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LIE THEORETIC ORIGIN OF SOME GENERATING FUNCTIONS OF FOX'S H-FUNCTION

RENU JAIN, D. K. JAIN AND P. K. CHATURVEDI

Abstract. The group theoretic method for achieving unification of diverse mass of literature of special functions is most recent of such efforts and is definitely the most elegant one. In this method the special functions emerge as basis vectors and matrix elements of local multiplier representation of some well known groups. This dual role played by special functions affords a powerful technique for derivation of several generating functions and addition theorems for them.

The present paper aims at harnessing this technique to generate, derive and interpret certain expansion of Fox's H-function in series of H-function. In the special case these expansions reduce to corresponding results for G-function.

1. Introduction

Lie theoretic techniques are being employed since 1950's to achieving unification in heterogeneous growth of literature on special functions.

One of the foremost applications of Lie theory to special functions is for derivation of their generating functions. On the basis of differential recurrence relations satisfied by a given special function a Lie algebra of differential operators is constructed. The local multiplier representation for the corresponding Lie group then yields generating relation for corresponding special functions.

Miller W. [4, 5] has done most noteworthy work in this direction and has obtained generating functions for hypergeometric functions, Lauricella functions and Meijer's G-functions, besides undertaking exhaustive theoretical investigations which have gone a long way in establishing the significance of group theoretic techniques. In the present paper a Lie algebra has been constructed on the basis of differential recurrence relations satisfied by H-function. Then multiplier representation for one parameter subgroup is employed to derive a generating relation for those functions.

Corresponding author: Renu Jain.
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2. The H-function

The Fox's H-function is defined as [2]

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}\{(a_j,\alpha_j)_{1,p}\}\\\{(b_j,\beta_j)_{1,q}\}\end{matrix}\right]$$
$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)} Z^s ds \tag{2.1}$$

Here ω is square root of -1 the detailed conditions of formation are referred to in the literature [2].

Following differential recurrence relations are obeyed by Fox's H-function [6, 8].

$$\left(z\partial z-\frac{(a_j-1)}{\alpha_j}\right)H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] = \pm\frac{1}{\alpha_j}H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_j-1,\alpha_j)\\(b_q,\beta_q)\end{matrix}\right] \tag{2.2}$$

(signs are positive or negative according to $1\le j\le n$ or $n+1\le j\le p$).

$$\left(z\partial z-\frac{b_k}{\beta_k}\right)H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] = \mp\frac{1}{\beta_k}H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_k+1,\beta_q)\end{matrix}\right] \tag{2.3}$$

(if $1\le k\le m$ sign would be negative and if $m+1\le k\le$ q sign would be positive). Also

$$zH_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p+\alpha_p,\alpha_p)\\(b_q+\beta_q,\beta_q)\end{matrix}\right]. \tag{2.4}$$

Introducing new parameters $t_1,t_2,\dots,t_p;u_1,u_2,\dots,u_q$, we define the following basis functions.

$$F_{p,q}^{m,n}\left[z\middle|\begin{matrix}t_p\\u_q\end{matrix}\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] t_1^{a_1}\cdots t_p^{a_p}\cdot u_1^{b_1}\cdots u_q^{b_q}. \tag{2.5}$$

With the help of relations (2.1) and (2.2), we construct partial differential operators

$$\begin{aligned}
T_j &= \frac{t_j}{\alpha_j}\partial t_j; \qquad L_j = t_j^{-1}\left(z\partial z-\frac{t_j}{\alpha_j}\partial t_j+\frac{1}{\alpha_j}\right) \quad \text{for } 1\le j\le p;\\
U_k &= \frac{u_k}{\beta_k}\partial u_k; \qquad R_k = u_k\left(z\partial z-\frac{u_k}{\beta_k}\partial u_k\right) \qquad \text{for } 1\le k\le q \text{ and}\\
V &= zt_1^{a_1}\cdots t_p^{a_p}\cdot u_1^{b_1}\cdots u_q^{b_q}
\end{aligned} \tag{2.6}$$

These operators satisfy following commutation relations

$$[T_j,L_j] = -\frac{L_j}{\alpha_j}; \qquad [U_k,R_k] = \frac{R_K}{\beta_K}; \qquad [T_j,V] = [U_k,V] = V. \tag{2.7}$$

All other commutator relations of any two operators are zero. It can be easily verified that operators generate a $2(p+q)+1$ dimensional Lie-algebra. The actions of these operators on basis functions defined by equation (2.5) is given by

$$T_j F_{p,q}^{m,n}[z] = \frac{a_j}{\alpha_j} F_{p,q}^{m,n}[z]$$

$$U_k F_{p,q}^{m,n}[z] = \frac{b_k}{\beta_k} F_{p,q}^{m,n}[z]$$

$$L_j F_{p,q}^{m,n}[z] = \pm\frac{1}{\alpha_j} F_{p,q}^{m,n}\left[z\Big| \begin{matrix} t_p \\ u_q \end{matrix} \Big| \begin{matrix} (a_1,\alpha_1),\dots,(a_j-1,\alpha_j),\dots,(a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right]$$

(plus sign corresponds to $1 \le j \le n$ and minus sign to $n+1 \le j \le p$).

$$R_k F_{p,q}^{m,n}[z] = \mp\frac{1}{\beta_k} F_{p,q}^{m,n}\left[z\Big| \begin{matrix} t_p \\ u_q \end{matrix} \Big| \begin{matrix} (a_p,\alpha_p) \\ (b_1,\beta_1),\dots,(b_k+1,\beta_k),\dots,(b_q,\beta_q) \end{matrix} \right] \tag{2.8}$$

(minus sign to be taken if $1 \le k \le m$ and plus sign to be taken if $m+1 \le k \le q$).

3. Generating functions resulting from the multiplier representation for one parameter subgroup exp (λL_j)

By standard multiplier representation technique for obtaining generating relations for H-functions we need computation of action of one parametric subgroup (λL_j) . This can be done by usual techniques of multiplier representation theory [4]. Thus we obtain resulting transformations in the form

$$\begin{bmatrix} z \\ t_j \\ u_k \end{bmatrix} \to \begin{bmatrix} z(1-\frac{\lambda}{\alpha_j t_j})^{-\alpha_j} \\ t_j - \frac{\lambda}{\alpha_j} \\ u_k \end{bmatrix} \tag{3.1}$$

Also the multiplier v is given by

$$v = \left(1 - \frac{\lambda}{\alpha_j t_j}\right)^{-1} \tag{3.2}$$

(i) Now we drive the multiplication theorems for Fox's H-functions corresponding to operatorL_j, when $1 \le j \le n$.

By direct expansion, we get

$$\exp(\lambda L_j) F_{p,q}^{m,n}\left[z\Big| \begin{matrix} t_p \\ u_q \end{matrix} \Big| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} (L_j)^r F_{p,q}^{m,n}\left[z\Big| \begin{matrix} t_p \\ u_q \end{matrix} \Big| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right]$$

$$= \sum_{r=0}^{\infty} \frac{\lambda^r}{r!(\alpha_j)^r} F_{p,q}^{m,n}\left[z\Big| \begin{matrix} t_p \\ u_q \end{matrix} \Big| \begin{matrix} (a_1,\alpha_1)\cdots(a_j-r,\alpha_j)\cdots(a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right]$$

$$= \sum_{r=0}^{\infty} \frac{\lambda^r}{r!(\alpha_j)^r} H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1,\alpha_1),\dots,(a_j-r,\alpha_j),\dots,(a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] t_1^{a_1}\cdots t_j^{a_j-r}\cdots t_p^{a_p}\cdot u_1^{b_1}\cdots u_q^{b_q}. \quad (3.3)$$

On the other hand evaluating $\exp(\lambda L_j)F_{p,q}^{m,n}$ by the transformation formulae (3.1) and (3.2), we get

$$\exp(\lambda L_j)F_{p,q}^{m,n}\left[z \middle| \begin{matrix} t_p \\ u_q \end{matrix} \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right]$$
$$= (1-\frac{\lambda}{\alpha_j t_j})^{-1} H_{p,q}^{m,n}\left[z(1-\frac{\lambda}{\alpha_j t_j})^{-\alpha_j} \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] t_1^{a_1}\cdots(t_j-\frac{\lambda}{\alpha_j})^{-a_j}\cdots t_p^{a_p}\cdot u_1^{b_1}\cdots u_q^{b_q}. \quad (3.4)$$

Comparing these two values of $exp(\lambda L_j)F_{p,q}^{m,n}[z]$, we arrive at the identity

$$(1-\frac{\lambda}{\alpha_j t_j})^{-1} H_{p,q}^{m,n}\left[z(1-\frac{\lambda}{\alpha_j t_j})^{-\alpha_j} \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] (t_j-\frac{\lambda}{\alpha_j})^{-\alpha_j}$$
$$= \sum_{r=0}^{\infty} \frac{\lambda^r}{r!(\alpha_j)^r} H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1,\alpha_1),\dots,(a_j-r,\alpha_j),\dots,(a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] \times t_j^{a_j-r} \quad (3.5)$$

Employing substitution $(1-\frac{\lambda}{\alpha_j t_j})^{-\alpha_j} = \eta$, we get the generating relation

$$H_{p,q}^{m,n}\left[z\eta \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] = \eta^{\frac{a_j-1}{\alpha_j}} \sum_{r=0}^{\infty} \frac{(1-\eta^{\frac{-1}{\alpha_j}})^r}{r!} H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1,\alpha_1),\dots,(a_j-r,\alpha_j),\dots,(a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right]. \quad (3.6)$$

For $j=1$, this general result reduces to well known generating relation for Fox's H-function [7, (2.9.11); p. 25].

$$H_{p,q}^{m,n}\left[z\eta] \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] = \eta^{\frac{a_1-1}{\alpha_1}} \sum_{r=0}^{\infty} \frac{(1-\eta^{\frac{-1}{\alpha_1}})^r}{r!} H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1-r,\alpha_1),(a_j,\alpha_j)_{2,p} \\ (b_k,\beta_k)_{1,q} \end{matrix} \right] \quad (3.7)$$

Again particularizing (3.7), by substitution $\alpha_j = \beta_k = 1$ and $\eta = \lambda$ it takes the form of a well known generating relation for Meijer's G-function [1, (3); p.213]

$$G_{p,q}^{m,n}\left[z\lambda \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right] = \lambda^{(a_1-1)} \sum_{r=0}^{\infty} \frac{(1-\frac{1}{\lambda})^r}{r!} G_{p,q}^{m,n}\left[z \middle| \begin{matrix} a_1-r,a_2,\dots,a_p \\ b_1,\dots,b_q \end{matrix} \right]. \quad (3.8)$$

(ii) Similarly, considering the action of one parameter subgroup $exp(\lambda L_j)$ generated by operator L_j when $n+1 \le j \le p$, we arrive at the identity

$$(1-\frac{\lambda}{\alpha_j t_j})^{-1} H_{p,q}^{m,n}\left[z(1-\frac{\lambda}{\alpha_j t_j})^{-\alpha_j} \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] (t_j-\frac{\lambda}{\alpha_j})^{a_j}$$

$$= \sum_{r=0}^{\infty} \frac{(-\lambda)^r}{(\alpha_j)^r r!} H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1,\alpha_1),\ldots,(a_j-r,\alpha_j),\ldots,(a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] \times t_j^{a_j-r}. \tag{3.9}$$

On substituting $(1-\frac{\lambda}{\alpha_j t_j})^{-\alpha_j} = \eta$, equation (3.9) takes the form of the following generating function for Fox's H-function

$$H_{p,q}^{m,n}\left[z\eta \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] = \eta^{\frac{(a_j-1)}{\alpha_j}} \sum_{r=0}^{\infty} \frac{(\eta^{\frac{-1}{\alpha_j}}-1)^r}{r!} H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1,\alpha_1),\ldots,(a_j-r,\alpha_j),\ldots,(a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right]. \tag{3.10}$$

Now for $j = p$, the relati on (3.10) reduces to a well- known generating function [7, (2.9.9); p.25] for Fox's H-function

$$H_{p,q}^{m,n}\left[z\eta \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] = \eta^{\frac{a_p-1}{\alpha_p}} \sum_{r=0}^{\infty} \frac{\eta^{\frac{-1}{\alpha_p}}-1^r}{r!} H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,p-1},(a_p-r,\alpha_p) \\ (b_k,\beta_k)_{1,q} \end{matrix} \right]. \tag{3.11}$$

Further if we take $\alpha_j = \beta_k = 1$ and $\eta = \lambda$ then result (20) reduces to well known generating function [1, (4); p. 213] for Meijer's G-function.

$$G_{p,q}^{m,n}\left[z\lambda \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right] = \lambda^{a_p-1} \sum_{r=0}^{\infty} \frac{1}{r!}(\frac{1}{\lambda}-1)^r G_{p,q}^{m,n}\left[z \middle| \begin{matrix} a_1,a_2,\ldots,a_{p-1},a_p-r \\ b_1,\ldots,b_q \end{matrix} \right]. \tag{3.12}$$

4. Generating functions resulting from multiplier representation for one parameter subgroup $\exp(\mu R_k)$

The action of one parameter subgroup $exp(\mu R_k)$ on analytic function R_k can be computed by employing usual techniques of multiplier representation theory [4].

The induced transformations are

$$\begin{bmatrix} z \\ t_j \\ u_k \end{bmatrix} \to \begin{bmatrix} z(1+\frac{\mu u_k}{\beta_k})^{\beta_k} \\ t_j \\ u_k(1+\frac{\mu u_k}{\beta_k})^{-1} \end{bmatrix}. \tag{4.1}$$

Evaluating $\exp(\mu R_k) F_{p,q}^{m,n}\left[z \right]$ by direct expansion on one hand and using above transformation on the other, as was done in the previous section, we arrive at the identities.

$$H_{p,q}^{m,n}\left[z(1+\frac{\mu u_k}{\beta_k})^{\beta_k} \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] u_k^{b_k}(1+\frac{\mu u_k}{\beta_k})^{-b_k}$$
$$= \sum_{r=0}^{\infty} \frac{(-\mu)^r}{r!(\beta_k)^r} H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_p,\alpha_p) \\ (b_1,\beta_1),\ldots,(b_k+r,\beta_k),\ldots,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),\ldots,(b_q,\beta_q) \end{matrix} \right] \times u_k^{b_k+r}, \tag{4.2}$$

when $1 \le k \le m$ and

$$H^{m,n}_{p,q}\left[z(1+\frac{\mu u_k}{\beta_k})^{\beta_k}\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right](u_k)^{b_k}(1+\frac{\mu u_k}{\beta_k})^{-b_k}$$
$$=\sum_{r=0}^{\infty}\frac{(\mu)^r}{r!(\beta_k)^r}H^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),(b_k+r,\beta_k),\ldots,(b_q,\beta_q)\end{matrix}\right]\times u_k^{b_k+r}, \tag{4.3}$$

when $m+1 \le k \le q$.

Using substitution $\eta = (1+\frac{\mu u_k}{\beta_k})^{\beta_k}$ these identities take the form

$$H^{m,n}_{p,q}\left[z\eta\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] \tag{4.4}$$
$$=\eta^{\frac{b_k}{\beta_k}}\sum_{r=0}^{\infty}\frac{(1-\eta^{\frac{1}{\beta_k}})^r}{r!}H^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_k+r,\beta_k),(b_m,\beta_m),(b_{m+1},\beta_{m+1}),\ldots,(b_q,\beta_q)\end{matrix}\right]$$

when $1 \le k \le m$ and

$$H^{m,n}_{p,q}\left[z\eta\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] \tag{4.5}$$
$$=\eta^{\frac{b_k}{\beta_k}}\sum_{r=0}^{\infty}\frac{(\eta^{\frac{1}{\beta_k}}-1)^r}{r!}H^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_m,\beta_m),(b_{m+1},\beta_{m+1}),\ldots,(b_k+r,\beta_k),\ldots,(b_q,\beta_q)\end{matrix}\right],$$

when $m+1 \le k \le q$.

Identities (**??**) and (4.5) reduce to a well known generating functions for Fox's H-functions [7, (2.9.10), (2.9.12); p. 25] in the special case when $k = 1$ and $k = q$ respectively.

Further if we take $\alpha_j = \beta_k$ and $\eta = \lambda$ then the result (**??**) and (4.5) reduce to a well known generating functions for Meijer's G-function [1, (1),(2); p. 213] respectively.

5. Conclusion

Besides obtaining the generating relations for the functions in terms of H-functions, the present paper attempts to uncover the group theoretic structure of H-functions. It illustrates how a systematic development can be accorded to the study of H-functions, and then from them particular cases to less general special functions.

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School of Mathematics and Allied Sciences, Jiwaji University, Gwalior - 474 011 (M.P.), India.

E-mail: renujain3@rediffmail.com

Department of Applied Mathematics, Madhav Institute of Technology & Science, Gwalior - 474005 (M.P.), India.

E-mail: jain_dkj@yahoo.co.in

3Department of Mathematics, Rishi Galav College, Morena - 476 001 (M.P.), India.