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AN UNIVALENT CONDITION FOR A FAMILY OF INTEGRAL OPERATORS

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Abstract. The object of the present paper is to derive an univalent condition for a family of integral operators.

1. Introduction

Let A be the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$U = \{z : z \in C \text{ and } |z| < 1\}.$$

We denote by S the subclass of A consisting of functions which are also univalent in U. For some recent investigations of various interesting subclasses of the univalent function class S, see the works by (for example) Altintas et al. [1], Gao et al. [5], and Owa et al. [6].

The univalent conditions involving the integral operators have been studied by several authors (see, e.g., [2, 3, 4, 7, 8]). In [7] Pescar obtained the following:

Theorem 1.(see [7]). *Let*

$$\alpha \in C \quad (Re\alpha > 0)$$

and

$$c \in C \quad (|c| \leq 1; c \neq -1).$$

Suppose also that the function $f(z)$ given by (1.1) *is analytic in U. If*

$$\left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1 \quad (z \in U), \tag{1.2}$$

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then the function $F_\alpha(z)$ *defined by*

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t)\,dt\right)^{\frac{1}{\alpha}} = z + \cdots \tag{1.3}$$

is analytic and univalent in U.

Very recently, another univalent condition associated with the general family of integral operators was given by Breaz et al. [4] as follows.

Theorem 2. (see [4]) *Let* $M \geq 1$ *and suppose that each of the functions* $g_j \in A$ $(j = 1, 2, \cdots, n)$ *satisfies the inequality*

$$\left|\frac{z^2 g_j'(z)}{[g_j(z)]^2} - 1\right| \leq 1 \quad (z \in U). \tag{1.4}$$

Also let

$$\alpha \in R \quad \left(\alpha \in \left[1, \frac{(2M+1)n}{(2M+1)n-1}\right]\right) \quad \textit{and} \quad c \in C.$$

If

$$|c| \leq 1 + \left(\frac{1-\alpha}{\alpha}\right)(2M+1)n$$

and

$$|g_j(z)| \leq M \quad (z \in U; j = 1, 2, \cdots, n),$$

then the function $G_{n,\alpha}(z)$ *defined by*

$$G_{n,\alpha}(z) = \left([n(\alpha-1)+1] \int_0^z [g_1(t)]^{\alpha-1} \cdots [g_n(t)]^{\alpha-1}\,dt\right)^{\frac{1}{n(\alpha-1)+1}} \quad (g_1, \cdots, g_n \in A) \tag{1.5}$$

is in the univalent function class S.

In this note we propose to investigate further univalent conditions involving the general family of integral operators defined by (1.5).

2. The main univalent condition

Theorem 3. *Let*

$$\alpha \in C \quad (Re\alpha > 0)$$

and

$$c \in C \quad (|c| < 1).$$

Suppose that each of the functions $g_j \in A$ $(j = 1, 2, \cdots, n)$ *satisfies the condition*

$$\left|\frac{z g_j'(z)}{g_j(z)} - 1\right| \leq \frac{|\alpha|}{n(1+|\alpha|)}(1-|c|), \tag{1.6}$$

then the function $G_{n,\alpha}(z)$ *defined by* (1.5) *is in the univalent function class* S.

Proof. From (1.6) and the hypothesis of the theorem, one can see that

$$\left|\frac{zg_j'(z)}{g_j(z)}-1\right|\leq 1 \quad (z\in U; j=1,2,\cdots,n),$$

which indicates that each function g_j ($j=1,2,\cdots,n$) is an univalent starlike function. So the function $G_{n,\alpha}(z)$ in (1.5) is well defined.

Put

$$f(z)=\int_0^z\prod_{j=1}^n\left(\frac{g_j(t)}{t}\right)^{\alpha-1}dt. \tag{1.7}$$

Then it follows from (1.7) that

$$f'(z)=\prod_{j=1}^n\left(\frac{g_j(z)}{z}\right)^{\alpha-1} \tag{1.8}$$

and

$$f''(z)=(\alpha-1)\sum_{j=1}^n\left(\left(\frac{g_j(z)}{z}\right)^{\alpha-2}\left(\frac{zg_j'(z)-g_j(z)}{z^2}\right)\cdot\prod_{k=1(k\neq j)}^n\left(\frac{g_k(z)}{z}\right)^{\alpha-1}\right). \tag{1.9}$$

We thus find from (1.8) and (1.9) that

$$\frac{zf''(z)}{f'(z)}=(\alpha-1)\sum_{j=1}^n\left(\frac{zg_j'(z)}{g_j(z)}-1\right),$$

which readily shows that

$$\begin{aligned}\left|c|z|^{2\alpha}+(1-|z|^{2\alpha})\frac{zf''(z)}{\alpha f'(z)}\right| &= \left|c|z|^{2\alpha}+(1-|z|^{2\alpha})\frac{\alpha-1}{\alpha}\sum_{j=1}^n\left(\frac{zg_j'(z)}{g_j(z)}-1\right)\right|\\ &\leq |c|+\frac{1+|\alpha|}{|\alpha|}\sum_{j=1}^n\left|\frac{zg_j'(z)}{g_j(z)}-1\right|.\end{aligned}$$

Now by using the hypothesis (1.6), we obtain

$$\left|c|z|^{2\alpha}+(1-|z|^{2\alpha})\frac{zf''(z)}{\alpha f'(z)}\right|\leq 1 \quad (z\in U).$$

Finally, by applying Theorem 1, we conclude that the function $G_{n,\alpha}(z)$ defined by (1.5) is in the univalent function class S. This evidently completes the proof of the theorem.

Setting $c=0$ in Theorem 3, we immediately arrive at the following application of Theorem 3.

Corollary 1. *Let* $\alpha\in C$ ($Re\alpha>0$) *and each of the functions* $g_j\in A$ ($j=1,2,\cdots,n$) *satisfy the condition*

$$\left|\frac{zg_j'(z)}{g_j(z)}-1\right|\leq\frac{|\alpha|}{n(1+|\alpha|)},$$

then the function $G_{n,\alpha}(z)$ *defined by* (1.5) *is in the univalent function class* S.

References

[1] O. Altintas, H. Irmak, S. Owa and H. M. Srivastava, *Coefficient bounds for some families of starlike and convex functions of complex order*, Appl. Math. Lett., **20**(2007), 1218–1222.

[2] D. Breaz, Integral Operators on Spaces of Univalent Functions, Publishing House of the Romanian Academy of Sciences, Bucharest, in Romanian, 2004.

[3] D. Breaz and N. Breaz, *Univalence of an integral operator*, *Mathematica*(Cluj) **47**(70)(2005), 35–38.

[4] D. Breaz, N. Breaz and H. M. Srivastava, *An extension of the univalent condition for a family of integral operators*, Appl. Math. Lett., **22**(2009), 41–44.

[5] C.-Y. Gao, S.-M. Yuan and H. M. Srivastava, *Some functional inequalities and inclusion relationships associated with certain families of integral operators*, Comput. Math. Appl., **49**(2005), 1787–1795.

[6] S. Owa, M. Nunokawa, H. Saitoh and H. M. Srivastava, *Close-to-convexity of certain analytic functions*, Appl. Math. Lett., **15**(2002), 63–69.

[7] V. Pescar, *A new generalization of Ahlfors's and Becker's criterion of univalence*, Bull. Malaysian Math. Soc. (Ser.2) **19**(1996), 53–54.

[8] V. Pescar, *On the univalence of some integral operators*, J. Indian Acad. Math., **27**(2005), 239-243.

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