Common fixed point theorems for contraction type mappings in partially ordered metric spaces

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Abstract

The purpose of this paper to present the common fixed point theorems for a stronger Meir-Keeler type mapping in partially ordered metric space. Our results generalize some recent results.

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1 Introduction and Preliminaries

Throughout this paper, by $\Re^+$ we denote the set of all nonnegative numbers, while $\mathcal{N}$ is the set of all natural numbers. Existence of fixed point for contraction type maps in partially ordered sets has been considered recently in [1, 2, 3, 4, 5, 6].

In this paper, we define a stronger Meir-Keeler type mapping $\psi : \Re^+ \to [0, 1)$ and we then present the common fixed point theorem for this generalized nonlinear contraction in partially ordered complete metric spaces.

In 1973, Geraghty[7] introduced the following generalization of Banach's contraction principle.

Let S denote the class of the functions $\beta : \Re^+ \to [0, 1)$ which satisfy the condition

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

Theorem 1 *[7] Let (X, d) be a complete metric space, and let $f : X \to X$ be a mapping satisfying*

$$d(fx, fy) \leq \beta(d(x, y)) \cdot d(x, y), \text{ for } x, y \in X,$$

where $\beta \in S$. Then f has a unique fixed point $z \in X$ and $\{f^n(x)\}$ converges to z for each $x \in X$.

Recently, A. Amini-Harandi and H. Emami[2] had proved a version of Theorem 1 in the context of partially ordered complete metric spaces, as follows:

Theorem 2 *[2] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exists $\beta \in S$ such that*

$$d(fx, fy) \leq \beta(d(x, y)) \cdot d(x, y), \text{ for } x, y \in X \text{ with } x \succeq y.$$

Assume that either f is continuous or X is such that

$$\text{if an inereasing sequence } \{x_n\} \to x \text{ in } X, \text{ then } x_n \preceq x, \forall n.$$

Besides, if

$$\text{for each } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y.$$

Then f has a unique fixed point.

Recall the notion of the Meir-Keeler type mapping. A function $\psi : \Re^+ \to \Re^+$ is said to be a Meir-Keeler type mapping(see [8]), if for each $\eta \in \Re^+$, there exists $\delta > 0$ such that for $t \in \Re^+$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$. We now define a new stronger Meir-Keeler cone-type mapping, as follows:

Definition 1 *A mapping $\psi : \Re^+ \to [0, 1)$ is called a stronger Meir-Keeler type mapping, if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in \Re^+$ with $\eta \leq t < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(t) < \gamma_\eta$.*

Example 1 *If $\psi : \Re^+ \to [0, 1)$, $\psi(t) = \frac{t}{t+1}$, then ψ is a stronger Meir-Keeler type mapping.*

2 Main results

Definition 2 *[3] Let $(X, \preceq)$ be a partially ordered set and $f, g : X \to X$. Then f is said to be g-nondecreasing if for $x, y \in X$,*

$$gx \preceq gy \Longrightarrow fx \preceq fy.$$

In the sequal, we prove our main results of this paper.

Theorem 3 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f, g : X \to X$ be such that $fX \subset gX$, and f be a g-nondecreasing mapping. Suppose that there exist a stronger Meir-Keeler type mapping $\psi : \Re^+ \to [0, 1)$ such that*

$$d(fx, fy) \leq \psi(d(gx, gy)) \cdot d(gx, gy), \text{ for } x, y \in X \text{ with } gx \succeq gy. \tag{1}$$

Also suppose,

$$\text{if } \{gx_n\} \subset X \text{ is a nondecreasing sequence with } gx_n \to gz \in gX,$$

$$\text{then } gx_n \preceq gz \text{ and } gz \preceq g^2 z \text{ for all } n \in N. \tag{2}$$

And suppose that gX is closed. If there exists an $x_0 \in X$ with $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Further, if f, g commute their coincidence points, then f, g have a common fixed point in X.

Proof. Given $x_0 \in X$ such that $gx_0 \preceq fx_0$. Since $fX \subset gX$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Continuing this process, we define the sequence $\{x_n\}$ in X recusively as follows:

$$fx_n = gx_{n+1}, \text{ for all } n \in \mathcal{N}.$$

Since $gx_0 \preceq fx_0$ and f is a g-nondecreasing mapping, we have

$$fx_0 \preceq fx_1 \preceq fx_2 \cdots \preceq fx_n \preceq fx_{n+1} \preceq \cdots$$

In what follows we will suppose that $d(fx_n, fx_{n+1}) > 0$ for all $n \in N$, since if $fx_{n+1} = fx_n$ for some n, then $fx_{n+1} = gx_{n+1}$, that is , f, g have a coincidence point x_{n+1}, and so we complete the proof.

By (1), we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq \psi(d(gx_n, gx_{n+1})) \cdot d(gx_n, gx_{n+1}) \qquad (*) \\ &\leq d(gx_n, gx_{n+1}) \\ &\leq d(fx_{n-1}, fx_n). \end{aligned}$$

Then the sequence $\{d(fx_{n+1}, fx_n)\}$ is descreasing and bounded below. Let $\lim_{n\to\infty} d(fx_{n+1}, fx_n) = \eta \geq 0$. Then there exists $\kappa_0 \in \mathcal{N}$ and $\delta > 0$ such that for all $n > \kappa_0$

$$\eta \leq d(fx_{n+1}, fx_n) < \eta + \delta.$$

For each $n \in \mathcal{N}$, since $\psi : \Re^+ \to [0, 1)$ is a stronger Meir-Keeler type mapping, for these η and $\delta > 0$ we have that for $d(fx_{\kappa_0+n+1}, fx_{\kappa_0+n}) \in \Re^+$ with $\eta \leq d(fx_{\kappa_0+n+1}, fx_{\kappa_0+n}) < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(d(fx_{\kappa_0+n+1}, fx_{\kappa_0+n})) < \gamma_\eta$. Thus, by $(*)$, we can deduce

$$\begin{aligned} d(fx_{\kappa_0+n+1}, fx_{\kappa_0+n}) &= \psi(d(fx_{\kappa_0+n}, fx_{\kappa_0+n-1})) \cdot d(fx_{\kappa_0+n}, fx_{\kappa_0+n-1}) \\ &< \gamma_\eta d(fx_{\kappa_0+n}, fx_{\kappa_0+n-1}), \end{aligned}$$

and it follows that for each $n \in \mathcal{N}$

$$\begin{aligned} d(fx_{\kappa_0+n+1}, fx_{\kappa_0+n}) &< \gamma_\eta d(d(fx_{\kappa_0+n}, fx_{\kappa_0+n-1}) \\ &< \cdots \\ &< \gamma_\eta^n d(fx_{\kappa_0+2}, fx_{\kappa_0+1}). \end{aligned}$$

So

$$\lim_{n\to\infty} d(fx_{\kappa_0+n+1}, fx_{\kappa_0+n}) = 0, \text{ since } \gamma_\eta < 1.$$

We now claim that $\lim_{n\to\infty} d(fx_{\kappa_0+m}, fx_{\kappa_0+n}) = 0$ for $m > n$. For $m, n \in \mathcal{N}$ with $m > n$, we have

$$d(fx_{\kappa_0+m}, fx_{\kappa_0+n}) \leq \sum_{i=n}^{m-1} d(fx_{\kappa_0+i+1}, fx_{\kappa_0+i}) < \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} d(fx_{\kappa_0+2}, fx_{\kappa_0+1}),$$

and hence $d(fx_m, fx_n) \to 0$, since $0 < \gamma_\eta < 1$. So $\{fx_n\}$ is a Cauchy sequence.

Since gX is closed and $\{fx_n\} = \{g_x n + 1\} \subset gX$, there exists $\mu \in X$ such that

$$\lim_{n\to\infty} gx_n = g\mu = \lim_{n\to\infty} fx_n.$$

Now we claim that μ is a coincidence point of f and g. By (2), we have

$$\begin{aligned} d(g\mu, f\mu) &\leq d(g\mu, fx_n) + d(fx_n, f\mu) \\ &\leq d(g\mu, fx_n) + \psi(d(gx_n, g\mu)) \cdot d(gx_n, g\mu) \\ &< d(g\mu, fx_n) + d(gx_n, g\mu). \end{aligned}$$

Letting $n \to \infty$, we get $g\mu = f\mu$.

Suppose that f and g commute at μ. Let $\nu = g\mu = f\mu$. Then

$$f\nu = fg\mu = gf\mu = g\nu.$$

By (2), since $g\mu \preceq g^2\mu = g\nu$, $g\mu = f\mu$ and $g\nu = f\nu$, we deduce

$$\begin{aligned} d(f\mu, f\nu) &\leq \psi(d(g\mu, g\nu)) \cdot d(g\mu, g\nu) \\ &< d(f\mu, f\nu). \end{aligned}$$

Hence $d(f\mu, f\nu) = 0$, that is $d(\nu, f\nu) = 0$. Therefore,

$$f\nu = g\nu = \nu.$$

So we proved that f and g have a common fixed point in X.□

By Theorem 3, we let $g = I =$ the identity mapping, then we immediate obtain the following corollaries.

Corollary 1 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping. Suppose that there exist a stronger Meir-Keeler type mapping $\psi : \Re^+ \to [0, 1)$ such that*

$$d(fx, fy) \leq \psi(d(x, y)) \cdot d(x, y), \text{ for } x, y \in X \text{ with } x \preceq y.$$

Also suppose,

$$\text{if } \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \to z \in X,$$
$$\text{then } x_n \preceq z \text{ forall } n \in N.$$

If there exists an $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X.

Corollary 2 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Suppose $f : X \to X$ be a nondecreasing mapping and*

$$d(fx, fy) \leq kd(x, y), \text{ for } x, y \in X \text{ with } x \preceq y \text{ where } k \in (0, 1).$$

Also suppose,

$$\text{if } \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \to z \in X,$$

$$\text{then } x_n \preceq z \text{ forall } n \in N.$$

If there exists an $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X.

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