuous, by Corollary 2.12, $P_sClf^{-1}(V) \subseteq f^{-1}(P_sClV)$. Since $V$ is semi-open set of $Y$, by Lemma 1.12, $P_sClV = ClV$ which implies that $f^{-1}(P_sClV) \subseteq f^{-1}(ClV)$. Therefore, by Theorem 1.17, we have $f^{-1}(P_sClV) \subseteq f^{-1}(ClV) \subseteq Clf^{-1}(V)$ and hence $P_sClf^{-1}(V) \subseteq Clf^{-1}(V)$. Since $V$ is semi-open set of $X$ and $f$ is irresolute, so $f^{-1}(V)$ is semi-open in $X$. Then by Lemma 1.12, we obtain that $f^{-1}(P_sClV) \subseteq P_sClf^{-1}(V)$. Therefore, we have $P_sClf^{-1}(V) = f^{-1}(P_sClV)$. Sufficiency. Follows from Theorem 2.12. □

From the above theorem and Lemma 1.12 we obtain the following results:

Corollary 3.28. If a function $f : X \to Y$ is almost $p_s$-continuous, irresolute and preopen, then $P_sIntf^{-1}(F) = f^{-1}(P_sIntF)$ for each semi-closed set $F$ of $Y$.

Corollary 3.29. If a function $f : X \to Y$ is almost $p_s$-continuous, irresolute and preopen, then $Clf^{-1}(V) = f^{-1}(ClV)$ for each semi-open set $V$ of $Y$. 
References


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