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NORMALITY CRITERIA CONCERNING SHARING HOLOMORPHIC FUNCTIONS

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Abstract. In this paper, we study the problem of normal families of meromorphic functions that share holomorphic functions with their linear differential polynomials. Meanwhile, some results about normal family are derived which improve and generalize several related theorems obtained by Chang, Fang and Zalcman [2], Pang [5] and Schwick [9].

1. Introduction and main results

In order to state our main results, we need the following definitions and notations.

Let f, g and a, b be holomorphic functions on a domain D in $\mathbb{C}$. We denote the condition that $f(z) - a(z) = 0$ implies $g(z) - b(z) = 0$ by $f(z) = a(z) \Rightarrow g(z) = b(z)$. If $f(z) = a(z) \Rightarrow g(z) = b(z)$ and $g(z) = b(z) \Rightarrow f(z) = a(z)$, we write $f(z) = a(z) \Leftrightarrow g(z) = b(z)$. Moreover, we say that f and g share a provide that $f(z) = a(z) \Leftrightarrow g(z) = a(z)$. In what follows, we assume that the reader is familiar with the basic notation and results in the Nevanlinna value distribution theory (see, [11, 12]).

One important subject in the theory of normal family is to find the normal families. According to Bloch's principle, a lot of normality criteria have been obtained by starting from Picard type theorems [8].

In 1992, Schwick [9] drawn a connection between values shared by functions in $\mathscr{F}$ and the normality of the family $\mathscr{F}$. In fact, he proved the following results.

Theorem A. *Let $\mathscr{F}$ be a family of meormophic functions in a domain D, and let a_1, a_2, a_3 be three distinct complex numbers. If for each $f \in \mathscr{F}$, f and f' share a_j ($j = 1, 2, 3$), then $\mathscr{F}$ is normal in D.*

From then on, many normality criteria have been obtained in this direction (see, [1, 3, 4, 6, 7, 10]). In 2000, Pang and Zalcman [5] obtained the following result.

Theorem B. *Let $\mathcal{F}$ be a family of meormophic functions in a domain D, and let a, b, c and d be complex numbers such that $c \neq a$ and $d \neq b$. If for each $f \in \mathcal{F}$, $f = a \Leftrightarrow f' = b$ and $f = c \Leftrightarrow f' = d$, then $\mathcal{F}$ is normal in D.*

In 2005, Chang, Fang and Zalcman [2] replaced the condition $f = c \Leftrightarrow f' = d$ by $f = c \Rightarrow f' = d$ and improved Theorem B as follows.

Theorem C. *Let $\mathcal{F}$ be a family of meormorphic functions in a domain D, and let a, b, c and d be complex numbers such that $b \neq 0$, $c \neq a$ and $d \neq b$. If for each $f \in \mathcal{F}$, $f = a \Leftrightarrow f' = b$ and $f = c \Rightarrow f' = d$, then $\mathcal{F}$ is normal in D.*

Remark 1. As a matter of fact, if the condition $d \neq b$ is omitted, Theorem B and C still hold.

From Theorem C, some questions are proposed as follows.

Question 1. Can the values a, b, c and d be weakened to holomorphic functions ?

Question 2. Can f' be extended to a linear differential polynomial in f ?

In the following, we use the notation

$$L[f] = a_0 f' + a_1 f \tag{1.1}$$

for a linear differential polynomial in f, where a_0, a_1 are two holomorphic functions with $a_0(z) \neq 0$.

In the paper, by considering the above questions, we obtain a result as follows, which is an improvement of the precious theorems.

Theorem 1.1. *Let $\mathcal{F}$ be a family of meormorphic functions in a domain D, let $L[f]$ be defined as (1.1), and let a, b, c, d be four holomorphic functions in D. For each $f \in \mathcal{F}$, if*

(1) $a(z) \neq c(z)$;

(2) $b(z) - a_1(z)a(z) - a_0(z)c'(z) \neq 0$;

(3) $b(z) - a_1(z)c(z) - a_0(z)c'(z) \neq 0$;

(4) $f(z) = a(z) \Leftrightarrow L[f](z) = b(z)$ *and* $f(z) = c(z) \Rightarrow L[f](z) = d(z)$, *then $\mathcal{F}$ is normal in D.*

Suppose that $a_0 = 1$ and $a_1 = 0$ in (1.1), we find that the condition (2) coincides with (3) in Theorem 1.1, and both of them reduce to $b(z) \neq c'(z)$. So, the following corollary is an immediately consequence of Theorem 1.1.

Corollary 1.2. *Let $\mathcal{F}$ be a family of meormorphic functions in a domain D, let a, b, c, d be four holomorphic functions in D. For each $f \in \mathcal{F}$, if*

(1) $a(z) \neq c(z)$;

(2) $b(z) \neq c'(z)$;

(3) $f(z) = a(z) \Leftrightarrow f'(z) = b(z)$ and $f(z) = c(z) \Rightarrow f'(z) = d(z)$, then $\mathscr{F}$ is normal in D.

Remark 2. If the functions a, b, c, d reduces to constants, then Corollary 1.2 becomes Theorem C. In [2, Theorem 2], the authors given an example to show the necessary of the condition $b \neq 0$, which also shows the necessary of the assumption (3) in Theorem 1.1 and (2) in Corollary 1.2. Now, we state the example here.

Example. Let

$$f_n(z) = \frac{(nz)^2}{(nz)^2 - 1}, \ n = 1, 2, \cdots$$

and let $\mathscr{F} = \{f_n\}_{n=1}^{\infty}$, $D = \{z : |z| < 1\}$. Then

$$f_n'(z) = \frac{-2n^2 z}{[(nz)^2 - 1]^2}.$$

Clearly, if $f \in \mathscr{F}$, f and f' vanish only at 0, also $f \neq 1$. Thus, we have $f(z) = 0 \Leftrightarrow f'(z) = 0$ and $f(z) = 1 \Rightarrow f'(z) = d$ for any d. However, $\mathscr{F}$ is not normal in D.

Remark 3. For families of holomorphic functions, Theorem 1.1 still valid if the condition (2) is deleted. The proofs of our results have roots in [2].

2. The lemma

To prove our result, we need the following lemma, which is essential to our proofs.

Lemma 2.1 ([7])**.** *Let $\mathscr{F}$ be a family of functions meromorphic in the unit disc Δ, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathscr{F}$ is not normal at z_0 in the unit disc, then there exist, for each $0 \leq \alpha \leq k$*

(a) *points $z_n \in \Delta$, $z_n \to z_0$;*

(b) *functions $f_n \in \mathscr{F}$ and*

(c) *positive number $\rho_n \to 0$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a non-constant meromorphic function in $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\sharp}(\xi) \leq g^{\sharp}(0) = kA + 1$. In particular, g has order at most two.*

3. Proof of the Theorem 1.1

Since normality is a local property, we assume that $D = \Delta$, the unit disc. Set $\mathscr{G} = \{F : F = f - c, \ f \in \mathscr{F}\}$. In view of the form of $L[f]$ and the assumption, we can easily deduce $F(z) = 0 \Rightarrow F'(z) = \phi(z)$, where

$$\phi = \frac{d - a_1 c - a_0 c'}{a_0}.$$

Suppose that $\mathscr{F}$ is not normal at $z_0 \in \Delta$. We have $A = \max_{z \in D_1}\{|\phi(z)|\}$, where $r > 0$ is a constant and $D_1 = \{z : |z - z_0| \leq r\} \subset \Delta$. Then, in domain $D_2 = \{z : |z - z_0| < \frac{r}{2}\}$, we have $F(z) = 0 \Rightarrow |F'(z)| = |\phi(z)| \leq M$, where

$$M = \max\{A, 1\} \geq 1.$$

Obviously, $\mathscr{G}$ is not normal at z_0 as well. Then, by Lemma 1, we can find a sequence of functions $F_n \in \mathscr{G}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\xi) = \rho_n^{-1} F_n(z_n + \rho_n \xi) = \rho_n^{-1}[f_n(z_n + \rho_n \xi) - c(z_n + \rho_n \xi)] \to g(\xi) \tag{3.1}$$

converges locally uniformly in $\mathbb{C}$, where g is a non-constant meromorphic function, which satisfies $g^\sharp(\xi) \leq g^\sharp(0) = M + 1$.

In the following, we claim that

(I) $g(\xi) = 0 \Rightarrow g'(\xi) = \phi(z_0)$,

(II) $g'(\xi) \neq B$,

(III) $g(\xi) \neq \infty$,

where $B = \frac{b(z_0) - a_0(z_0)c'(z_0) - a_1(z_0)c(z_0)}{a_0(z_0)} \neq 0$ is a constant.

We prove the claim as follows.

By (3.1), we have

$$g_n'(\xi) = f_n'(z_n + \rho_n \xi) - c'(z_n + \rho_n \xi) \to g'(\xi). \tag{3.2}$$

Furthermore

$$\begin{aligned}
\frac{L[f_n](z_n + \rho_n \xi)}{a_0(z_n + \rho_n \xi)} &= f_n'(z_n + \rho_n \xi) + \frac{a_1(z_n + \rho_n \xi)}{a_0(z_n + \rho_n \xi)} f_n(z_n + \rho_n \xi) \\
&= f_n'(z_n + \rho_n \xi) + \frac{a_1(z_n + \rho_n \xi)}{a_0(z_n + \rho_n \xi)}[\rho_n g_n(\xi) + c(z_n + \rho_n \xi)] \\
&\to g'(\xi) + c'(z_0) + \frac{a_1(z_0)}{a_0(z_0)} c(z_0).
\end{aligned} \tag{3.3}$$

Suppose that $g(\eta_0) = 0$. Note that $g \not\equiv 0$, by Hurwitz's theorem and (3.1), there exists a sequence $\eta_n \to \eta_0$ such that (for n large enough)

$$f_n(z_n + \rho_n \eta_n) = c(z_n + \rho_n \eta_n).$$

Since $f(z) = c(z) \Rightarrow L[f](z) = d(z)$, we have $L[f_n](z_n + \rho_n\eta_n) = d(z_n + \rho_n\eta_n)$. Then, it follows from (3.3) that

$$
\begin{aligned}
g'(\eta_0) &= \lim_{n\to\infty} \frac{L[f_n](z_n + \rho_n\eta_n)}{a_0(z_n + \rho_n\eta_n)} - c'(z_0) - \frac{a_1(z_0)}{a_0(z_0)}c(z_0) \\
&= \lim_{n\to\infty} \frac{d(z_n + \rho_n\eta_n)}{a_0(z_n + \rho_n\eta_n)} - c'(z_0) - \frac{a_1(z_0)}{a_0(z_0)}c(z_0) \\
&= \frac{d(z_0) - a_0(z_0)c'(z_0) - a_1(z_0)c(z_0)}{a_0(z_0)} = \phi(z_0),
\end{aligned}
$$

which implies (I).

With (3.3), we deduce

$$
\begin{aligned}
\frac{L[f_n](z_n + \rho_n\xi) - b(z_n + \rho_n\xi)}{a_0(z_n + \rho_n\xi)} &\to g'(\xi) + c'(z_0) + \frac{a_1(z_0)}{a_0(z_0)}c(z_0) - \frac{b(z_0)}{a_0(z_0)} \\
= g'(\xi) - \frac{b(z_0) - a_0(z_0)c'(z_0) - a_1(z_0)c(z_0)}{a_0(z_0)} &= g'(\xi) - B.
\end{aligned}
\tag{3.4}
$$

Suppose that $g'(\theta_0) = B$, obviously $g' \neq B$. Otherwise $g' = B$ and $g(\xi) = B\xi + C$, where C is a constant. From (I), we deduce $B = \phi(z_0)$. Then, a simple calculation shows that

$$
M + 1 = g^\sharp(0) \leq |\phi(z_0)| < M + 1,
$$

which is a contradiction.

Then, by Hurwitz's theorem and (3.4), there exists a sequence $\theta_n \to \theta_0$, such that (for n sufficiently large)

$$
L[f_n](z_n + \rho_n\theta_n) - b(z_n + \rho_n\theta_n) = 0
$$

From the assumption, we have $f_n(z_n + \rho_n\theta_n) = a(z_n + \rho_n\theta_n)$. Then, by (3.1) and $a(z) \neq c(z)$, we derive that

$$
\begin{aligned}
g(\theta_0) &= \lim_{n\to\infty} \rho_n^{-1}[f_n(z_n + \rho_n\rho_n) - c(z_n + \rho_n\rho_n)] \\
&= \lim_{n\to\infty} \rho_n^{-1}[a(z_n + \rho_n\eta_n) - c(z_n + \rho_n\eta_n)] = \infty,
\end{aligned}
$$

a contradiction. Thus, we prove (II).

Now, we prove (III). Suppose that $g(\xi_0) = \infty$. Since $g \neq \infty$, there exists a closed disc $K = \{\xi : |\xi - \xi_0| \leq \delta\}$ on which $1/g$ and $1/g_n$ are holomorphic (for n large enough) and $1/g_n \to 1/g$ uniformly. From $a(z) \neq c(z)$, we deduce

$$
\frac{1}{g_n(\xi)} - \frac{\rho_n}{a(z_n + \rho_n\xi) - c(z_n + \rho_n\xi)} \to \frac{1}{g(\xi)}
\tag{3.5}
$$

uniformly on K. Assume that the multiplicity of the zero of $1/g$ at ξ_0 be m. Then, $(\frac{1}{g})^{(m)}(\xi_0) \neq 0$. Noting that $1/g$ is not a constant, then, by Hurwitz's theorem and (3.5), there exist $\xi_{n,j} \to$

ξ_0 $(j = 1, 2 \cdots, m)$ such that

$$\frac{1}{g_n(\xi_{n,j})} - \frac{\rho_n}{a(z_n + \rho_n \xi_{n,j}) - c(z_n + \rho_n \xi_{n,j})} = 0. \tag{3.6}$$

It is not difficult from (3.6) to deduce

$$f_n(z_n + \rho_n \xi_{n,j}) = a(z_n + \rho_n \xi_{n,j}), \ (j = 1, 2 \cdots, m). \tag{3.7}$$

Thus, we have

$$L[f_n](z_n + \rho_n \xi_{n,j}) = b(z_n + \rho_n \xi_{n,j}). \tag{3.8}$$

Observing that the form of $L[f]$, we derive

$$f_n'(z_n + \rho_n \xi_{n,j}) = \frac{b - a_1 a}{a_0}|_{z=z_n+\rho_n\xi_{n,j}}.$$

Furthermore,

$$\begin{aligned} g_n'(z_n + \rho_n \xi_{n,j}) &= f_n'(z_n + \rho_n \xi_{n,j}) - c'(z_n + \rho_n \xi_{n,j}) \\ &= \frac{b - a_1 a - a_0 c'}{a_0}|_{z=z_n+\rho_n\xi_{n,j}}, \ (j = 1, 2 \cdots, m). \end{aligned} \tag{3.9}$$

It follows from $b(z) - a_1(z)a(z) - a_0(z)c'(z) \neq 0$ that, for $j = 1, 2 \cdots, m$,

$$(\frac{1}{g_n})'|_{\xi=\xi_{n,j}} = -\frac{g_n'(\xi_{n,j})}{g_n^2(\xi_{n,j})} = -\rho_n^2 \frac{b - a_1 a - a_0 c'}{(a-c)^2}|_{z=z_n+\rho_n\xi_{n,j}} \neq 0.$$

That is $\xi_{n,j} \neq \xi_{n,i} (1 \leq i \neq j \leq m)$. Set

$$H_n(\xi) = (\frac{1}{g_n(\xi)})' + \rho_n^2 \frac{b - a_1 a - a_0 c'}{(a-c)^2}|_{z=z_n+\rho_n\xi}.$$

For $j = 1, 2 \cdots, m$, we have $H_n(\xi_{n,j}) = 0$ and

$$H_n(\xi) \rightarrow (\frac{1}{g(\xi)})' \tag{3.10}$$

uniformly on K. Combining (3.10), Hurwitz's theorem and $H_n(\xi_{n,j}) = 0$ $(j = 1, 2 \cdots, m)$ yields that ξ_0 is a zero of $(\frac{1}{g})'$ with multiplicity m, and thus $(\frac{1}{g})^{(m)}(\xi_0) = 0$, which is a contradiction. Thus, we prove (III).

Hence, we finish the proof of the claim.

It follows from (III) that g is an entire function and therefore of exponential type. By (II), we have

$$g'(\xi) = B + \lambda \mu e^{\mu \xi}, \tag{3.11}$$

where $\lambda(\neq 0)$, μ are two constant. Suppose that $\mu = 0$, then $g' = B$. From $g^{\sharp}(0) = M + 1$, it is easy to deduce a contradiction. Now, we assume that $\mu \neq 0$. Integral (3.11) yields

$$g(\xi) = B\xi + D + \lambda e^{\mu\xi}, \tag{3.12}$$

where D is a constant. Let β_0 be a zero of $g(\xi)$. It means that

$$B\beta_0 + D + \lambda e^{\mu\beta_0} = 0. \tag{3.13}$$

By (I), we have $g'(\beta_0) = \phi(z_0)$. That is

$$B + \lambda\mu e^{\mu\beta_0} = \phi(z_0). \tag{3.14}$$

Combining (3.13) and (3.14) yields $\beta_0 = \frac{B - \phi(z_0) - D\mu}{\mu B}$, which indicates that g has the unique zero β_0. But it is obvious from (3.12) that g has infinitely many zeros.

Thus, $\mathcal{G}$ is normal at z_0, and $\mathcal{F}$ is normal at z_0 as well. Furthermore, $\mathcal{F}$ is normal in D.

Hence, we complete the proof of the Theorem 1.1.

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