AN INVERSE SPECTRAL PROBLEM FOR DIFFERENTIAL OPERATORS WITH INTEGRAL DELAY

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Abstract. The uniqueness theorem is proved for the solution of the inverse spectral problem for second-order integro-differential operators on a finite interval. These operators are perturbations of the Sturm-Liouville operator with convolution and one-dimensional operators. The main tool is an integral transform connected with solutions of integro-differential operators.

1. Introduction

Consider the problem $L = L(q, M, R, V)$ of the form

$$
\ell y(x) := -y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t)\, dt + R(x)\int_0^\pi V(t)y(t)\, dt = \lambda y(x), \quad 0 \leq x \leq \pi,
$$

(1)

$$
y(0) = y'(0) = 0,
$$

(2)

where $\lambda$ is the spectral parameter, and $q, M, R, V$ are continuous functions. The operator $\ell$ is a perturbation of the Sturm-Liouville operator with convolution and one-dimensional operators. We study the inverse spectral problem for $L$. Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in many branches of natural sciences and engineering. For the Sturm-Liouville differential operator, inverse spectral problems have been studied fairly completely (see the monographs [1]-[5] and the references therein). Inverse problems for integro-differential and integral operators are much more difficult for investigating, and nowadays there are only several isolated results related to these "non-local" inverse problems (see [6]-[9] and the references therein). The integro-differential operator $\ell$, considered in the present paper, is the one-dimensional perturbation of the Volterra integro-differential operator. We study the inverse problem of recovering the perturbation provided that the Volterra part is known a priori. In order to formulate the inverse problem for (1)-(2) we first introduce the spectral data for $L$.

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Let \( u(x, \lambda) \) be the solution of the Cauchy problem
\[
-u''(x, \lambda) + q(x)u(x, \lambda) + \int_{0}^{x} M(x - t)u(t, \lambda) \, dt + R(x) = \lambda u(x, \lambda), \quad u(0, \lambda) = u'(0, \lambda) = 0. \tag{3}
\]
Denote
\[
\Delta(\lambda) := 1 - \int_{0}^{\pi} V(t)u(t, \lambda) \, dt. \tag{4}
\]
The function \( \Delta(\lambda) \) is entire in \( \lambda \), and its zeros \( \{\lambda_n\} \) coincide with the eigenvalues of \( L \). The function \( \Delta(\lambda) \) is called the characteristic function of \( L \).

We note that if \( V(x) \equiv 0 \) or/and \( R(x) \equiv 0 \), then \( \Delta(\lambda) \equiv 1 \) (if \( R(x) \equiv 0 \), then \( u(x, \lambda) \equiv 0 \), since the Cauchy problem is equivalent to the homogeneous integral Volterra equation of second kind), and \( L \) has no eigenvalues. In order to avoid this trivial case we will assume in the sequel that
\[
R(x) \sim d_0 x^\alpha, \quad V(\pi - x) \sim d_1 x^\beta, \quad x \to 0, \quad d_0 d_1 \neq 0, \tag{5}
\]
where \( \alpha, \beta \geq 0 \). In this case \( L \) has a countable set of eigenvalues \( \{\lambda_n\}_{n \geq 1} \). Moreover, if \( \kappa_n \geq 1 \) is a multiplicity of the zero \( \lambda_n \) of \( \Delta(\lambda) \), then the functions
\[
u_{jn}(x) := \frac{\partial^j u(x, \lambda)}{\partial \lambda^j} \bigg|_{\lambda = \lambda_n}, \quad j = 0, \kappa_n - 1
\]
are root functions of \( L \). Denote \( \alpha_{jn} := \nu_{jn}(\pi), \quad n \geq 1, \quad j = 0, \kappa_n - 1 \). The data \( S := \{\lambda_n, \alpha_{jn}\}_{n \geq 1, j = 0, \kappa_n - 1} \) are called the spectral data of \( L \).

The inverse problem is formulated as follows: Let \( q \) and \( M \) be known a priori and fixed. Given the spectral data \( S \), construct \( R \) and \( V \).

The goal of the present paper is to prove the uniqueness theorem for the solution of this inverse problem. For this purpose together with \( L \) we consider a problem \( \tilde{L} := L(q, M, \tilde{R}, \tilde{V}) \). We agree that if a certain symbol \( \tilde{a} \) denotes an object related to \( L \), then \( \tilde{a} \) will denote a similar object related to \( \tilde{L} \). Now we formulate the main result of the paper.

**Theorem 1.** If \( S = \tilde{S} \), then \( R = \tilde{R} \) and \( V = \tilde{V} \). Thus, the specification of the spectral data \( S \) uniquely determines the functions \( R \) and \( V \).

In Section 2 we establish some auxiliary propositions, and in Section 3 we provide the proof of Theorem 1.

2. **Auxiliary propositions**

Denote
\[
\ell_1 y(x) := -y''(x) + q(x)y(x) + \int_{0}^{x} M(x - t)y(t) \, dt.
\]
Let \( S(x, \lambda) \) be the solution of the Cauchy problem

\[
\ell_1 S(x, \lambda) = \lambda S(x, \lambda), \quad S(0, \lambda) = 0, \quad S'(0, \lambda) = 1. \tag{6}
\]

By the same arguments as in [7] and [9] one gets that the following representation holds

\[
S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x K(x, t) \frac{\sin \rho t}{\rho} \, dt, \quad \lambda = \rho^2, \tag{7}
\]

where \( K(x, t) \) is a twice continuously differentiable function which does not depend on \( \lambda \), and \( K(x, 0) = 0 \).

**Lemma 1.** The following relations are valid

\[
K_{tt}(x, t) - K_{xx}(x, t) + q(x)K(x, t) + M(x - t) + \int_t^x M(x - \xi)K(\xi, t) \, d\xi = 0, \tag{8}
\]

\[
q(x) = 2 \frac{dK(x, x)}{dx}. \tag{9}
\]

**Proof.** Differentiating (7) twice with respect to \( x \), we get

\[
S'(x, \lambda) = \cos \rho x + K(x, x) \frac{\sin \rho x}{\rho} + \int_0^x K_x(x, t) \frac{\sin \rho t}{\rho} \, dt,
\]

\[
S''(x, \lambda) = -\rho \sin \rho x + \frac{d}{dx} \left( K(x, x) \frac{\sin \rho x}{\rho} \right) + K_x(x, t) \left( \frac{\sin \rho x}{\rho} \right)_{t=x} + \int_0^x K_{xx}(x, t) \frac{\sin \rho t}{\rho} \, dt.
\]

Substituting into (6) we calculate

\[
\rho \sin \rho x - \frac{d}{dx} \left( K(x, x) \frac{\sin \rho x}{\rho} \right) - K_x(x, t) \left( \frac{\sin \rho x}{\rho} \right)_{t=x} + \int_0^x K_{xx}(x, t) \frac{\sin \rho t}{\rho} \, dt + q(x) \frac{\sin \rho x}{\rho} + q(x) \int_0^x K(x, t) \frac{\sin \rho t}{\rho} \, dt + \int_0^x M(x - t) \frac{\sin \rho t}{\rho} \, dt
\]

\[
+ \int_0^x M(x - t) \left( \int_0^t K(t, \zeta) \frac{\sin \rho \zeta}{\rho} \, d\zeta \right) dt = \rho \sin \rho x + \rho \int_0^x K(x, t) \sin \rho t \, dt.
\]

Integrating twice by parts the last integral we obtain

\[
\int_0^x \left( -K_{xx}(x, t) + q(x)K(x, t) + M(x - t) + \int_t^x M(x - \xi)K(\xi, t) \, d\xi \right) \frac{\sin \rho t}{\rho} \, dt
\]

\[
- \left. K_x(x, t) \frac{\sin \rho x}{\rho} \right|_{t=x} - \frac{dK(x, x)}{dx} \frac{\sin \rho x}{\rho} - K(x, x) \cos \rho x + q(x) \frac{\sin \rho x}{\rho}
\]

\[
= -K(x, x) \cos \rho x + K(x, 0) + \left. \frac{\sin \rho x}{\rho} \right|_{t=x} - \int_0^x K_{tt}(x, t) \frac{\sin \rho t}{\rho} \, dt.
\]

Since

\[
K_x(x, t) |_{t=x} + K_t(x, t) |_{t=x} = \frac{dK(x, x)}{dx}, \quad K(x, 0) = 0,
\]
it follows that
\[ \int_0^x A(x, t) \sin \rho t \, dt + B(x) \sin \rho x = 0, \] (10)
where
\[ A(x, t) = K_{tt}(x, t) - K_{xx}(x, t) + q(x)K(x, t) + M(x - t) + \int_t^x M(x - \xi)K(\xi, t) \, d\xi, \]
\[ B(x) = q(x) - 2 \frac{dK(x, x)}{dx}. \]
Fix \( x \) and take \( \rho = \rho_n := (2\pi n + \pi/2)x^{-1} \) in (10). Then
\[ \lim_{n \to \infty} \int_0^x A(x, t) \sin(\rho_n t x^{-1}) \, dt = 0, \]
and consequently, (10) yields \( B(x) \equiv 0 \). Therefore, \( \int_0^x A(x, t) \sin \rho t \, dt = 0 \) for all \( \rho \), and hence \( A(x, t) = 0 \) for \( 0 \leq t \leq x \). Lemma 1 is proved.

**Lemma 2.** Let \( f \in C^2[0, \pi] \), and let
\[ h(x) := f(x) + \int_0^x K(x, t) f(t) \, dt. \] (11)
Then
\[ \ell_1 h(x) = -f''(x) - \int_0^x K(x, t) f''(t) \, dt + f(0)K_f(x, t)_{|t=0}, \quad x \in [0, \pi]. \] (12)

**Proof.** Differentiating (11) twice we get
\[ h'(x) = f'(x) + K(x, x) f(x) + \int_0^x K_x(x, t) f(t) \, dt, \]
\[ h''(x) = f''(x) + \frac{d}{dx}\left(K(x, x) f(x)\right) + K_x(x, x)f(x) + \int_0^x K_{xx}(x, t) f(t) \, dt, \]
and consequently,
\[ \ell_1 h(x) = -f''(x) - \frac{d}{dx}\left(K(x, x) f(x)\right) - K_x(x, x)f(x) - \int_0^x K_{xx}(x, t) f(t) \, dt \\
+ q(x)f(x) + \int_0^x q(x)K(x, t) f(t) \, dt + \int_0^x M(x - t)f(t) \, dt \\
+ \int_0^x M(x - t)\left(\int_0^t K(t, \xi) f(\xi) \, d\xi\right) \, dt. \]
Taking (9) into account we infer
\[ \ell_1 h(x) = -f''(x) + \int_0^x f(t) \left(-K_{xx}(x, t) + q(x)K(x, t) + M(x - t) + \int_t^x M(x - \xi)K(\xi, t) \, d\xi\right) \, dt \\
- K_{xx}(x, t)_{|t=x}f(x) - \frac{dK(x, x)}{dx}f(x) = K(x, x)f'(x) + 2 \frac{dK(x, x)}{dx}f(x). \]
Together with (8) this yields
\[ \ell_1 h(x) = -f''(x) - \int_0^x f(t)K_{tt}(x, t) \, dt - K_x(x, t)|_{t=x}f(x) + \frac{dK(x, x)}{dx}f(x) - K(x, x)f'(x). \]

Integrating the integral by parts twice we calculate
\[ \ell_1 h(x) = -f''(x) - \int_0^x K(x, t)f''(t) \, dt + f(0)K_t(x, t)|_{t=0}, \]
i.e. (12) is valid. Lemma 2 is proved.

### 3. The proof of Theorem 1

For each fixed \( t \), let \( g(x, t, \lambda), \ x \geq t \) be the solution of the Cauchy problem
\[ \begin{align*}
- &g''(x, t, \lambda) + q(x)g(x, t, \lambda) - \lambda g(x, t, \lambda) + \int_t^x M(x - \xi)g(\xi, t, \lambda) \, d\xi = 0, \quad \text{(13)} \\
&g(t, t, \lambda) = 0, \quad g_x(x, t, \lambda)|_{x=t} = 1 \quad \text{(14)}
\end{align*} \]
with respect to \( x \). It is easy to check that
\[ u(x, \lambda) = \int_0^x g(x, t, \lambda)R(t) \, dt, \quad \text{(15)} \]
where \( u(x, \lambda) \) was defined by (3). Fix \( t \) in (13)-(14) and make the substitution \( x = z + t \), \( g_1(z, \lambda) := g(z + t, \lambda) \). Then (13)-(14) take the form
\[ \begin{align*}
- &g_1''(z, \lambda) + q(z + t)g_1(z, \lambda) - \lambda g_1(z, \lambda) + \int_0^z M(z - \tau)g_1(\tau, \lambda) \, d\tau = 0, \quad \text{(16)} \\
&g_1(0, \lambda) = 0, \quad g_1'(0, \lambda) = 1. \quad \text{(17)}
\end{align*} \]

Similar to (7) one gets the representation
\[ g_1(z, \lambda) = \frac{\sin \rho z}{\rho} + \int_0^z P_1(z, \tau, t)\frac{\sin \rho \tau}{\rho} \, d\tau, \]
where \( P_1(z, \tau, t) \) is a smooth function. Therefore,
\[ g(x, t, \lambda) = \frac{\sin \rho (x - t)}{\rho} + \int_0^{x-t} P(x, \tau, t)\frac{\sin \rho \tau}{\rho} \, d\tau, \quad \text{(18)} \]
where \( P(x, \tau, t) = P_1(z, \tau, t) \). Substituting (18) into (15) we get
\[ u(x, \lambda) = \int_0^x \left( \frac{\sin \rho (x - t)}{\rho} + \int_0^{x-t} P(x, \tau, t)\frac{\sin \rho \tau}{\rho} \, d\tau \right)R(t) \, dt, \]
and consequently,
\[ u(x, \lambda) = \int_0^x \left( R(x - t) + \int_0^{x-t} P(x, \tau, t)R(\tau) \, d\tau \right)\frac{\sin \rho t}{\rho} \, dt. \quad \text{(19)} \]
Substituting (19) into (4) we calculate
\[ \Delta(\lambda) = 1 - \int_0^{\pi} V(t) \, dt \int_0^t \left( R(t-s) + \int_0^{t-s} P(t,s,\tau) \, d\tau \right) \frac{\sin \rho s}{\rho} \, ds. \]

This yields
\[ \Delta(\lambda) = 1 - \int_0^{\pi} \frac{\sin \rho s}{\rho} \left( \int_s^{\pi} V(t) \left( R(t-s) + \int_0^{t-s} P(t,s,\tau) \, d\tau \right) \, dt \right) \, ds \]
or
\[ \Delta(\lambda) = 1 + \int_0^{\pi} B(t) \frac{\sin \rho t}{\rho} \, dt, \quad (20) \]
where
\[ B(t) = -\int_t^{\pi} V(s) \left( R(s-t) + \int_0^{s-t} P(s,t,\tau) \, d\tau \right) \, ds \quad (21) \]

Changing the variable \( s = \pi - \xi \) in (21) we infer
\[ B(t) = -\int_0^{\pi-t} V(\pi-\xi) \left( R(\pi-\xi-t) + \int_0^{\pi-\xi-t} P(\pi-\xi,t,\tau) \, d\tau \right) \, d\xi. \]

Taking \( x = \pi - t \), we rewrite the last relation in the following form
\[ B(\pi-x) = -\int_0^x V(\pi-x) \left( R(x-\xi) + \int_0^{x-\xi} P(\pi-x,\pi-x,\tau) \, d\tau \right) \, d\xi. \quad (22) \]

It follows from (5) and (22) that \( B(t) \in C[0,\pi] \), and
\[ B(\pi-x) = dx^{\alpha+\beta+1}(1 + o(1)), \quad x \to 0, \quad (23) \]

where \( d = -d_0 d_1 \neq 0 \). Using (20) and (23) by the well-known method (see, for example, [10]) one gets that the function \( \Delta(\lambda) \) is entire in \( \lambda \) of order 1/2, and
\[ |\Delta(\lambda)| \geq C_\delta |\rho|^{-\gamma} \exp(|\text{Im} \rho|/\pi), \quad \rho \in G_\delta, \quad (24) \]

where \( \gamma = \alpha + \beta + 3, \quad C_\delta = |\rho_0 : |\rho - \rho_n| \geq \delta \quad \forall n \geq 1|, \quad \lambda = \rho^2, \quad \lambda_\rho = \rho_\rho^2, \quad \delta > 0 \). According to (24) and Hadamard's factorization theorem, the function \( \Delta(\lambda) \) is uniquely determined by its zeros. Under the assumptions of the theorem this yields
\[ \Delta(\lambda) \equiv \tilde{\Delta}(\lambda). \quad (26) \]

We consider the function
\[ F(\lambda) := \frac{u(\pi,\lambda) - \tilde{u}(\pi,\lambda)}{\Delta(\lambda)}. \]

Using (19), (25) and the assumptions of the theorem we conclude that the function \( F(\lambda) \) is entire in \( \lambda \), and \( F(\lambda) = O(\lambda^p) \) as \( |\lambda| \to \infty \), with \( p \geq 0 \). By Liouville's theorem one gets that \( F(\lambda) \)
is a polynomial. On the other hand, taking (19) and (24) into account we infer that $F(\lambda)$ tends to zero as $\lambda > 0, \lambda \to +\infty$. Therefore $F(\lambda) \equiv 0$, i.e.

$$u(\pi, \lambda) \equiv \tilde{u}(\pi, \lambda).$$  \quad (27)

Furthermore, it follows from (19) and (27) that

$$\int_0^\pi \left( R(\pi - t) - \tilde{R}(\pi - t) + \int_0^{\pi - t} P(\pi, t, \tau)(R(\tau) - \tilde{R}(\tau)) \, d\tau \right) \sin \rho t \, dt \equiv 0,$$

and consequently,

$$R(\pi - t) - \tilde{R}(\pi - t) + \int_0^{\pi - t} P(\pi, t, \tau)(R(\tau) - \tilde{R}(\tau)) \, d\tau \equiv 0,$$

or

$$R(x) - \tilde{R}(x) + \int_0^x P(\pi, \pi - x, \tau)(R(\tau) - \tilde{R}(\tau)) \, d\tau \equiv 0.$$

Since this homogeneous integral equation has only the trivial solution it follows that

$$R(x) = \tilde{R}(x), \quad x \in [0, \pi].$$  \quad (28)

Let the functions $u_0(x, \lambda)$ and $R_0(x)$ be the solutions of the integral equations

$$u(x, \lambda) = u_0(x, \lambda) + \int_0^x K(x, t)u_0(t, \lambda) \, dt, \quad R(x) = R_0(x) + \int_0^x K(x, t)R_0(t) \, dt.  \quad (29)$$

In view of (3) and (29),

$$u_0(0, \lambda) = u'_0(0, \lambda) = 0, \ell_1 u(x, \lambda) = \lambda u(x, \lambda) - R(x).$$

Moreover, applying Lemma 2 one gets

$$\ell_1 u(x, \lambda) = -u''_0(x, \lambda) - \int_0^x K(x, t)u'_0(t, \lambda) \, dt.  \quad (30)$$

It follows from (29) and () that

$$\ell_1 u(x, \lambda) = \lambda u_0(x, \lambda) + \lambda \int_0^x K(x, t)u_0(t, \lambda) \, dt - R_0(x) - \int_0^x K(x, t)R_0(t) \, dt.$$

Comparing this relation with (30) we obtain

$$u''_0(x, \lambda) + \lambda u_0(x, \lambda) - R_0(x) + \int_0^x K(x, t)(u'_0(t, \lambda) + \lambda u_0(t, \lambda) - R_0(t)) \, dt = 0,$$

and consequently,

$$u''_0(x, \lambda) + \lambda u_0(x, \lambda) = R_0(x).  \quad (31)$$
The solution of the Cauchy problem (31), () has the form
\[ u_0(x, \lambda) = \int_0^x \frac{\sin \rho(x - t)}{\rho} R_0(t) \, dt. \] (32)

Denote
\[ V_0(x) = V(x) + \int_x^\pi K(t, x) V(t) \, dt. \] (33)

It follows from (4), (29) and (33) that
\[ \Delta(\lambda) = 1 - \int_0^\pi V_0(t) u_0(t, \lambda) \, dt. \] (34)

Taking (28), (29) and (32) into account we obtain \( u_0(x, \lambda) \equiv \tilde{u}_0(x, \lambda). \) Together with (26) and (34) this yields
\[ \int_0^\pi (V_0(t) - \tilde{V}_0(t)) u_0(t, \lambda) \, dt \equiv 0. \]

Using (32) again we calculate
\[ \int_0^\pi \left( \int_s^\pi (V_0(t) - \tilde{V}_0(t)) R_0(t - s) \, dt \right) \sin \rho s \, ds \equiv 0, \]
and consequently,
\[ \int_s^\pi (V_0(t) - \tilde{V}_0(t)) R_0(t - s) \, dt \equiv 0, \quad s \in [0, \pi], \]
or
\[ \int_0^\pi (V_0(\pi - \xi) - \tilde{V}_0(\pi - \xi)) R_0(x - \xi) \, d\xi \equiv 0, \quad x \in [0, \pi]. \]

Applying Titchmarsh’s theorem we infer \( V_0(\pi - \xi) \equiv \tilde{V}_0(\pi - \xi), \quad \xi \in [0, \pi] \) or
\[ V_0(x) \equiv \tilde{V}_0(x), \quad x \in [0, \pi]. \]

By virtue of (33) this yields
\[ V(x) \equiv \tilde{V}(x), \quad x \in [0, \pi], \]
and Theorem 1 is proved. \( \square \)

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References


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