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EQUIVALENCE OF THE CONVERGENCE OF MANN ITERATION WITH MODIFIED ERRORS AND ISHIKAWA ITERATION WITH MODIFIED ERRORS FOR THE CLASS OF UNIFORMLY CONTINUOUS, STRONGLY PSEUDOCONTRACTIVE MAPS

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Abstract. We prove that the convergence of the Mann iteration with modified errors is equivalent to the convergence of the Ishikawa iteration with modified errors for the class of uniformly continuous and strongly pseudocontractive maps. Our results improve the results of Soltuz [14] and extend the results of Rhoades and Soltuz [12] to the iterations with modified errors.

1. Introduction

Throught this paper, E denotes a real Banach space and E^*, the dual of E; and I denotes the identity operator on E.

We denote by J, the duality map from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle \cdot, \cdot \rangle$ is the generalized duality pairing of E and E^*.

A map T from E to E is said to be *strongly pseudocontractive* if there exists $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{1.1.1}$$

for all x, y in E and $r > 0$.

If $t = 1$, then T is called a *pseudocontrative* map.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *accretive* if for all $x, y \in D(T)$, there exists $t > 1$ such that

$$\|x - y\| \leq \|(x - y) + k(Tx - Ty)\| \tag{1.1.2}$$

for every $k > 0$, and T is said to be *strongly accretive* if there is a positive constant k such that $T - kI$ is accretive.

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The pseudocontractive maps are characterized by the fact that the mapping T is pseudocontractive if and only if $I-T$ is accretive.

As a consequence of the result of Kato [9], T is pseudocontractive if and only if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that $\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq 0$.

Furthermore, T is strongly pseudocontractive if and only if there exists a $k>0$ such that

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq k\|x-y\|^2 \tag{1.1.3}$$

for all x, y in $D(T)$.

Also, from Bogin [1], it follows that the inequality (1.1.3) inplies that T is strongly pseudocontractive with $k \in (0,1)$ if and only if for all $x, y \in D(T)$ the following inequality holds:

$$\|x-y\| \leq \|x-y+s[(I-T-kI)x-(I-T-kI)y]\| \tag{1.1.4}$$

for all $s>0$.

According to Deimling [5], if K is a closed convex subset of E and $T : K \to K$ is continuous, strongly pseudocontractive then T has a unique fixed point in K. Further, it is proved ([6], Theorem 13.1, p.125) that if $T : E \to E$ is continuous and strongly accretive then T is surjective; i.e., for a give $f \in E$ the equation $Tx=f$ has a unique solution.

For a selfmap T of K, we denote $F(T)$, the fixed point set of T.

Let K be a nonempty convex subset of E and T a selfmap of K. For $u_0 \in K$, the sequence $\{u_n\}_{n=0}^{\infty} \subset K$ defined by

$$u_{n+1}=(1-\alpha_n)u_n+\alpha_n T u_n, \qquad n=0,1,2,\ldots \tag{1.1.5}$$

is called the *Mann iteration*, where $\{\alpha_n\}$ is a sequence of positive numbers in $(0,1)$ satisfying

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \Sigma\alpha_n=\infty. \tag{1.1.6}$$

For $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty} \subset K$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n+\alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n+\beta_n T x_n, \qquad n=0,1,2,\ldots, \end{aligned} \tag{1.1.7}$$

is called the *Ishikawa iteration*, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $(0,1)$ satisfying

$$\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \beta_n = 0 \text{ and } \Sigma\alpha_n=\infty. \tag{1.1.8}$$

In recent years, Mann and Ishikawa iterations have been studied extensively by several authors [2, 3, 4, 7, 13] for the class of strongly pseudocontractive operators.

Rhoades and Soltuz [12] studied the equivalence between the convergences of Mann and Ishikawa iterations and established the following theorem.

Theorem 1.1. *Let K be a closed, convex subset of E and T a Lipschitz, strongly pseudocontractive selfmap of K. Consider the Mann and Ishikawa iterations with the same initial point as in* (1.1.5) *and* (1.1.7)*, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $(0,1)$ satisfying* (1.1.8)*. Then the following are equivalent:*

(1) *The Mann iteration* (1.1.5) *converges to* $x^* \in F(T)$.
(2) *The Ishikawa iteration* (1.1.7) *converges to* $x^* \in F(T)$.

In 1995, Liu [10] introduced iterations with errors as follows:
For a nonempty convex subset K of E and $T : K \to E$, for $u_0 \in K$, the sequence $\{u_n\}_{n=0}^{\infty} \subset K$ defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n + e_n, \qquad n = 0, 1, 2, \dots \tag{A}$$

where (i) $\{\alpha_n\}$ is a sequence of positive numbers in $(0, 1)$ satisfying (1.1.6) and (ii) $\Sigma\|e_n\| < \infty$, is called the *Mann iteration with errors.*

For $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty} \subset K$ defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n + p_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n + q_n, \qquad n = 0, 1, 2, \dots \end{aligned} \tag{B}$$

where (i) $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers satisfying (1.1.8) and (ii) $\Sigma\|p_n\| < \infty$, $\Sigma\|q_n\| < \infty$, is called the *Ishikawa iteration with errors.*

Soltuz [14] obtained the following result, to establish the equivalence between the Mann and Ishikawa iterations with errors.

Theorem 1.2. *Let K be a nonempty, bounded, convex and closed subset of E and $T : K \to E$ a strongly pseudocontractive and Lipschitzian map with Lipschitz constant $L > 1$ with $T(K)$ bounded and $F(T) \neq \varnothing$. If $x_0 = u_0 \in K$, then the following assertions are equivalent:*

(1) *The Mann iteration with errors* (*A*) *converges to* $x^* \in F(T)$.
(2) *The Ishikawa iteration with errors* (*B*) *converges to* $x^* \in F(T)$.

In particular, the conditions $\Sigma\|e_n\| < \infty$, $\Sigma\|p_n\| < \infty$ and $\Sigma\|q_n\| < \infty$ imply that the error terms in (A) and (B) tend to zero as $n \to \infty$. This is incompatible with the randomness of the occurrence of errors. Afterwards, in 1998, Xu [15] introduced a more satisfactory error terms in iterations in the following way.

Let K be a nonempty convex subset of E and $T : K \to K$ a mapping. For given $u_0 \in E$, the sequence $\{u_n\}_{n=0}^{\infty} \subset K$ defined by

$$u_{n+1} = a_n u_n + b_n T u_n + c_n e_n, \qquad n = 0, 1, 2, \dots \tag{M}$$

where $\{e_n\}$ is a bounded sequence in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in $(0, 1)$ such that $a_n + b_n + c_n = 1$, for all $n \geq 0$, is called the *Mann iteration with modified errors.*

For a given $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty} \subset K$ defined by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n p_n, \\ y_n &= a'_n x_n + b'_n T x_n + c'_n q_n, \qquad n = 0, 1, 2, \dots \end{aligned} \tag{I}$$

where $\{p_n\}$, $\{q_n\}$ are bounded sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are sequences in $(0,1)$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n = 1$, for all $n \geq 0$, is called the *Ishikawa iteration with modified errors.* If K is bounded then the sequences $\{p_n\}$, $\{q_n\}$ and $\{e_n\}$ in the iterations can be arbitrarily.

In 1998, Chidume [3] established the following convergence theorems.

Theorem 1.3. *Let K be a closed, convex and bounded subset of E, $T : K \to K$ a uniformly continuous and strongly pseudocontractive mapping. For a given $x_0, p_0, q_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n p_n, \\ y_n &= a'_n x_n + b'_n T x_n + c'_n q_n, \qquad n = 0, 1, 2, \dots \end{aligned} \tag{1.3.1}$$

where $\{p_n\}$, $\{q_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are sequences in $(0,1)$ satisfying

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n = 1$, *for all* $n \geq 0$,

(ii) $\Sigma b_n = \infty$,

(iii) $\Sigma c_n < \infty$, *and*

(iv) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = 0$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $x^* \in F(T)$.

Theorem 1.4. *Let E, K and T be as in Theorem* 1.3. *For a given $x_0, p_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n p_n, \qquad n = 0, 1, 2, \dots$$

where $\{p_n\}$ is an arbitrary sequence in K and $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences in $(0,1)$ satisfying

(i) $a_n + b_n + c_n = 1$, *for all* $n \geq 0$,

(ii) $\Sigma b_n = \infty$,

(iii) $\Sigma c_n < \infty$, *and*

(iv) $\lim_{n\to\infty} b_n = 0$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $x^* \in F(T)$.

In this paper, we prove that the convergence of Mann iteration with modified errors and Ishikawa iteration with modified errors are equivalent for the class of uniformly continuous and strongly pseudocontractive maps. Also, we obtain the equivalence between the convergences of Mann and Ishikawa iterations with modified errors to the solution of the operator equation $Ax = f$ (Corollary 2.5) and $x + Tx = f$ (Corollary 2.6).

To prove our main results, we use the following lemma.

Lemma 1.5.([10]). *Let ρ_n be a non-negative sequence of reals satisfying*

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n + \gamma_n,$$

with $\{\lambda_n : n = 0, 1, 2, \ldots\} \subset [0, 1]$, $\Sigma\lambda_n = \infty$, $\sigma_n = o(\lambda_n)$, *and* $\Sigma\gamma_n < \infty$. *Then* $\lim_{n\to\infty} \rho_n = 0$.

2. Main Results

Theorem 2.1. *Let* E, K *and* T *be as in Theorem* 1.3. *For a given* x_0, p_0, q_0, u_0 *and* $e_0 \in E$, *we define the sequences* $\{u_n\}_{n=0}^{\infty}$ *and* $\{x_n\}_{n=0}^{\infty}$ *by*

$$u_{n+1} = a_n u_n + b_n T u_n + c_n e_n, \quad n = 0, 1, 2, 3, \ldots \tag{2.1.1}$$

and

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n p_n, \\ y_n &= a'_n x_n + b'_n T x_n + c'_n q_n, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{2.1.2}$$

where $\{p_n\}$, $\{q_n\}$ *and* $\{e_n\}$ *are arbitrary sequences in* K *and* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ *and* $\{c'_n\}$ *are sequences in* $(0, 1)$ *satisfying*

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n = 1$, *for all* $n \geq 0$,
(ii) $\Sigma b_n = \infty$,
(iii) $\Sigma c_n < \infty$, *and*
(iv) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = 0$.

If $u_0 = x_0 \in K$, *then the following are equivalent:*

(1) *The Mann iteration with modified errors* $\{u_n\}$ *defined by* (2.1.1) *converges to* $x^* \in F(T)$.
(2) *The Ishikawa iteration with modified errors* $\{x_n\}$ *defined by* (2.1.2) *converges to* $x^* \in F(T)$.

Proof. Set $\alpha_n = b_n + c_n$. From (2.1.2), we have

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n - c_n(T y_n - p_n).$$

Thus

$$\begin{aligned} x_n = {} & (1 + \alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (1 - k)\alpha_n x_n \\ & +(2 - k)\alpha_n^2(x_n - T y_n) + \alpha_n(T x_{n+1} - T y_n) \\ & +c_n[1 + (2 - k)\alpha_n][T y_n - p_n]. \end{aligned} \tag{2.1.3}$$

Similarly from (2.1.1), we get

$$\begin{aligned} u_n = {} & (1 + \alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} - (1 - k)\alpha_n u_n \\ & +(2 - k)\alpha_n^2(u_n - T u_n) + \alpha_n(T u_{n+1} - T u_n) \\ & +c_n[1 + (2 - k)\alpha_n][T u_n - e_n]. \end{aligned} \tag{2.1.4}$$

Hence from (2.1.3) and (2.1.4), we have

$$\begin{aligned} x_n - u_n = &(1+\alpha_n)(x_{n+1} - u_{n+1}) + \alpha_n(I - T - kI)x_{n+1} - \alpha_n(I - T - kI)u_{n+1} \\ &-(1-k)\alpha_n(x_n - u_n) + (2-k)\alpha_n^2(x_n - Ty_n - u_n + Tu_n) \\ &+\alpha_n(Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n) \\ &+c_n[1 + (2-k)\alpha_n](Ty_n - p_n - Tu_n + e_n). \end{aligned}$$

Now on taking norms on both sides, we have

$$\begin{aligned} \|x_n - u_n\| \geq &\|(1+\alpha_n)(x_{n+1} - u_{n+1}) + \alpha_n(I - T - kI)x_{n+1} \\ &-\alpha_n(I - T - kI)u_{n+1}\| - (1-k)\alpha_n\|x_n - u_n\| \\ &-(2-k)\alpha_n^2\|x_n - Ty_n - u_n + Tu_n\| \\ &-\alpha_n\|Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n\| \\ &-c_n[1 + (2-k)\alpha_n]\|Ty_n - p_n - Tu_n + e_n\|. \end{aligned} \tag{2.1.5}$$

By using the inequality (1.1.4) with $x = x_{n+1}$, $y = u_{n+1}$ and $s = \frac{\alpha_n}{1+\alpha_n}$ we get

$$\begin{aligned} \|x_n - u_n\| \geq &(1+\alpha_n)\|x_{n+1} - u_{n+1}\| - (1-k)\alpha_n\|x_n - u_n\| \\ &-(2-k)\alpha_n^2\|x_n - Ty_n - u_n + Tu_n\| \\ &-\alpha_n\|Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n\| \\ &-c_n[1 + (2-k)\alpha_n]\|Ty_n - p_n - Tu_n + e_n\|. \end{aligned}$$

Thus

$$\begin{aligned} (1+\alpha_n)\|x_{n+1} - u_{n+1}\| \leq &(1 + (1-k)\alpha_n)\|x_n - u_n\| \\ &+(2-k)\alpha_n^2\|x_n - Ty_n - u_n + Tu_n\| \\ &+\alpha_n\|Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n\| \\ &+c_n[1 + (2-k)\alpha_n]\|Ty_n - p_n - Tu_n + e_n\|. \end{aligned}$$

This implies

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| \leq &\frac{(1 + (1-k)\alpha_n)}{(1+\alpha_n)}\|x_n - u_n\| + (2-k)\alpha_n^2[\|x_n - Ty_n\| + \|u_n - Tu_n\|] \\ &+\alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] \\ &+c_n[1 + (2-k)\alpha_n][\|Ty_n - p_n\| + \|Tu_n - e_n\|] \\ \leq &(1 - k\alpha_n + \alpha_n^2)\|x_n - u_n\| + (2-k)\alpha_n^2[\|x_n - Ty_n\| + \|u_n - Tu_n\|] \\ &+\alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] \\ &+c_n[1 + (2-k)\alpha_n][\|Ty_n - p_n\| + \|Tu_n - e_n\|] \\ \leq &(1 - k\alpha_n)\|x_n - u_n\| + (3-k)\alpha_n^2\|x_n - Ty_n\| \\ &+(2-k)\alpha_n^2\|u_n - Tu_n\| + \alpha_n^2\|u_n - Ty_n\| \\ &+\alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] \\ &+2c_nl_n[\|Ty_n - p_n\| + \|Tu_n - e_n\|] \end{aligned} \tag{2.1.6}$$

where $l_n = (1 + (2-k)\alpha_n) \le M_1$, with $M_1 = 3-k$. Thus

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| \le\ & (1-k\alpha_n)\|x_n - u_n\| + (4-k)\alpha_n^2 D \\ & +(2-k)\alpha_n^2\|u_n - Tu_n\| \\ & +\alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] \\ & +2c_n l_n D \end{aligned} \tag{2.1.7}$$

where D is the diameter of K.

Write $\beta_n = b'_n + c'_n$ in (2.1.2), then we have

$$y_n = (1-\beta_n)x_n + \beta_n T x_n - c'_n(Tx_n - q_n)$$

Consider

$$\begin{aligned}\|x_{n+1} - y_n\| &\le \|x_{n+1} - x_n\| + \|x_n - y_n\| \\ &\le (\alpha_n + c_n)D + (\beta_n + c'_n)D \to 0 \text{ as } n \to \infty.\end{aligned}$$

Now by using uniform continuity of T, we get

$$\|Tx_{n+1} - Ty_n\| \to 0 \text{ as } n \to \infty. \tag{2.1.8}$$

Since T is uniformly continuous and strongly pseudocontractive on K, it follows by Deimling [5] that $F(T) \neq \varnothing$.

We now prove the conclusion of the theorem.

First suppose that $u_n \to x^*$, $x^* \in F(T)$. Then

$$\|Tu_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{2.1.9}$$

Also, $\|u_n - u_{n+1}\| \to 0$ as $n \to \infty$. Again by using the uniform continuity of T, we get

$$\|Tu_n - Tu_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{2.1.10}$$

Write
$\rho_n = \|x_n - u_n\|$, $\lambda_n = k\alpha_n$,

$$\begin{aligned}\sigma_n =\ & (4-k)\alpha_n^2 D + (2-k)\alpha_n^2\|u_n - Tu_n\| \\ & +\alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|], \text{ and}\end{aligned}$$

$\gamma_n = 2c_n l_n D$ in (2.1.7), we get

$$\rho_{n+1} \le (1-\lambda_n)\rho_n + \sigma_n + \gamma_n.$$

From (2.1.8), (2.1.9) and (2.1.10), we have $\frac{\sigma_n}{\lambda_n} \to 0$ as $n \to \infty$, so that $\sigma_n = o(\lambda_n)$ and $\Sigma\gamma_n < \infty$. Now on using Lemma 1.5, we get

$$\lim_{n\to\infty} \rho_n = 0. \tag{2.1.11}$$

Since $u_n \to x^*$ and from (2.1.11), we have

$$\|x_n - x^*\| \le \|x_n - u_n\| + \|u_n - x^*\| \to 0 \text{ as } n \to 0$$

which implies $x_n \to x^*$ as $n \to \infty$.

Conversely suppose that $x_n \to x^*$ as $n \to \infty$. Then

$$\begin{aligned}\|u_n - u_{n+1}\| &\le \alpha_n\|u_n - Tu_n\| + c_n\|Tu_n - e_n\| \\ &\le (\alpha_n + c_n)D \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus by the uniform continuity of T, we get

$$\|Tu_n - Tu_{n+1}\| \to \infty \text{ as } n \to \infty. \tag{2.1.12}$$

Now, write

$\rho_n = \|x_n - u_n\|, \quad \lambda_n = k\alpha_n,$
$\sigma_n = (6-2k)\alpha_n^2 D + \alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|]$ and
$\gamma_n = 2c_n l_n D,$ in (2.1.7), we get

$$\rho_{n+1} \le (1-\lambda_n)\rho_n + \sigma_n + \gamma_n, \quad n = 1, 2, 3, \ldots.$$

From (2.1.12) and (2.1.8), we have
$\sigma = o(\lambda_n)$ and $\Sigma\gamma_n < \infty$. Thus by using Lemma 1.5, we get
$\|x_n - u_n\| \to 0$ as $n \to \infty$.

Since $x_n \to x^*$, we have
$\|u_n - x^*\| \le \|x_n - u_n\| + \|x_n - x^*\| \to 0$ as $n \to \infty$ which implies that
$u_n \to x^*$ as $n \to \infty$.

This completes the proof of the theorem.

Remarks 2.2.

(i) Theorem 2.1 is proved under the assumption that T is uniformly continuous, by relaxing the Lipschitz property of T in Theorem 1.2.

(ii) From Theorem 2.1, it follows that Theorem 1.3 and Theorem 1.4 are equivalent, provided the initial points of the iterations are same.

Theorem 2.3. *Let K be a closed and convex subset of E. Assume that $T : K \to K$ is uniformly continuous and strongly pseudocontractive map with $k \in (0,1)$ with $T(K)$ bounded. For x_0, u_0, p_0, q_0 and e_0 in K, we define the sequence $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ iteratively by $x_0 = u_0 \in K$ as* (2.1.1) *and* (2.1.2) *respectively, where $\{p_n\}$, $\{q_n\}$ and $\{e_n\}$ are bounded sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are sequences in $(0,1)$ satisfying* (i) - (iv) *of Theorem* 2.1. *Then the following are equivalent:*

(1) *The Mann iteration with modified errors $\{u_n\}$ defined by* (2.1.1) *converges to $x^* \in F(T)$.*

(2) *The Ishikawa iteration with modified errors* $\{x_n\}$ *defined by* (2.1.2) *converges to* $x^* \in F(T)$.

Proof. By Deimling [5], $F(T) \neq \varnothing$.
Write

$$M_2 = \sup_{n\geq 1} \|Ty_n - p_n\| \text{ and } M_3 = \sup_{n\geq 1} \|Tu_n - e_n\|.$$

From (2.1.6), we have

$$\begin{aligned}\|x_n - u_n\| \leq & (1 - k\alpha_n)\|x_n - u_n\| + (3 - k)\alpha_n^2\|x_n - Ty_n\| \\ & +(2 - k)\alpha_n^2\|u_n - Tu_n\| + \alpha_n^2\|u_n - Ty_n\| \\ & +\alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] + c_n l_n(M_2 + M_3) \quad (2.3.1)\end{aligned}$$

Let

$$M = \|x_0 - u_0\| + \sup_{n\geq 1}\{\|Ty_n - Tu_n\| + \|Ty_n - p_n\| + \|Tu_n - e_n\|\}.$$

Obviously $0 \leq M < \infty$.
Clearly $\|x_0 - u_0\| \leq M$. Suppose that $\|x_n - u_n\| \leq M$. We prove that $\|x_{n+1} - u_{n+1}\| \leq M$.
From (2.1.1) and (2.1.2), we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| & \leq (1 - \alpha_n)\|x_n - u_n\| + \alpha_n[\|Ty_n - Tu_n\| + c_n[\|Ty_n - p_n\| + \|Tu_n - e_n\|]] \\ & \leq (1 - \alpha_n)\|x_n - u_n\| + \alpha_n[\|Ty_n - Tu_n\| + \|Ty_n - p_n\| + \|Ty_n - e_n\|] \\ & \leq (1 - \alpha_n)M + \alpha_n M = M.\end{aligned}$$

Hence

$$\|x_n - u_n\| \leq M \text{ for all } n \in N. \quad (2.3.2)$$

From (2.1.1) and (2.1.2), we have

$$\begin{aligned}\|x_{n+1} - y_n\| \leq & (\beta_n - \alpha_n)\|x_n\| + \alpha_n\|Ty_n\| + \beta_n\|Tx_n\| \\ & +c_n\|Ty_n - p_n\| + c'_n\|Tx_n - q_n\| \quad (2.3.3)\end{aligned}$$

First we suppose that $u_n \to x^*$. It is clear that from (2.3.2), the sequence $\{x_n\}$ is bounded. Then

$$\begin{aligned}\|x_{n+1} - y_n\| \leq & (\beta_n - \alpha_n)\|x_n\| + \alpha_n\|Ty_n\| + \beta_n\|Tx_n\| \\ & +c_n\|Ty_n - p_n\| + c'_n\|Tx_n - q_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Now by using the uniform continuity of T, we get

$$\|Tx_{n+1} - Ty_n\| \to 0 \text{ as } n \to \infty. \quad (2.3.4)$$

Since $u_n \to x^*$, clearly $\|u_n - u_{n+1}\| \to 0$ as $n \to \infty$.

Again by using the uniform continuity of T, we get

$$\|Tu_n - Tu_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{2.3.5}$$

Now

$$\begin{aligned}\|x_n - Ty_n\| &\leq \|x_n - u_n\| + \|u_n - Tu_n\| + \|Tu_n - Ty_n\| \\ &\leq \|x_n - u_n\| + \|u_n - Tu_n\| + D_1,\end{aligned} \tag{2.3.6}$$

where D_1 is the diameter of $T(K)$, and

$$\|Ty_n - u_n\| \leq \|Ty_n - Tu_n\| + \|Tu_n - u_n\|. \tag{2.3.7}$$

On substituting (2.3.6) and (2.3.7) in (2.3.1), we get

$$\begin{aligned}\|x_n - u_n\| &\leq (1 - k\alpha_n)\|x_n - u_n\| + (3-k)\alpha_n^2[\|x_n - u_n\| + \|u_n - Tu_n\| + D_1] \\ &\quad + (2-k)\alpha_n^2\|u_n - Tu_n\| + \alpha_n^2[\|Ty_n - Tu_n\| + \|u_n - Tu_n\|] \\ &\quad + \alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] \\ &\quad + c_n[1 + (2-k)\alpha_n][\|Ty_n - p_n\| + \|Tu_n - e_n\|] \\ &\leq (1 - k\alpha_n + (3-k)\alpha_n^2)\|x_n - u_n\| + (6-2k)\alpha_n^2\|u_n - Tu_n\| \\ &\quad + (4-k)\alpha_n^2 D_1 + \alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] \\ &\quad + c_n l_n(M_2 + M_3).\end{aligned} \tag{2.3.8}$$

Since $\alpha_n \to 0$ as $n \to \infty$, there exists $n_0 \in N$ such that $\alpha_n \leq \frac{5k}{8(3-k)}$ for all $n \geq n_0$. Thus

$$1 - k\alpha_n + (3-k)\alpha_n^2 \leq 1 - \frac{3k\alpha_n}{8} \quad \text{for all } n \geq n_0. \tag{2.3.9}$$

Hence from (2.3.8) and (2.3.9), we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leq \Big(1 - \frac{3k\alpha_n}{8}\Big)\|x_n - u_n\| + (6-2k)\alpha_n^2\|u_n - Tu_n\| + (4-k)\alpha_n^2 D_1 \\ &\quad + \alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] \\ &\quad + c_n l_n(M_2 + M_3) \quad \text{for all } n \geq n_0.\end{aligned} \tag{2.3.10}$$

Now by applying Lemma 1.5 with

$\rho_n = \|x_n - u_n\|, \quad \lambda_n = \dfrac{3k\alpha_n}{8},$

$\sigma_n = (6-2k)\alpha_n^2\|Tu_n - u_n\| + (4-k)\alpha_n^2 D_1$

$\quad + \alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|],$ and

$\gamma_n = c_n l_n(M_2 + M_3),$ we get $\sigma_n = o(\lambda_n)$ and $\gamma_n < \infty,$ and

$\|x_n - u_n\| \to 0$ as $n \to \infty,$ which implies $x_n \to x^*$ as $n \to \infty.$

Conversely suppose that $x_n \to x^*$ as $n \to \infty$.
Then by using (2.1.1), we get

$$\begin{aligned}\|u_{n+1} - u_n\| &\le \alpha_n \|Tu_n - u_n\| + c_n \|Tu_n - e_n\| \\ &\le \alpha_n(\|Tu_n\| + \|u_n\|) + c_n \|Tu_n - q_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus by the uniform continuity of T, we have

$$\|Tu_n - Tu_{n+1}\| \to \infty \text{ as } n \to \infty. \tag{2.3.11}$$

We use the following estimates:

$$\|x_n - Ty_n\| \le \|x_n - Tx_n\| + \|Tx_n - Ty_n\| \tag{2.3.12}$$

$$\|u_n - Tu_n\| \le \|u_n - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tu_n\| \tag{2.3.13}$$

$$\begin{aligned}\|u_n - Ty_n\| &\le \|u_n - Tu_n\| + \|Tu_n - Ty_n\| \\ &\le \|x_n - u_n\| + \|x_n - Tx_n\| \\ &\quad + \|Tx_n - Tu_n\| + \|Tu_n - Ty_n\|.\end{aligned} \tag{2.3.14}$$

On substituting (2.3.12), (2.3.13) and (2.3.14) in (2.3.10), we get

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\le (1 - k\alpha_n + (3-k)\alpha_n^2)\|x_n - u_n\| + (6-2k)\alpha_n^2 \|x_n - Tx_n\| \\ &\quad + (7-2k)\alpha_n^2 D_1 + \alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|] \\ &\quad + c_n l_n (M_1 + M_2) \qquad (2.3.15) \\ &\le \Big(1 - \frac{3k\alpha_n}{8}\Big)\|x_n - u_n\| + (6-2k)\alpha_n^2 \|x_n - Tx_n\| \\ &\quad + (7-2k)\alpha_n^2 D_1 + \alpha_n \|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\| \\ &\quad + c_n l_n (M_1 + M_2) \quad \text{for all } n \ge n_0. \qquad (2.3.16)\end{aligned}$$

Write

$$\begin{aligned}&\rho_n = \|x_n - u_n\|, \quad \lambda_n = \frac{3k\alpha_n}{8} \\ &\sigma_n = (6-2k)\alpha_n^2 \|x_n - Tx_n\| + (7-2k)\alpha_n^2 D_1 \\ &\qquad + \alpha_n[\|Tx_{n+1} - Ty_n\| + \|Tu_{n+1} - Tu_n\|], \quad \text{and} \\ &\gamma_n = c_n l_n (M_2 + M_3).\end{aligned}$$

Here we observe that $\sigma_n = o(\lambda_n)$.
Now on using Lemma 1.5, we get
$\|x_n - u_n\| \to 0$ as $n \to \infty$, which implies $u_n \to x^*$ as $n \to \infty$.

This completes the proof of the theorem.

Remarks 2.4.

(i) In Theorem 2.3, we assumed that the range of T is bounded whereas in Theorem 1.2, Soltuz [14] assumed that both the domain and range of T are bounded. Thus, boundedness of the domain of T of Theorem 1.2 is redundant.

(ii) Theorem 2.3 generalizes Theorem 1.2 in the sense that we relaxed the boundedness assumption on the set K and the Lipschitz property of the map T by the uniform continuity of T in Theorem 2.3.

Corollary 2.5. *Let E be a real Banach space and $A : E \to E$ be a uniformly continuous and strongly accretive map with the range of $(I - A)$ is bounded. For a fixed $f \in E$, define $A^* : E \to E$ by $A^*x = f + x - Ax$ for each $x \in E$. For a given x_0, p_0, q_0, u_0 and $e_0 \in E$, we define the sequences $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ iteratively by*

$$u_{n+1} = a_n u_n + b_n A^* u_n + c_n e_n, \quad n = 0, 1, 2, 3, \dots. \tag{2.5.1}$$

and

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n A^* y_n + c_n p_n, \\ y_n &= a'_n x_n + b'_n A^* x_n + c'_n q_n, \qquad n = 0, 1, 2, \dots \end{aligned} \tag{2.5.2}$$

where $\{p_n\}$, $\{q_n\}$ and $\{e_n\}$ are arbitrary bounded sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are sequences in $(0, 1)$ satisfying (i) - (iv) of Theorem 2.1 with $x_0 = u_0$ in E, then the following are equivalent:

(1) *The Mann iteration with modified errors (2.5.1) converges to the solution of the equation $Ax = f$.*

(2) *The Ishikawa iteration with modified errors (2.5.2) converges to the solution of the equation $Ax = f$.*

Proof. Existence of the solution follows from Deimling [5]. Since A is uniformly continuous it follows that A^* is uniformly continuous. Since the range of $(I - A)$ is bounded, we have $\|f + (I - A)x_n\|$ is bounded and also $\|f + (I - A)y_n\|$ is bounded. Hence Theorem 2.3 gives the conclusion of Corollary 2.5.

Corollary 2.6. *Let E be a real Banach space and $T : E \to E$ a uniformly continuous and strongly accretive map. Suppose the range of T is bounded. For a fixed $f \in E$, define $S : E \to E$ by $Sx = f - Tx$ for each $x \in E$. For a given x_0, p_0, q_0, u_0 and $e_0 \in E$, we define the sequences $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ iteratively by*

$$u_{n+1} = a_n u_n + b_n S u_n + c_n e_n, \quad n = 0, 1, 2, 3, \dots. \tag{2.6.1}$$

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n S y_n + c_n p_n, \\ y_n &= a'_n x_n + b'_n S x_n + c'_n q_n, \qquad n = 0, 1, 2, \dots \end{aligned} \tag{2.6.2}$$

where $\{p_n\}$, $\{q_n\}$ and $\{e_n\}$ are arbitrary bounded sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are sequences in $(0, 1)$ satisfying (i) - (iv) of Theorem 2.1 with $x_0 = u_0$ in E, then the following are equivalent:

(1) *The Mann iteration with modified errors* (2.6.1) *converges to the solution of the equation* $x + Tx = f$.
(2) *The Ishikawa iteration converges with modified errors* (2.6.2) *to the solution of the equation* $x + Tx = f$.

Proof. We define $A = I + T$, then clearly A is uniformly continuous and strongly accretive. Hence on using Deimling's result [6], the equation $Ax = f$ has a unique solution, say x^* in E. Since the range of T is bounded it follows that range of $I - A$ is bounded. Also,

$$Sx = f - Tx = f - (A - I)x = f + (I - A)x = A^*x,$$

where A^* is as defined in Corollary 2.5. Now from Corollary 2.5, the conclusion follows.

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