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SOME SUBCLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER DEFINED BY NEW DIFFERENTIAL OPERATOR

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Abstract. Let $\mathscr{A}(n)$ denote the class of analytic functions f in the open unit disk $U=\{z:|z|<1\}$ normalized by $f(0)=f'(0)-1=0$. In this paper, we introduce and study the classes $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ of functions $f\in\mathscr{A}(n)$ with $(\mu)z(D_\lambda^{\mho+2}(\alpha,\omega)f(z))'+(1-\mu)z(D_\lambda^{\mho+1}(\alpha,\omega)f(z))'\neq 0$ and satisfy some conditions available in literature, where $f\in\mathscr{A}(n),\alpha,\omega,\lambda,\mu\geq 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U$, and $D_\lambda^m(\alpha,\omega)f(z):\mathscr{A}\to\mathscr{A}$, is the linear fractional differential operator, newly defined as follows

$$D_\lambda^m(\alpha,\omega)f(z)=z+\sum_{k=2}^{\infty}a_k(1+(k-1)\lambda\omega^\alpha)^m z^k.$$

Several properties such as coefficient estimates, growth and distortion theorems, extreme points, integral means inequalities and inclusion for the functions included in the classes $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ are given.

1. Introduction and preliminaries

Let $\mathscr{H}$ be the class of functions analytic in $U=\{z:|z|<1\}$ and let $\mathscr{H}[a,n]$ be the subclass of $\mathscr{H}$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\cdots$. Let $\mathscr{A}$ be the subclass of $\mathscr{H}$ consisting of functions of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$ or

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k. \tag{1.1}$$

Let $\mathscr{A}(n)$ denote the class of functions $f(z)$ of the form

$$f(z)=z-\sum_{k=n}^{\infty}a_{k+1}z^{k+1} \tag{1.2}$$

$$a_{k+1}\geq 0,\ n\in\{1,2,3,\ldots\},$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$.

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Next we define (n,δ)-neighborhood for the functions belonging to class $\mathscr{A}(n)$ and also for identity function.

Definition 1.1. (n,δ)-neighborhood·

By following the earlier investigations by Goodman[2] and Ruscheweyh[30], for any $f(z) \in \mathscr{A}(n)$ and $\delta \geq 0$, we define the (n,δ)-neighborhood of f by

$$N_{n,\delta}(f) = \{g \in \mathscr{A}(n) : g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \text{ and } \sum_{k=n}^{\infty} (k+1)|a_{k+1} - b_{k+1}| \leq \delta\}. \tag{1.3}$$

In particular for the identity function

$$e(z) = z,$$

we have

$$N_{n,\delta}(e) = \{g \in \mathscr{A}(n) : g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \text{ and } \sum_{k=n}^{\infty} (k+1)|b_{k+1}| \leq \delta\}. \tag{1.4}$$

We say that a function $f(z) \in \mathscr{A}(n)$ is said to be starlike functions of complex order γ or $f(z) \in S_n^*(\gamma)$ if it satisfies the inequality

$$Re(1 + \frac{1}{\gamma}(\frac{zf'(z)}{f(z)} - 1)) > 0 \; z \in U, \; \gamma \in \mathbb{C} \setminus \{0\}. \tag{1.5}$$

Furthermore, a function $f(z) \in \mathscr{A}(n)$ is said to be convex functions of complex order γ or $f(z) \in C_n^*(\gamma)$ if it satisfies the inequality

$$Re(1 + \frac{1}{\gamma}(\frac{zf''(z)}{f'(z)})) > 0 \; z \in U, \; \gamma \in \mathbb{C} \setminus \{0\}. \tag{1.6}$$

The classes $S_n^*(\gamma)$ and $C_n^*(\gamma)$ are essentially from the classes of starlike and convex functions of complex order, which were considered by Nasr and Aouf[12] and Wiatrowski[23] respectively (Refer also [22]). Let $S_n(\gamma,\lambda,\beta)$ denote the subclass of $\mathscr{A}(n)$ consisting of functions $f(z)$ which satisfy the following inequality

$$|\frac{1}{\gamma}(\frac{\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)} - 1)| < \beta$$

$$(z \in U, \; \gamma \in \mathbb{C} \setminus \{0\}, \; 0 \leq \lambda \leq 1, \; 0 < \beta \leq 1).$$

Let $R_n(\gamma,\lambda,\beta)$ denote the subclass of $\mathscr{A}(n)$ consisting of functions $f(z)$ which satisfy the following inequality

$$|\frac{1}{\gamma}(\lambda z^2 f'''(z) + (1+2\lambda)zf''(z) + f'(z) - 1)| < \beta$$

$$(z \in U, \; \gamma \in \mathbb{C} \setminus \{0\}, \; 0 \leq \lambda \leq 1, \; 0 < \beta \leq 1).$$

The class $S_n(\gamma,\lambda,\beta)$ was studied by [15]. Since $\mathscr{A}$ is class of functions $f(z)$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For a function f in $\mathscr{A}$ we define the following differential operator

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1_\lambda(\alpha,\omega) f(z) &= (1-\lambda\omega^\alpha) f(z) + (\lambda\omega^\alpha) z f'(z), \\ D^2_\lambda(\alpha,\omega) f(z) &= D(D^1_\lambda(\alpha,\omega) f(z)), \\ &\vdots \\ D^m_\lambda(\alpha,\omega) f(z) &= D(D^{m-1}_\lambda(\alpha,\omega) f(z)). \end{aligned} \tag{1.7}$$

If f is given by (1.1) then from (1.7) we define the following new differential operator

$$D^m_\lambda(\alpha,\omega) f(z) = z + \sum_{k=2}^{\infty} a_k (1+(k-1)\lambda\omega^\alpha)^m z^k, \tag{1.8}$$

$$(f \in \mathscr{A}, \alpha, \omega, \lambda \geq 0)$$

which generalizes many differential operators. Indeed, if in the definition of $D^n_\lambda(\alpha,\omega) f(z)$ we put

- $\alpha = 1$, $\omega = 1$, we obtain $D^m_\lambda f(z) = z + \sum_{k=2}^{\infty} (1+\lambda(k-1))^m a_k z^k$ Al-Oboudi differential operator [6].
- $\alpha = 1$, $\omega = 1$, $\lambda = 1$, we obtain $D^m f(z) = z + \sum_{k=2}^{\infty} (k)^m a_k z^k$ Sălăgean's differential operator [7].
- $\alpha = 1$, $\omega = 1$, $\lambda = \frac{1}{2}$, we obtain $I^m f(z) = z + \sum_{k=2}^{\infty} (\frac{k+1}{2})^m a_k z^k$ Uralegaddi and Somanatha differential operator [3].

Similarly by using the same process we can write the following equalities for the functions $f(z)$ belonging to the class $\mathscr{A}(n)$.

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1_\lambda(\alpha,\omega) f(z) &= (1-\lambda\omega^\alpha) f(z) + (\lambda\omega^\alpha) z f'(z), \\ D^2_\lambda(\alpha,\omega) f(z) &= D(D^1_\lambda(\alpha,\omega) f(z)), \\ &\vdots \\ D^{\mho}_\lambda(\alpha,\omega) f(z) &= D(D^{\mho-1}_\lambda(\alpha,\omega) f(z)). \end{aligned} \tag{1.9}$$

If f is given by (1.2) then from (1.9) we define the following new differential operator

$$D^{\mho}_\lambda(\alpha,\omega) f(z) = z - \sum_{k=n}^{\infty} (1+k\lambda\omega^\alpha)^{\mho} a_{k+1} z^{k+1}, \qquad (f \in \mathscr{A}(n), \alpha, \omega, \lambda \geq 0, \mho \in \mathbb{N} \cup \{0\}). \tag{1.10}$$

Finally, in the terms of the generalized Sălăgean's differential operator, let $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$|\frac{1}{\gamma}(\frac{(\mu)z(D_{\lambda}^{\mho+3}(\alpha,\omega)f(z))'+(1-\mu)z(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'}{(\mu)z(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'+(1-\mu)z(D_{\lambda}^{\mho+1}(\alpha,\omega)f(z))'}-1)|<\beta, \tag{1.11}$$

$$(f\in\mathcal{A}(n),\ \gamma\in\mathbb{C}\setminus\{0\},\ \alpha,\omega,\lambda,\mu\geq 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Also, let $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$|\frac{1}{\gamma}\{\mu(D_{\lambda}^{\mho+3}(\alpha,\omega)f(z))'+(1-\mu)(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'-1\}|<\beta, \tag{1.12}$$

$$(f\in\mathcal{A}(n),\ \gamma\in\mathbb{C}\setminus\{0\},\ \alpha,\omega,\lambda,\mu\geq 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Our main work is to investigate the (n,δ)-neighborhood of the above said classes i.e. $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$. Similar work has been seen for different subclasses done by other authors (see for example [17, 18, 19].

2. Inclusion relations involving the $N_{n,\delta}(e)$-neighborhood

Here we proved the class relations as well as inclusion relations involving $N_{n,\delta}(e)$-neighborhood for the subclasses $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ which depends on the following lemmas.

Lemma 2.1. *Let the function $f(z)\in\mathcal{A}(n)$ be defined by (1.2), then $f(z)$ is in the class $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ if and only if*

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})a_{k+1}\leq\beta|\gamma|, \tag{2.1}$$

$$(f\in\mathcal{A}(n),\alpha,\omega,\lambda,\mu\geq 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Proof. Let $f(z)\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$, then from (1.11), we have

$$|\frac{1}{\gamma}(\frac{(\mu)z(D_{\lambda}^{\mho+3}(\alpha,\omega)f(z))'+(1-\mu)z(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'}{(\mu)z(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'+(1-\mu)z(D_{\lambda}^{\mho+1}(\alpha,\omega)f(z))'}-1)|<\beta,$$

$$(f\in\mathcal{A}(n),\alpha,\omega,\lambda,\mu\geq 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Or

$$Re(\frac{(\mu)z(D_{\lambda}^{\mho+3}(\alpha,\omega)f(z))'+(1-\mu)z(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'}{(\mu)z(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'+(1-\mu)z(D_{\lambda}^{\mho+1}(\alpha,\omega)f(z))'}-1)>-\beta|\gamma|,$$

$$(f\in\mathcal{A}(n),\alpha,\omega,\lambda,\mu\geq 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

This implies that

$$Re\left(\frac{-\sum_{k=n}^{\infty} k\lambda\omega^{\alpha}(1+\mu k\lambda\omega^{\alpha})(k+1)(1+k\lambda\omega^{\alpha})^{\mho+1}a_{k+1}z^{k+1}}{z-\sum_{k=n}^{\infty}(1+\mu k\lambda\omega^{\alpha})(k+1)(1+k\lambda\omega^{\alpha})^{\mho+1}a_{k+1}z^{k+1}}\right) > -\beta|\gamma|,$$

$$(f\in\mathscr{A}(n), \alpha,\omega,\lambda,\mu\geq 0, \mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

after taking limit when $z\to 1^{-}$ and simplifying we get

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})a_{k+1}\leq\beta|\gamma|,$$

$$(f\in\mathscr{A}(n), \alpha,\omega,\lambda,\mu\geq 0, \mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Conversely, by applying the hypothesis (2.1) and letting $|z|=1$ we get

$$\begin{aligned}
&\left|\frac{\mu z(D_{\lambda}^{\mho+3}(\alpha,\omega)f(z))'+(1-\mu)z(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'}{\mu z(D_{\lambda}^{\mho+2}(\alpha,\omega)f(z))'+(1-\mu)z(D_{\lambda}^{\mho+1}(\alpha,\omega)f(z))'}-1\right|\\
&\quad=\left|\left(\frac{-\sum_{k=n}^{\infty} k\lambda\omega^{\alpha}(1+\mu k\lambda\omega^{\alpha})(k+1)(1+k\lambda\omega^{\alpha})^{\mho+1}a_{k+1}z^{k+1}}{z-\sum_{k=n}^{\infty}(1+\mu k\lambda\omega^{\alpha})(k+1)(1+k\lambda\omega^{\alpha})^{\mho+1}a_{k+1}z^{k+1}}\right)\right|\\
&\qquad(f\in\mathscr{A}(n), \alpha,\omega,\lambda,\mu\geq 0, \mho\in\mathbb{N}\cup\{0\},\ z\in U).\\
&\quad\leq\left|\frac{\beta|\gamma|(1-\sum_{k=n}^{\infty}(1+\mu k\lambda\omega^{\alpha})(k+1)(1+k\lambda\omega^{\alpha})^{\mho+1}a_{k+1})}{1-\sum_{k=n}^{\infty}(1+\mu k\lambda\omega^{\alpha})(k+1)(1+k\lambda\omega^{\alpha})^{\mho+1}a_{k+1}}\right|\\
&\quad\leq\beta|\gamma|.
\end{aligned}$$

This implies that $f(z)\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$. □

Corollary 2.1. *Let the function f which is defined by (1.2) be in the class $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$. Then we have*

$$a_{k+1}\leq\frac{\beta|\gamma|}{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})},\ k\geq n,$$

$$(\alpha,\omega,\lambda,\mu\geq 0, \mho\in\mathbb{N}\cup\{0\}).$$

Lemma 2.2. *Let the function $f(z)\in\mathscr{A}(n)$ be defined by (1.2) then $f(z)$ is in the class $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ if and only if*

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+2}(2+\mu k\lambda\omega^{\alpha})(k+1)a_{k+1}\leq\beta|\gamma|, \tag{2.2}$$

$$(f\in\mathscr{A}(n), \alpha,\omega,\lambda,\mu\geq 0, \mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Proof. Same as Lemma 2.1. □

Theorem 2.3. *Let $f \in \mathscr{A}(n)$ then*

$$S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho) \subset N_{n,\delta}(e)$$

if

$$\delta = \frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(n\mu\lambda\omega^{\alpha}+1)(n\lambda\omega^{\alpha}+\beta|\gamma|)} \tag{2.3}$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Proof. Let $f(z) \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ then from (2.1) we get

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})a_{k+1} \leq \beta|\gamma|,$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Or

$$(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})\sum_{k=n}^{\infty}a_{k+1} \leq \beta|\gamma|,$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

This implies that

$$\sum_{k=n}^{\infty}a_{k+1} \leq \frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})} \tag{2.4}$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

By using (2.1) we have

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(k\lambda\omega^{\alpha}+1-1+\beta|\gamma|)a_{k+1} \leq \beta|\gamma|,$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Therefore

$$\begin{aligned}
&(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n\lambda\omega^{\alpha}+1)\sum_{k=n}^{\infty}(k+1)a_{k+1}\\
&\leq \beta|\gamma|+(1-\beta|\gamma|)(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)\sum_{k=n}^{\infty}a_{k+1},\\
&\leq \beta|\gamma|+\\
&(1-\beta|\gamma|)(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)\frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})},\\
&\leq \frac{\beta|\gamma|(1+n\lambda\omega^{\alpha})}{(\beta|\gamma|+n\lambda\omega^{\alpha})}, \qquad (f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).
\end{aligned}$$

Hence
$$\sum_{k=n}^{\infty}(k+1)a_{k+1}\leq\frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(\beta|\gamma|+n\lambda\omega^{\alpha})}=\delta.$$
Hence by using (1.4), we conclude that $f(z)\in N_{n,\delta}(e)$, this implies that

$$S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)\subset N_{n,\delta}(e)$$ □

Using the same technique of the proof of Theorem 2.3, we proved the following theorem.

Theorem 2.4. *Let $f\in\mathscr{A}(n)$, then*

$$R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)\subset N_{n,\delta}(e)$$

if

$$\delta=\frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+2}(2+n\mu\lambda\omega^{\alpha})}\qquad(f\in\mathscr{A}(n),\alpha,\omega,\lambda,\mu\geq0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).\tag{2.5}$$

Proof. Same as Theorem 2.3. □

3. Neighborhood properties for $S^{\tau}_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ and $R^{\tau}_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$.

In this section, we determine the neighborhood for each of the classes

$$S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho),\text{ and }R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho).$$

A function $f(z)\in\mathscr{A}(n)$ is said to be in the class $S^{\tau}_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$, if there exists a function $g(z)\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\tau,\ z\in U,\ \tau\geq0\cdot\tag{3.1}$$

Similarly A function $f(z)\in\mathscr{A}(n)$ is said to be in the class $R^{\tau}_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$, if there exists a function $g(z)\in R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ satisfying the same inequality

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\tau,\ z\in U,\ \tau\geq0.$$

Theorem 3.5. *Let $g\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$, and*

$$\tau=1-\frac{\delta(1+n\lambda\omega^{\alpha})^{\mho+1}(1+n\mu\lambda\omega^{\alpha})(\beta|\gamma|+n\lambda\omega^{\alpha})}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+n\mu\lambda\omega^{\alpha})(\beta|\gamma|+n\lambda\omega^{\alpha})-\beta|\gamma|}\tag{3.2}$$

$$(f\in\mathscr{A}(n),\alpha,\omega,\lambda,\mu\geq0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Then

$$N_{n,\delta}(g)\subset S^{\tau}_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho).$$

Proof. Let $f \in N_{n,\delta}(g)$, then from (1.3) we can write that

$$\sum_{k=n}^{\infty}(k+1)|a_{k+1}-b_{k+1}| \le \delta,$$

this implies that

$$\sum_{k=n}^{\infty}|a_{k+1}-b_{k+1}| \le \frac{\delta}{(n+1)}.$$

Since it is given that $g \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$, so from (2.1) we can write that

$$\sum_{k=n}^{\infty} b_{k+1} \le \frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}.$$

Now

$$\begin{aligned}
|\frac{f(z)}{g(z)}-1| &< \frac{\sum_{k=n}^{\infty}|a_{k+1}-b_{k+1}|}{1-\sum_{k=n}^{\infty}b_{k+1}},\\
&\le \frac{\delta}{(n+1)}\cdot\frac{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})-\beta|\gamma|},\\
&= \frac{\delta(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(\beta|\gamma|+n\lambda\omega^{\alpha})}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})-\beta|\gamma|}\\
&= 1-\tau,
\end{aligned} \tag{3.3}$$

this implies that $f \in S^{\tau}_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$, therefore

$$N_{n,\delta}(g) \subset S^{\tau}_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho).$$

□

Similarly by using the same technique of Theorem 3.5 we proved the following theorem.

Theorem 3.6. *Let $g \in R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho)$ and*

$$\tau = 1-\frac{\delta(1+n\lambda\omega^{\alpha})^{\mho+2}(2+\mu n\lambda\omega^{\alpha})}{(1+n\lambda\omega^{\alpha})^{\mho+2}(2+\mu n\lambda\omega^{\alpha})(n+1)-\beta|\gamma|} \tag{3.4}$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \ge 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Then

$$N_{n,\delta}(g) \subset R^{\tau}_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho).$$

Our next work is to investigate several new results like growth and distortion theorems, Hadamard Product, Extreme Points, Integral Means Inequalities and inclusion properties for the functions included in the classes $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)\cdot$ Similar work has been seen for different subclasses done by other authors (see for example [31, 21, 16, 29, 5, 4, 1, 9, 8]).

4. Growth and distortion theorems

A growth and distortion property for functions f in the respective classes $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ are given as follows:

Theorem 4.7. *If the function f defined by (1.2) belongs to the class $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ then for $|z|<1$, we have*

$$|f(z)| \le |z| + \frac{\beta|\gamma||z|^{n+1}}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}$$

$$|f(z)| \ge |z| - \frac{\beta|\gamma||z|^{n+1}}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})},$$

$$(f\in\mathscr{A}(n),\alpha,\omega,\lambda,\mu\ge 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

The extremal functions are

$$f(z) = z - \frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})} z^{k+1},\ k\ge n.$$

Proof. Let $f(z)\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ then from (2.1) we get

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})|a_{k+1}| \le \beta|\gamma|,$$

$$(f\in\mathscr{A}(n),\alpha,\omega,\lambda,\mu\ge 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Or

$$(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})\sum_{k=n}^{\infty}|a_{k+1}| \le \beta|\gamma|,$$

$$(f\in\mathscr{A}(n),\alpha,\omega,\lambda,\mu\ge 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

This implies that

$$\sum_{k=n}^{\infty}|a_{k+1}| \le \frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})} \tag{4.1}$$

$$(f\in\mathscr{A}(n),\alpha,\omega,\lambda,\mu\ge 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

From (1.2) we have

$$|f(z)| = |z - \sum_{k=n}^{\infty} a_{k+1}z^{k+1}|,$$

or

$$|f(z)| \geq |z| - \sum_{k=n}^{\infty} |a_{k+1}||z^{n+1}|,$$

this implies that

$$|f(z)| \geq |z| - \frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}|z^{n+1}|.$$

Similarly

$$|f(z)| = |z - \sum_{k=n}^{\infty} a_{k+1}z^{k+1}| \leq |z + \sum_{k=n}^{\infty} a_{k+1}z^{k+1}|$$

or

$$|f(z)| \leq |z| + \sum_{k=n}^{\infty} |a_{k+1}||z^{n+1}|,$$

$$|f(z)| \leq |z| + \frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}|z^{n+1}|.$$

□

Theorem 4.8. *If the function f defined by (1.2) belongs to the class $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ then for $|z|<1$, we have*

$$|f(z)| \leq |z| + \frac{\beta|\gamma||z|^{n+1}}{(1+n\lambda\omega^{\alpha})^{\mho+2}(2+\mu n\lambda\omega^{\alpha})(n+1)}$$

$$|f(z)| \geq |z| - \frac{\beta|\gamma||z|^{n+1}}{(1+n\lambda\omega^{\alpha})^{\mho+2}(2+\mu n\lambda\omega^{\alpha})(n+1)},$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

The extremal functions are

$$f(z) = z - \frac{\beta|\gamma|}{(1+n\lambda\omega^{\alpha})^{\mho+2}(2+\mu n\lambda\omega^{\alpha})(n+1)}z^{k+1},\ k \geq n.$$

Proof. Same as Theorem 4.7. □

Theorem 4.9. *If the function f defined by (1.2) belongs to the class $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ then for $|z|<1$,*

$$|D_{\lambda}^{\mho}(\alpha,\omega)f(z)| \leq |z| + \frac{\beta|\gamma|}{(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}|z^{k+1}|$$

$$|D_{\lambda}^{\mho}(\alpha,\omega)f(z)| \geq |z| - \frac{\beta|\gamma|}{(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}|z^{k+1}|,$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Proof. Let $f(z) \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ then from (1.11) we get

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})|a_{k+1}| \le \beta|\gamma|,$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \ge 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Or

$$(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})\sum_{k=n}^{\infty}|a_{k+1}| \le \beta|\gamma|,$$

$$(\alpha,\omega,\lambda,\mu \ge 0, \mho \in \mathbb{N}\cup\{0\}).$$

this implies that

$$(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})\sum_{k=n}^{\infty}|a_{k+1}| \le$$

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})|a_{k+1}| \le \beta|\gamma|\cdot$$

Or

$$(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho}|a_{k+1}| \le \beta|\gamma|,$$

$$(\alpha,\omega,\lambda,\mu \ge 0, \mho \in \mathbb{N}\cup\{0\}),$$

implies that

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho}|a_{k+1}| \le \frac{\beta|\gamma|}{(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}.$$

From (1.10), we have

$$|D_{\lambda}^{\mho}(\alpha,\omega)f(z)| = |z - \sum_{k=n}^{\infty}(1+(k+1-1)\lambda\omega^{\alpha})^{\mho}a_{k+1}z^{k+1}|,$$

$$|D_{\lambda}^{\mho}(\alpha,\omega)f(z)| \ge |z| - \sum_{k=n}^{\infty}(1+(k+1-1)\lambda\omega^{\alpha})^{\mho}|a_{k+1}||z^{k+1}|,$$

$$|D_{\lambda}^{\mho}(\alpha,\omega)f(z)| \ge |z| - \frac{\beta|\gamma|}{(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}|z^{k+1}|.$$

Similarly we can show that

$$|D_{\lambda}^{\mho}(\alpha,\omega)f(z)| \le |z| + \frac{\beta|\gamma|}{(1+\mu n\lambda\omega^{\alpha})(n+1)(\beta|\gamma|+n\lambda\omega^{\alpha})}|z^{k+1}|.$$

□

Theorem 4.10. *If the function f defined by (1.2) belongs to the class $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ then for $|z|<1$,*

$$|D_{\lambda}^{\mho}(\alpha,\omega)f(z)| \leq |z| + \frac{\beta|\gamma||z|^{n+1}}{(2+\mu n\lambda\omega^{\alpha})(n+1)}$$

$$|D_{\lambda}^{\mho}(\alpha,\omega)f(z)| \geq |z| - \frac{\beta|\gamma||z|^{n+1}}{(2+\mu n\lambda\omega^{\alpha})(n+1)},$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Proof. Use same technique of proof of Theorem 4.9 for solution. □

Theorem 4.11. *Let the hypotheses of Theorem 4.7 be satisfied, then*

$$|f'(z)| \leq 1 + \frac{\beta|\gamma||z|^{n}}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(\beta|\gamma|+n\lambda\omega^{\alpha})}$$

$$|f'(z)| \geq 1 - \frac{\beta|\gamma||z|^{n}}{(1+n\lambda\omega^{\alpha})^{\mho+1}(1+\mu n\lambda\omega^{\alpha})(\beta|\gamma|+n\lambda\omega^{\alpha})},$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Proof. The proof is similar to that of the proof of Theorem 4.7. □

Theorem 4.12. *Let the hypotheses of Theorem 4.7 be satisfied, then*

$$|f'(z)| \leq 1 + \frac{\beta|\gamma||z|^{n}}{(1+n\lambda\omega^{\alpha})^{\mho+2}(2+\mu n\lambda\omega^{\alpha})}$$

$$|f'(z)| \geq 1 - \frac{\beta|\gamma||z|^{n}}{(1+n\lambda\omega^{\alpha})^{\mho+2}(2+\mu n\lambda\omega^{\alpha})},$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Proof. The proof is similar to that of the proof of Theorem 4.8. □

Theorem 4.13. *Let the hypotheses of Theorem 4.7 be satisfied, then*

$$|(D_{\lambda}^{\mho}(\alpha,\omega)f(z))'| \leq 1 + \frac{\beta|\gamma|}{(1+\mu n\lambda\omega^{\alpha})(\beta|\gamma|+n\lambda\omega^{\alpha})}|z^{k}|$$

$$|(D_{\lambda}^{\mho}(\alpha,\omega)f(z))'| \geq 1 - \frac{\beta|\gamma|}{(1+\mu n\lambda\omega^{\alpha})(\beta|\gamma|+n\lambda\omega^{\alpha})}|z^{k}|,$$

$$(f \in \mathscr{A}(n), \alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\},\ z \in U).$$

Proof. The proof is similar to that of the proof of Theorem 4.9. □

Theorem 4.14. *Let the hypotheses of Theorem 4.7, be satisfied, then*

$$|(D_\lambda^{\mho}(\alpha,\omega)f(z))'| \le 1+\frac{\beta|\gamma||z|^n}{(2+\mu n\lambda\omega^\alpha)}$$

$$|(D_\lambda^{\mho}(\alpha,\omega)f(z))'| \ge 1-\frac{\beta|\gamma||z|^n}{(2+\mu n\lambda\omega^\alpha)},$$

$$(f\in\mathscr{A}(n),\alpha,\omega,\lambda,\mu\ge 0,\mho\in\mathbb{N}\cup\{0\},\ z\in U).$$

Proof. Use same technique of the proof of Theorem 4.9 for solution. □

5. Extreme points

In this section we have discussed extreme points for functions belonging to the classes $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$.

Theorem 5.15. *(a). If*

$$f_1(z)=z \text{ and}$$

$$f_i(z)=z-\frac{\beta|\gamma|}{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}z^{i+1}, k\ge n.$$

Then $f\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ if and only if it can be expressed in the form $f(z)=\sum_{i=1}^{\infty}\lambda_i f_i(z)$ where $\lambda_i\ge 0$ and $\sum_{i=1}^{\infty}\lambda_i=1$.

(b). If

$$f_1(z)=z \text{ and}$$

$$f_i(z)=z-\frac{\beta|\gamma|}{(1+k\lambda\omega^\alpha)^{\mho+2}(2+\mu k\lambda\omega^\alpha)(k+1)}z^{i+1}, k\ge n.$$

Then $f\in R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ if and only if it can be expressed in the form $f(z)=\sum_{i=1}^{\infty}\lambda_i f_i(z)$ where $\lambda_i\ge 0$ and $\sum_{i=1}^{\infty}\lambda_i=1$.

Proof. Let $f(z)=\sum_{i=1}^{\infty}\lambda_i f_i(z),\ i=1,2,3,\dots\ \lambda_i\ge 0$ with $\sum_{i=1}^{\infty}\lambda_i=1\cdot$

This implies that

$$f(z)=\sum_{i=1}^{\infty}\lambda_i f_i(z),$$

or

$$f(z)=\lambda_1(z)+\sum_{i=2}^{\infty}\lambda_i\Big(z-\frac{\beta|\gamma|}{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}z^{i+1}\Big),$$

$$f(z)=\lambda_1(z)+\sum_{i=2}^{\infty}\lambda_i(z)-\sum_{i=2}^{\infty}\lambda_i\frac{\beta|\gamma|\lambda_i}{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}z^{i+1},$$

$$f(z)=\sum_{i=1}^{\infty}\lambda_i(z)-\sum_{i=2}^{\infty}\lambda_i\Big(\frac{\beta|\gamma|}{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}z^{i+1}\Big),$$

$$f(z) = (z) - \sum_{i=2}^{\infty} \lambda_i \Big(\frac{\beta|\gamma|}{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})} z^{i+1}\Big)\cdot \tag{5.1}$$

Since

$$\sum_{i=2}^{\infty} \frac{\lambda_i \beta|\gamma|(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})}{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})}$$
$$= \sum_{i=2}^{\infty} \lambda_i \beta|\gamma|$$
$$= \beta|\gamma| \sum_{i=2}^{\infty} \lambda_i$$
$$= \beta|\gamma|(1-\lambda_1) < \beta|\gamma|.$$

The condition (2.1) for $f(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z)$ is satisfied $\cdot$ Thus $f \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$.

Conversely, we suppose that $f \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ since

$$|a_{k+1}| \leq \frac{\beta|\gamma|}{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})},\ k \geq n,$$
$$(\alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\})\cdot$$

We put

$$\lambda_i = \frac{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})}{\beta|\gamma|} a_i,\ k \geq n,$$
$$(\alpha,\omega,\lambda,\mu \geq 0, \mho \in \mathbb{N}\cup\{0\})$$

and

$$\lambda_1 = 1 - \sum_{i=2}^{\infty} \lambda_i,$$

then

$$f(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z).$$

The proof of the second part of the Theorem 5.15 is similar to 1st part$\cdot$ □

6. Integral means inequalities

For any two functions f and g analytic in U, f is said to be subordinate to g in U denoted by $f \prec g$ if there exists an analytic function w defined U satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $z \in U$.

In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)\cdot$ In 1925, Littlewood[10] proved the following Subordination Theorem.

Theorem 6.16. *[10] If f and g are any two functions, analytic in U with $f \prec g$ then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \le \int_0^{2\pi} |g(z)|^{\mu} d\theta.$$

Theorem 6.17. *(a)· Let $f \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ and f_k be defined by*

$$f_k(z) = z - \frac{\beta|\gamma|}{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})} z^{k+1} \; k \ge n.$$

If there exists an analytic function $w(z)$ given by

$$[w(z)]^k = \frac{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})}{\beta|\gamma|} \sum_{k=n}^{\infty} a_{k+1} z^k,$$

then for $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta \le \int_0^{2\pi} |f_k(re^{i\theta})|^{\mu} d\theta.$$

(b)· Let $f \in R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ and f_k be defined by

$$f_k(z) = z - \frac{\beta|\gamma|}{(1+k\lambda\omega^{\alpha})^{\mho+2}(2+\mu k\lambda\omega^{\alpha})(k+1)} z^{k+1} \; k \ge n.$$

If there exists an analytic function $w(z)$ given by

$$[w(z)]^k = \frac{(1+k\lambda\omega^{\alpha})^{\mho+2}(2+\mu k\lambda\omega^{\alpha})(k+1)}{\beta|\gamma|} \sum_{k=n}^{\infty} a_{k+1} z^k,$$

then for $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta \le \int_0^{2\pi} |f_k(re^{i\theta})|^{\mu} d\theta.$$

Proof. (a) We have to show that

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta \le \int_0^{2\pi} |f_k(re^{i\theta})|^{\mu} d\theta,$$

or

$$\int_0^{2\pi} |z - \sum_{k=n}^{\infty} a_{k+1} z^{k+1}|^{\mu} d\theta \le \int_0^{2\pi} |z - \frac{\beta|\gamma|}{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})} z^{k+1}|^{\mu} d\theta,$$

or

$$\int_0^{2\pi} |1 - \sum_{k=n}^{\infty} a_{k+1} z^{k}|^{\mu} d\theta \le \int_0^{2\pi} |1 - \frac{\beta|\gamma|}{(1+k\lambda\omega^{\alpha})^{\mho+1}(1+\mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha})} z^{k}|^{\mu} d\theta.$$

By using Theorem 6.16 it is enough to show that

$$1-\sum_{k=n}^{\infty} a_{k+1}z^k \prec 1-\frac{\beta|\gamma|}{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}z^k.$$

Now by taking

$$1-\sum_{k=n}^{\infty} a_{k+1}z^k = 1-\frac{\beta|\gamma|}{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}(w(z))^k.$$

After simplification we get

$$[w(z)]^k = \frac{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}{\beta|\gamma|}\sum_{k=n}^{\infty} a_{k+1}z^k,$$

this implies that $w(0)=0$ and

$$|[w(z)]^k| = |\frac{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}{\beta|\gamma|}\sum_{k=n}^{\infty} a_{k+1}z^k|,$$

or

$$|[w(z)]^k| = \frac{(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)}{\beta|\gamma|}\sum_{k=n}^{\infty} |a_{k+1}||z^k|.$$

By using (2.1) we get

$$|[w(z)]^k| \le |z| < 1.$$

□

The proof of the second part of the Theorem 6.17 is similar to 1st part.

7. Inclusion properties

Here we have discussed the Inclusion Properties of the subclasses of analytic function of complex order denoted by $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$.

Theorem 7.18. *(a)· If* $0\le\alpha_1\le\alpha_2\le 1$, $0\le\beta_1\le\beta_2\le$, $0\le\lambda_1\le\lambda_2\le 1$ *and* $0\le\omega_1\le\omega_2\le 1$. *Let a function f be in the class* $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ *satisfying* $\sum_{k=n}^{\infty}(1+k\lambda\omega^\alpha)^{\mho+1}(1+\mu k\lambda\omega^\alpha)(k+1)(\beta|\gamma|+k\lambda\omega^\alpha)a_{k+1}\le\beta|\gamma|$ *then show that*

- $S_{n,\mu}(\gamma,\alpha_1,\beta,\lambda,\mho,\omega)\subseteq S_{n,\mu}(\gamma,\alpha_2,\beta,\lambda,\mho,\omega)$.
- $S_{n,\mu}(\gamma,\alpha,\beta,\lambda_2,\mho,\omega)\subseteq S_{n,\mu}(\gamma,\alpha,\beta,\lambda_1,\mho,\omega)$.
- $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega_2)\subseteq S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega_1)$.
- $S_{n,\mu_2}(\gamma,\alpha,\beta,\lambda,\mho,\omega)\subseteq S_{n,\mu_1}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$.

(b)· If $0 \le \alpha_1 \le \alpha_2 \le 1,\ 0 \le \beta_1 \le \beta_2 \le,\ 0 \le \lambda_1 \le \lambda_2 \le 1$ *and* $0 \le \omega_1 \le \omega_2 \le 1$*. Let a function* f *be in the class* $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ *satisfying* $\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha})^{\mho+2}(2+\mu k\lambda\omega^{\alpha})(k+1)a_{k+1} \le \beta|\gamma|$ *then show that*

- $R_{n,\mu}(\gamma,\alpha_1,\beta,\lambda,\mho,\omega) \subseteq R_{n,\mu}(\gamma,\alpha_2,\beta,\lambda,\mho,\omega)$.
- $R_{n,\mu}(\gamma,\alpha,\beta,\lambda_2,\mho,\omega) \subseteq R_{n,\mu}(\gamma,\alpha,\beta,\lambda_1,\mho,\omega)$.
- $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega_2) \subseteq R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega_1)$.
- $R_{n,\mu_2}(\gamma,\alpha,\beta,\lambda,\mho,\omega) \subseteq R_{n,\mu_1}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$.

Proof. (a) To prove

- $S_{n,\mu}(\gamma,\alpha_1,\beta,\lambda,\mho,\omega) \subseteq S_{n,\mu}(\gamma,\alpha_2,\beta,\lambda,\mho,\omega)$.

Since it is given that $0 \le \alpha_1 \le \alpha_2 \le 1$ this implies that

$$\sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha_2})^{\mho+1}(1+\mu k\lambda\omega^{\alpha_2})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha_2})a_{k+1}$$
$$\le \sum_{k=n}^{\infty}(1+k\lambda\omega^{\alpha_1})^{\mho+1}(1+\mu k\lambda\omega^{\alpha_1})(k+1)(\beta|\gamma|+k\lambda\omega^{\alpha_1})a_{k+1},$$

therefore if $f \in S_{n,\mu}(\gamma,\alpha_1,\beta,\lambda,\mho,\omega)$ implies $f \in S_{n,\mu}(\gamma,\alpha_2,\beta,\lambda,\mho,\omega)$. This show that

$$S_{n,\mu}(\gamma,\alpha_1,\beta,\lambda,\mho,\omega) \subseteq S_{n,\mu}(\gamma,\alpha_2,\beta,\lambda,\mho,\omega).$$

Similarly, to prove that

- $S_{n,\mu}(\gamma,\alpha,\beta,\lambda_2,\mho,\omega) \subseteq S_{n,\mu}(\gamma,\alpha,\beta,\lambda_1,\mho,\omega)\cdot$

Since it is also given that $0 \le \lambda_1 \le \lambda_2 \le 1$ therefore

$$\sum_{k=n}^{\infty}(1+k\lambda_1\omega^{\alpha_2})^{\mho+1}(1+\mu k\lambda_1\omega^{\alpha_2})(k+1)(\beta|\gamma|+k\lambda_1\omega^{\alpha_2})a_{k+1}$$
$$\le \sum_{k=n}^{\infty}(1+k\lambda_2\omega^{\alpha_1})^{\mho+1}(1+\mu k\lambda_2\omega^{\alpha_1})(k+1)(\beta|\gamma|+k\lambda_2\omega^{\alpha_1})a_{k+1},$$

therefore if $f \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda_2,\mho,\omega)$ implies $f \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda_1,\mho,\omega)$. This implies that

$$S_{n,\mu}(\gamma,\alpha,\beta,\lambda_2,\mho,\omega) \subseteq S_{n,\mu}(\gamma,\alpha,\beta,\lambda_1,\mho,\omega). \qquad \square$$

The proof of the remaining parts of the theorem is similar.

8. Hadamard Product

Let $f,g \in \mathscr{A}(n)$ where $f(z)$ is given in (1.2) and $g(z) = z - \sum_{k=n}^{\infty} b_{k+1}z^{k+1}$ then the modified Hadamard product $f * g$ is defined by $(f*g) = z - \sum_{k=n}^{\infty} a_{k+1}b_{k+1}z^{k+1}$.

Theorem 7.19. (a). *If* $f(z) = z - \sum_{k=n}^{\infty} a_{k+1} z^{k+1} \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ *and* $g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1}$ $\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ *then prove that* $(f * g)(z) \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$.
(b). *If* $f(z) = z - \sum_{k=n}^{\infty} a_{k+1} z^{k+1} \in R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$
and $g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \in R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$
then prove that $(f * g)(z) \in R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$.

Proof. (a) Since it is given that $f(z) = z - \sum_{k=n}^{\infty} a_{k+1} z^{k+1} \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$, this implies that

$$\sum_{k=n}^{\infty} (1 + k\lambda\omega^{\alpha})^{\mho+1}(1 + \mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma| + k\lambda\omega^{\alpha}) a_{k+1} \le \beta|\gamma|,$$

similarly $g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \in R_{n,\mu}(\gamma,\alpha,\beta,\lambda,\mho,\omega)$ implies

$$\sum_{k=n}^{\infty} (1 + k\lambda\omega^{\alpha})^{\mho+1}(1 + \mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma| + k\lambda\omega^{\alpha}) b_{k+1} \le \beta|\gamma|,$$

because

$$\begin{aligned}
&\sum_{k=n}^{\infty} (1 + k\lambda\omega^{\alpha})^{\mho+1}(1 + \mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma| + k\lambda\omega^{\alpha}) a_{k+1} b_{k+1} \\
&\quad \le \sum_{k=n}^{\infty} (1 + k\lambda\omega^{\alpha})^{\mho+1}(1 + \mu k\lambda\omega^{\alpha})(k+1)(\beta|\gamma| + k\lambda\omega^{\alpha}) a_{k+1} \\
&\quad \le \beta|\gamma|.
\end{aligned}$$

□

Other work regarding differential operators for various problems can be found in ([11], [24]-[28], [20], [14], [13]).

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