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AN INVERSE SPECTRAL PROBLEM FOR DIFFERENTIAL OPERATORS WITH INTEGRAL DELAY

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Abstract. The uniqueness theorem is proved for the solution of the inverse spectral problem for second-order integro-differential operators on a finite interval. These operators are perturbations of the Sturm-Liouville operator with convolution and one-dimensional operators. The main tool is an integral transform connected with solutions of integro-differential operators.

1. Introduction

Consider the problem $L = L(q, M, R, V)$ of the form

$$\ell y(x) := -y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t)\,dt + R(x)\int_0^\pi V(t)y(t)\,dt = \lambda y(x),\ 0 \le x \le \pi, \quad (1)$$

$$y(0) = y'(0) = 0, \quad (2)$$

where λ is the spectral parameter, and q, M, R, V are continuous functions. The operator ℓ is a perturbation of the Sturm-Liouville operator with convolution and one-dimensional operators. We study the inverse spectral problem for L. Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in many branches of natural sciences and engineering. For the Sturm-Liouville differential operator, inverse spectral problems have been studied fairly completely (see the monographs [1]-[5] and the references therein). Inverse problems for integro-differential and integral operators are much more difficult for investigating, and nowadays there are only several isolated results related to these "non-local" inverse problems (see [6]-[9] and the references therein). The integro-differential operator ℓ, considered in the present paper, is the one-dimensional perturbation of the Volterra integro-differential operator. We study the inverse problem of recovering the perturbation provided that the Volterra part is known a priori. In order to formulate the inverse problem for (1)-(2) we first introduce the spectral data for L.

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Let $u(x,\lambda)$ be the solution of the Cauchy problem

$$-u''(x,\lambda)+q(x)u(x,\lambda)+\int_0^x M(x-t)u(t,\lambda)\,dt+R(x)=\lambda u(x,\lambda),\quad u(0,\lambda)=u'(0,\lambda)=0. \tag{3}$$

Denote

$$\Delta(\lambda):=1-\int_0^\pi V(t)u(t,\lambda)\,dt. \tag{4}$$

The function $\Delta(\lambda)$ is entire in λ, and its zeros $\{\lambda_n\}$ coincide with the eigenvalues of L. The function $\Delta(\lambda)$ is called the characteristic function of L.

We note that if $V(x)\equiv 0$ or/and $R(x)\equiv 0$, then $\Delta(\lambda)\equiv 1$ (if $R(x)\equiv 0$, then $u(x,\lambda)\equiv 0$, since the Cauchy problem is equivalent to the homogeneous integral Volterra equation of second kind), and L has no eigenvalues. In order to avoid this trivial case we will assume in the sequel that

$$R(x)\sim d_0x^\alpha,\quad V(\pi-x)\sim d_1x^\beta,\ x\to 0,\ d_0d_1\neq 0, \tag{5}$$

where $\alpha,\beta\geq 0$. In this case L has a countable set of eigenvalues $\{\lambda_n\}_{n\geq 1}$. Moreover, if $\kappa_n\geq 1$ is a multiplicity of the zero λ_n of $\Delta(\lambda)$, then the functions

$$u_{jn}(x):=\frac{\partial^j u(x,\lambda)}{\partial\lambda^j}_{|\lambda=\lambda_n},\quad j=\overline{0,\kappa_n-1}$$

are root functions of L. Denote $\alpha_{jn}:=u_{jn}(\pi)$, $n\geq 1$, $j=\overline{0,\kappa_n-1}$. The data $S:=\{\lambda_n,\alpha_{jn}\}_{n\geq 1,\ j=\overline{0,\kappa_n-1}}$ are called the spectral data of L.

The inverse problem is formulated as follows: Let q and M be known a priori and fixed. Given the spectral data S, construct R and V.

The goal of the present paper is to prove the uniqueness theorem for the solution of this inverse problem. For this purpose together with L we consider a problem $\tilde{L}:=L(q,M,\tilde{R},\tilde{V})$. We agree that if a certain symbol a denotes an object related to L, then $\tilde{a}$ will denote a similar object related to $\tilde{L}$. Now we formulate the main result of the paper.

Theorem 1. *If $S=\tilde{S}$, then $R=\tilde{R}$ and $V=\tilde{V}$. Thus, the specification of the spectral data S uniquely determines the functions R and V.*

In Section 2 we establish some auxiliary propositions, and in Section 3 we provide the proof of Theorem 1.

2. Auxiliary propositions

Denote

$$\ell_1 y(x):=-y''(x)+q(x)y(x)+\int_0^x M(x-t)y(t)\,dt.$$

Let $S(x,\lambda)$ be the solution of the Cauchy problem

$$\ell_1 S(x,\lambda) = \lambda S(x,\lambda), \quad S(0,\lambda) = 0,\ S'(0,\lambda) = 1. \tag{6}$$

By the same arguments as in [7] and [9] one gets that the following representation holds

$$S(x,\lambda) = \frac{\sin\rho x}{\rho} + \int_0^x K(x,t)\frac{\sin\rho t}{\rho}\,dt, \quad \lambda = \rho^2, \tag{7}$$

where $K(x,t)$ is a twice continuously differentiable function which does not depend on λ, and $K(x,0) = 0$.

Lemma 1. *The following relations are valid*

$$K_{tt}(x,t) - K_{xx}(x,t) + q(x)K(x,t) + M(x-t) + \int_t^x M(x-\xi)K(\xi,t)\,d\xi = 0, \tag{8}$$

$$q(x) = 2\frac{dK(x,x)}{dx}. \tag{9}$$

Proof. Differentiating (7) twice with respect to x, we get

$$S'(x,\lambda) = \cos\rho x + K(x,x)\frac{\sin\rho x}{\rho} + \int_0^x K_x(x,t)\frac{\sin\rho t}{\rho}\,dt,$$

$$S''(x,\lambda) = -\rho\sin\rho x + \frac{d}{dx}\Big(K(x,x)\frac{\sin\rho x}{\rho}\Big) + K_x(x,t)_{|t=x}\frac{\sin\rho x}{\rho} + \int_0^x K_{xx}(x,t)\frac{\sin\rho t}{\rho}\,dt.$$

Substituting into (6) we calculate

$$\begin{aligned}
&\rho\sin\rho x - \frac{d}{dx}\Big(K(x,x)\frac{\sin\rho x}{\rho}\Big) - K_x(x,t)_{|t=x}\frac{\sin\rho x}{\rho} - \int_0^x K_{xx}(x,t)\frac{\sin\rho t}{\rho}\,dt \\
&\quad + q(x)\frac{\sin\rho x}{\rho} + q(x)\int_0^x K(x,t)\frac{\sin\rho t}{\rho}\,dt + \int_0^x M(x-t)\frac{\sin\rho t}{\rho}\,dt \\
&\quad + \int_0^x M(x-t)\Big(\int_0^t K(t,\xi)\frac{\sin\rho\xi}{\rho}\,d\xi\Big)dt = \rho\sin\rho x + \rho\int_0^x K(x,t)\sin\rho t\,dt.
\end{aligned}$$

Integrating twice by parts the last integral we obtain

$$\begin{aligned}
&\int_0^x \Big(-K_{xx}(x,t) + q(x)K(x,t) + M(x-t) + \int_t^x M(x-\xi)K(\xi,t)\,d\xi\Big)\frac{\sin\rho t}{\rho}\,dt \\
&\quad - K_x(x,t)_{|t=x}\frac{\sin\rho x}{\rho} - \frac{dK(x,x)}{dx}\frac{\sin\rho x}{\rho} - K(x,x)\cos\rho x + q(x)\frac{\sin\rho x}{\rho} \\
&= -K(x,x)\cos\rho x + K(x,0) + K_t(x,t)_{|t=x}\frac{\sin\rho x}{\rho} - \int_0^x K_{tt}(x,t)\frac{\sin\rho t}{\rho}\,dt.
\end{aligned}$$

Since

$$K_x(x,t)_{|t=x} + K_t(x,t)_{|t=x} = \frac{dK(x,x)}{dx}, \quad K(x,0) = 0,$$

it follows that

$$\int_0^x A(x,t)\sin\rho t\,dt + B(x)\sin\rho x = 0, \tag{10}$$

where

$$\begin{aligned} A(x,t) &= K_{tt}(x,t) - K_{xx}(x,t) + q(x)K(x,t) + M(x-t) + \int_t^x M(x-\xi)K(\xi,t)\,d\xi, \\ B(x) &= q(x) - 2\frac{dK(x,x)}{dx}. \end{aligned}$$

Fix x and take $\rho = \rho_n := (2\pi n + \pi/2)x^{-1}$ in (10). Then

$$\lim_{n\to\infty}\int_0^x A(x,t)\sin(\rho_n t x^{-1})\,dt = 0,$$

and consequently, (10) yields $B(x) \equiv 0$. Therefore, $\int_0^x A(x,t)\sin\rho t\,dt = 0$ for all ρ, and hence $A(x,t) = 0$ for $0 \le t \le x$. Lemma 1 is proved. □

Lemma 2. *Let $f \in C^2[0,\pi]$, and let*

$$h(x) := f(x) + \int_0^x K(x,t)f(t)\,dt. \tag{11}$$

Then

$$\ell_1 h(x) = -f''(x) - \int_0^x K(x,t)f''(t)\,dt + f(0)K_t(x,t)_{|t=0},\ x \in [0,\pi]. \tag{12}$$

Proof. Differentiating (11) twice we get

$$\begin{aligned} h'(x) &= f'(x) + K(x,x)f(x) + \int_0^x K_x(x,t)f(t)\,dt, \\ h''(x) &= f''(x) + \frac{d}{dx}\Big(K(x,x)f(x)\Big) + K_x(x,t)_{|t=x}f(x) + \int_0^x K_{xx}(x,t)f(t)\,dt, \end{aligned}$$

and consequently,

$$\begin{aligned} \ell_1 h(x) = &-f''(x) - \frac{d}{dx}\Big(K(x,x)f(x)\Big) - K_x(x,t)_{|t=x}f(x) - \int_0^x K_{xx}(x,t)f(t)\,dt \\ &+ q(x)f(x) + \int_0^x q(x)K(x,t)f(t)\,dt + \int_0^x M(x-t)f(t)\,dt \\ &+ \int_0^x M(x-t)\Big(\int_0^t K(t,\xi)f(\xi)\,d\xi\Big)\,dt. \end{aligned}$$

Taking (9) into account we infer

$$\begin{aligned} \ell_1 h(x) = -f''(x) + \int_0^x f(t)\Big(-K_{xx}(x,t) + q(x)K(x,t) + M(x-t) + \int_t^x M(x-\xi)K(\xi,t)\,d\xi\Big)\,dt \\ -K_x(x,t)_{|t=x}f(x) - \frac{dK(x,x)}{dx}f(x) - K(x,x)f'(x) + 2\frac{dK(x,x)}{dx}f(x). \end{aligned}$$

Together with (8) this yields

$$\ell_1 h(x) = -f''(x) - \int_0^x f(t)K_{tt}(x,t)\,dt - K_x(x,t)_{|t=x} f(x) + \frac{dK(x,x)}{dx} f(x) - K(x,x)f'(x).$$

Integrating the integral by parts twice we calculate

$$\ell_1 h(x) = -f''(x) - \int_0^x K(x,t) f''(t)\,dt + f(0)K_t(x,t)_{|t=0},$$

i.e. (12) is valid. Lemma 2 is proved. □

3. The proof of Theorem 1

For each fixed t, let $g(x,t,\lambda)$, $x \ge t$ be the solution of the Cauchy problem

$$-g''(x,t,\lambda) + q(x)g(x,t,\lambda) - \lambda g(x,t,\lambda) + \int_t^x M(x-\xi)g(\xi,t,\lambda)\,d\xi = 0, \tag{13}$$

$$g(t,t,\lambda) = 0, \quad g_x(x,t,\lambda)_{|x=t} = 1 \tag{14}$$

with respect to x. It is easy to check that

$$u(x,\lambda) = \int_0^x g(x,t,\lambda)R(t)\,dt, \tag{15}$$

where $u(x,\lambda)$ was defined by (3). Fix t in (13)-(14) and make the substitution $x = z + t$, $g_1(z,\lambda) := g(z+t,t,\lambda)$. Then (13)-(14) take the form

$$-g_1''(z,\lambda) + q(z+t)g_1(z,\lambda) - \lambda g_1(z,\lambda) + \int_0^z M(z-\tau)g_1(\tau,\lambda)\,d\tau = 0, \tag{16}$$

$$g_1(0,\lambda) = 0, \quad g_1'(0,\lambda) = 1. \tag{17}$$

Similar to (7) one gets the representation

$$g_1(z,\lambda) = \frac{\sin\rho z}{\rho} + \int_0^z P_1(z,\tau,t)\frac{\sin\rho\tau}{\rho}\,d\tau,$$

where $P_1(z,\tau,t)$ is a smooth function. Therefore,

$$g(x,t,\lambda) = \frac{\sin\rho(x-t)}{\rho} + \int_0^{x-t} P(x,\tau,t)\frac{\sin\rho\tau}{\rho}\,d\tau, \tag{18}$$

where $P(x,\tau,t) = P_1(z,\tau,t)$. Substituting (18) into (15) we get

$$u(x,\lambda) = \int_0^x \Big(\frac{\sin\rho(x-t)}{\rho} + \int_0^{x-t} P(x,\tau,t)\frac{\sin\rho\tau}{\rho}\,d\tau\Big)R(t)\,dt,$$

and consequently,

$$u(x,\lambda) = \int_0^x \Big(R(x-t) + \int_0^{x-t} P(x,t,\tau)R(\tau)\,d\tau\Big)\frac{\sin\rho t}{\rho}\,dt. \tag{19}$$

Substituting (19) into (4) we calculate

$$\Delta(\lambda)=1-\int_0^{\pi}V(t)\,dt\int_0^{t}\Big(R(t-s)+\int_0^{t-s}P(t,s,\tau)R(\tau)\,d\tau\Big)\frac{\sin\rho s}{\rho}\,ds.$$

This yields

$$\Delta(\lambda)=1-\int_0^{\pi}\frac{\sin\rho s}{\rho}\Big(\int_s^{\pi}V(t)\Big(R(t-s)+\int_0^{t-s}P(t,s,\tau)R(\tau)\,d\tau\Big)dt\Big)ds$$

or

$$\Delta(\lambda)=1+\int_0^{\pi}B(t)\frac{\sin\rho t}{\rho}\,dt, \tag{20}$$

where

$$B(t)=-\int_t^{\pi}V(s)\Big(R(s-t)+\int_0^{s-t}P(s,t,\tau)R(\tau)\,d\tau\Big)ds \tag{21}$$

Changing the variable $s=\pi-\xi$ in (21) we infer

$$B(t)=-\int_0^{\pi-t}V(\pi-\xi)\Big(R(\pi-\xi-t)+\int_0^{\pi-\xi-t}P(\pi-\xi,t,\tau)R(\tau)\,d\tau\Big)d\xi.$$

Taking $x=\pi-t$, we rewrite the last relation in the following form

$$B(\pi-x)=-\int_0^{x}V(\pi-\xi)\Big(R(x-\xi)+\int_0^{x-\xi}P(\pi-\xi,\pi-x,\tau)R(\tau)\,d\tau\Big)d\xi. \tag{22}$$

It follows from (5) and (22) that $B(t)\in C[0,\pi]$, and

$$B(\pi-x)=dx^{\alpha+\beta+1}(1+o(1)),\ x\to 0, \tag{23}$$

where $d=-d_0d_1\neq 0$. Using (20) and (23) by the well-known method (see, for example, [10]) one gets that the function $\Delta(\lambda)$ is entire in λ of order $1/2$, and

$$\Delta(\lambda)=1+O(\rho^{-1}),\quad \lambda=\rho^2>0,\ \lambda\to+\infty, \tag{24}$$

$$|\Delta(\lambda)|\geq C_\delta|\rho|^{-\gamma}\exp(|Im\,\rho|\pi),\ \rho\in G_\delta, \tag{25}$$

where $\gamma=\alpha+\beta+3$, $G_\delta=\{\rho:\ |\rho-\rho_n|\geq\delta\ \ \forall n\geq 1\}$, $\lambda=\rho^2$, $\lambda_n=\rho_n^2$, $\delta>0$. According to (24) and Hadamard's factorization theorem, the function $\Delta(\lambda)$ is uniquely determined by its zeros. Under the assumptions of the theorem this yields

$$\Delta(\lambda)\equiv\tilde{\Delta}(\lambda). \tag{26}$$

We consider the function

$$F(\lambda):=\frac{u(\pi,\lambda)-\tilde{u}(\pi,\lambda)}{\Delta(\lambda)}.$$

Using (19), (25) and the assumptions of the theorem we conclude that the function $F(\lambda)$ is entire in λ, and $F(\lambda)=O(\lambda^p)$ as $|\lambda|\to\infty$, with $p\geq 0$. By Liouville's theorem one gets that $F(\lambda)$

is a polynomial. On the other hand, taking (19) and (24) into account we infer that $F(\lambda)$ tends to zero as $\lambda > 0$, $\lambda \to +\infty$. Therefore $F(\lambda) \equiv 0$, i.e.

$$u(\pi,\lambda) \equiv \tilde{u}(\pi,\lambda). \tag{27}$$

Furthermore, it follows from (19) and (27) that

$$\int_0^{\pi} \Big(R(\pi - t) - \tilde{R}(\pi - t) + \int_0^{\pi - t} P(\pi, t, \tau)(R(\tau) - \tilde{R}(\tau))\, d\tau \Big) \sin \rho t \, dt \equiv 0,$$

and consequently,

$$R(\pi - t) - \tilde{R}(\pi - t) + \int_0^{\pi - t} P(\pi, t, \tau)(R(\tau) - \tilde{R}(\tau))\, d\tau \equiv 0,$$

or

$$R(x) - \tilde{R}(x) + \int_0^{x} P(\pi, \pi - x, \tau)(R(\tau) - \tilde{R}(\tau))\, d\tau \equiv 0.$$

Since this homogeneous integral equation has only the trivial solution it follows that

$$R(x) = \tilde{R}(x), \quad x \in [0,\pi]. \tag{28}$$

Let the functions $u_0(x,\lambda)$ and $R_0(x)$ be the solutions of the integral equations

$$u(x,\lambda) = u_0(x,\lambda) + \int_0^x K(x,t) u_0(t,\lambda)\, dt, \; R(x) = R_0(x) + \int_0^x K(x,t) R_0(t)\, dt. \tag{29}$$

In view of (3) and (29),

$$u_0(0,\lambda) = u_0'(0,\lambda) = 0, \ell_1 u(x,\lambda) = \lambda u(x,\lambda) - R(x).$$

Moreover, applying Lemma 2 one gets

$$\ell_1 u(x,\lambda) = -u_0''(x,\lambda) - \int_0^x K(x,t) u_0''(t,\lambda)\, dt. \tag{30}$$

It follows from (29) and () that

$$\ell_1 u(x,\lambda) = \lambda u_0(x,\lambda) + \lambda \int_0^x K(x,t) u_0(t,\lambda)\, dt - R_0(x) - \int_0^x K(x,t) R_0(t)\, dt.$$

Comparing this relation with (30) we obtain

$$u_0''(x,\lambda) + \lambda u_0(x,\lambda) - R_0(x) + \int_0^x K(x,t)(u_0''(t,\lambda) + \lambda u_0(t,\lambda) - R_0(t))\, dt = 0,$$

and consequently,

$$u_0''(x,\lambda) + \lambda u_0(x,\lambda) = R_0(x). \tag{31}$$

The solution of the Cauchy problem (31), () has the form

$$u_0(x,\lambda) = \int_0^x \frac{\sin\rho(x-t)}{\rho} R_0(t)\,dt. \tag{32}$$

Denote

$$V_0(x) = V(x) + \int_x^\pi K(t,x)V(t)\,dt. \tag{33}$$

It follows from (4), (29) and (33) that

$$\Delta(\lambda) = 1 - \int_0^\pi V_0(t)u_0(t,\lambda)\,dt. \tag{34}$$

Taking (28), (29) and (32) into account we obtain $u_0(x,\lambda) \equiv \tilde{u}_0(x,\lambda)$. Together with (26) and (34) this yields

$$\int_0^\pi (V_0(t) - \tilde{V}_0(t))u_0(t,\lambda)\,dt \equiv 0.$$

Using (32) again we calculate

$$\int_0^\pi \Big(\int_s^\pi (V_0(t) - \tilde{V}_0(t))R_0(t-s)\,dt\Big)\sin\rho s\,ds \equiv 0,$$

and consequently,

$$\int_s^\pi (V_0(t) - \tilde{V}_0(t))R_0(t-s)\,dt \equiv 0,\ s \in [0,\pi],$$

or

$$\int_0^x (V_0(\pi-\xi) - \tilde{V}_0(\pi-\xi))R_0(x-\xi)\,d\xi \equiv 0,\ x \in [0,\pi].$$

Applying Titchmarsh's theorem we infer $V_0(\pi-\xi) \equiv \tilde{V}_0(\pi-\xi)$, $\xi \in [0,\pi]$ or

$$V_0(x) \equiv \tilde{V}_0(x),\ x \in [0,\pi].$$

By virtue of (33) this yields

$$V(x) \equiv \tilde{V}(x),\ x \in [0,\pi],$$

and Theorem 1 is proved. □

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