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NOTE ON OSCILLATION THEOREMS FOR THIRD ORDER NON-LINEAR DIFFERENCE EQUATIONS

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Abstract. In this paper some sufficient conditions for oscillation of all solutions of certain difference equations are obtained. Examples are given to illustrate the results.

1. Introduction

We are concerned with the oscillatory properies of all solutions of third order linear and non-linear difference equations of the form

$$\Delta^2(a_n\Delta x_n) - p_n\Delta x_n + q_n f(x_{n+1}) = 0, \quad n = 0,1,2\ldots, \tag{1.1}$$

$$\Delta^2(a_n\phi(x_n)\Delta x_n) - p_n\Delta x_n + q_n f(x_{n+1}) = 0, \quad n = 0,1,2\ldots, \tag{1.2}$$

$$\Delta^2(a_n\Delta x_n) + q_n f(x_{n+1}) = 0, \quad n = 0,1,2\ldots, \tag{1.3}$$

$$\Delta^2(a_n\phi(x_n)\Delta x_n) + q_n f(x_{n+1}) = 0, \quad n = 0,1,2\ldots, \tag{1.4}$$

where the following conditions are assumed to hold.

(H1) $\{a_n\}$,$\{p_n\}$ and $\{q_n\}$ are real positive sequences where $n \in N = \{0,1,2,3.,...\}$.

(H2) $f : R \to R$ is continous and $xf(x) > 0$ for all $x \neq 0$.

(H3) there exixts a real valued function g such that $f(u) - f(v) = g(u,v)(u-v)$, for all $u \neq 0$ and $v \neq 0$ and $g(u,v) \geq L > 0 \in R$.

(H4) $\phi : R \to R$ is continous for all $x \neq 0, \phi(x) > 0$

(H5) $\sum_{n=M}^{\infty}(n+1)p_n^2 < \infty$.

(H6) $\sum_{n=0}^{\infty} a_n^2 < \infty$.

(H7) $\sum_{n=0}^{\infty}(n+1)q_n = \infty$.

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By a solution of equation (1.1), we mean a real sequecne $\{x_n\}$ satsifying (1.1) for $n = 0, 1, \ldots$. A solution $\{x_n\}$ of (1.1) is said to be oscillatory if it is neither eventually positve nor eventually negative. Otherwise, it is called non-oscillatory. Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.This definition holds for other equations.

In recent years, much research is going on the study of oscillatory behavior of solutions of third order difference equations.

For more details on oscillatory behavior of difference equations, one may refer to [1-24].

2. Main results

In this section, we present some sufficient conditions for the oscillation of all solutions of (1.1)-(1.4). We begin with the following lemma.

Lemma 1. *Let $P(n,s,x)$ be defined on $N \times N \times R^+, N = \{0,1,2,\ldots\}$, $R^+ = [0,\infty)$, such that for fixed n and s,the function $P(n,s,x)$ is non-decreasing in x. Let $\{r_n\}$ be a given sequence and the sequences $\{x_n\}$ and $\{z_n\}$ be defined on N satisfying, for all $n \in N$,*

$$x_n \geq r_n + \sum_{s=0}^{n-1} P(n,s,x_s), \tag{2.1}$$

and

$$z_n = r_n + \sum_{s=0}^{n-1} P(n,s,z_s) \tag{2.2}$$

respectively. Then $z_n \leq x_n$ for all $n \in N$.

The proof can be found in [15].

Theorem 1. *In addition to* (H1), (H2) and (H3), *assume that* (H5), (H6) *and* (H7) *hold.Then,every solution of* (1.1) *is oscillatory.*

Proof. Suppose the contrary.Then we may assume that$\{x_n\}$ be a non-oscillatory solution of (1.1),such that $x_n > 0 (or\, x_n < 0)$ for all $n \geq M-1$, $M > 0$ is an integer.

Equation (1.1) implies

$$\Delta(a_{n+1}\Delta x_{n+1}) - \Delta(a_n \Delta x_n) - p_n \Delta x_n + q_n f(x_{n+1}) = 0 \tag{2.3}$$

Multiplying (2.3) by $\frac{n+1}{f(x_{n+1})}$ and summing from M to $(n-1)$, we obtain

$$\sum_{s=M}^{n-1}\left(\frac{s+1}{f(x_{s+1})}\right)\Delta(a_{s+1}\Delta x_{s+1}) - \sum_{s=M}^{n-1}\left(\frac{s+1}{f(x_{s+1})}\right)\Delta(a_s \Delta x_s) - \sum_{s=M}^{n-1}\left(\frac{s+1}{f(x_{s+1})}\right)p_s \Delta x_s + \sum_{s=M}^{n-1}(s+1)q_s = 0. \tag{2.4}$$

But

$$\begin{aligned}
&\sum_{s=M}^{n-1}\left(\frac{s+1}{f(x_{s+1})}\right)\Delta(a_{s+1}\Delta x_{s+1})\\
&\quad=\left(\left(\frac{s+1}{f(x_{s+1})}\right)(a_{s+1}\Delta x_{s+1})\right)_{s=M}^{n}-\sum_{s=M}^{n-1}\Delta\left(\frac{s+1}{f(x_{s+1})}\right)(a_{s+2}\Delta x_{s+2}).\\
&\quad=\frac{(n+1)a_{n+1}\Delta x_{n+1}}{f(x_{n+1})}-\frac{(M+1)a_{M+1}\Delta x_{M+1}}{f(x_{M+1})}\\
&\qquad-\sum_{s=M}^{n-1}\frac{a_{s+2}\Delta x_{s+2}}{f(x_{s+2})}+\sum_{s=M}^{n-1}\frac{(s+1)a_{s+2}g(x_{s+2},x_{s+1})\Delta x_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})}.
\end{aligned}\tag{2.5}$$

Also,

$$\begin{aligned}
\sum_{s=M}^{n-1}\left(\frac{s+1}{f(x_{s+1})}\right)\Delta(a_s\Delta x_s)&=\left(\left(\frac{s+1}{f(x_{s+1})}\right)(a_s\Delta x_s)\right)_{s=M}^{n}-\sum_{s=M}^{n-1}\Delta\left(\frac{s+1}{f(x_{s+1})}\right)(a_{s+1}\Delta x_{s+1})\\
&=\frac{(n+1)a_n\Delta x_n}{f(x_{n+1})}-\frac{(M+1)a_M\Delta x_M}{f(x_{M+1})}\\
&\quad-\sum_{s=M}^{n-1}\frac{a_{s+1}\Delta x_{s+1}}{f(x_{s+2})}+\sum_{s=M}^{n-1}\frac{(s+1)a_{s+1}g(x_{s+2},x_{s+1})(\Delta x_{s+1})^2}{f(x_{s+1})f(x_{s+2})}.
\end{aligned}\tag{2.6}$$

Subsituting (2.5) and (2.6) in (2.4), we have

$$\begin{aligned}
&\left(\frac{(n+1)a_{n+1}\Delta x_{n+1}}{f(x_{n+1})}-\frac{(n+1)a_n\Delta x_n}{f(x_{n+1})}\right)-\left(\sum_{s=M}^{n-1}\frac{a_{s+2}\Delta x_{s+2}}{f(x_{s+2})}-\sum_{s=M}^{n-1}\frac{a_{s+1}\Delta x_{s+1}}{f(x_{s+2})}\right)\\
&\quad+\left(\sum_{s=M}^{n-1}\frac{(s+1)a_{s+2}g(x_{s+2},x_{s+1})\Delta x_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})}-\sum_{s=M}^{n-1}\frac{(s+1)a_{s+1}g(x_{s+2},x_{s+1})(\Delta x_{s+1})^2}{f(x_{s+1})f(x_{s+2})}\right)\\
&\quad-\sum_{s=M}^{n-1}\left(\frac{s+1}{f(x_{s+1})}\right)p_s\Delta x_s+\sum_{s=M}^{n-1}(s+1)q_s\\
&=\left(\frac{(M+1)a_{M+1}\Delta x_{M+1}}{f(x_{M+1})}-\frac{(M+1)a_M\Delta x_M}{f(x_{M+1})}\right).
\end{aligned}\tag{2.7}$$

Using Schwarz's inequality,we have

$$\sum_{s=M}^{n-1}\left(\frac{a_{s+2}\Delta x_{s+2}}{f(x_{s+2})}\right)\le\left(\sum_{s=M}^{n-1}a_{s+2}^2\right)^{\frac{1}{2}}\cdot\left(\sum_{s=M}^{n-1}\frac{(\Delta x_{s+2})^2}{f^2(x_{s+2})}\right)^{\frac{1}{2}}.\tag{2.8}$$

$$\sum_{s=M}^{n-1}\left(\frac{a_{s+1}\Delta x_{s+1}}{f(x_{s+2})}\right)\le\left(\sum_{s=M}^{n-1}a_{s+1}^2\right)^{\frac{1}{2}}\cdot\left(\sum_{s=M}^{n-1}\frac{(\Delta x_{s+1})^2}{f^2(x_{s+2})}\right)^{\frac{1}{2}}.\tag{2.9}$$

$$\begin{aligned}
&\sum_{s=M}^{n-1}\left(\frac{(s+1)a_{s+2}g(x_{s+2},x_{s+1})\Delta x_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})}\right)\\
&\qquad\le\left(\sum_{s=M}^{n-1}a_{s+2}^2\right)^{\frac{1}{2}}\cdot\left(\sum_{s=M}^{n-1}\frac{(s+1)^2g^2(x_{s+2},x_{s+1})(\Delta x_{s+1})^2(\Delta x_{s+2})^2}{f^2(x_{s+1})f^2(x_{s+2})}\right)^{\frac{1}{2}}.
\end{aligned}\tag{2.10}$$

$$\sum_{s=M}^{n-1}\left(\frac{(s+1)a_{s+1}g(x_{s+2},x_{s+1})(\Delta x_{s+1})^2}{f(x_{s+1})f(x_{s+2})}\right)$$

$$\leq\left(\sum_{s=M}^{n-1}a_{s+1}^2\right)^{\frac{1}{2}}\cdot\left(\sum_{s=M}^{n-1}\frac{(s+1)^2g^2(x_{s+2},x_{s+1})(\Delta x_{s+1})^4}{f^2(x_{s+1})f^2(x_{s+2})}\right)^{\frac{1}{2}}. \tag{2.11}$$

$$\sum_{s=M}^{n-1}\left(\frac{(s+1)p_s\Delta x_s}{f(x_{s+1})}\right)\leq\left(\sum_{s=M}^{n-1}(s+1)p_s^2\right)^{\frac{1}{2}}\cdot\left(\sum_{s=M}^{n-1}\frac{(s+1)(\Delta x_s)^2}{f^2(x_{s+1})}\right)^{\frac{1}{2}}. \tag{2.12}$$

In view of (2.8), (2.9), (2.10), (2.11) and (2.12),the summation in (2.7) is bounded, we have

$$\left(\frac{(n+1)a_{n+1}\Delta x_{n+1}}{f(x_{n+1})}-\frac{(n+1)a_n\Delta x_n}{f(x_{n+1})}\right)-\left(\sum_{s=M}^{n-1}a_{s+2}^2\right)^{\frac{1}{2}}\left(\sum_{s=M}^{n-1}\frac{(\Delta x_{s+2})^2}{f^2(x_{s+2})}\right)^{\frac{1}{2}}$$

$$+\left(\sum_{s=M}^{n-1}a_{s+1}^2\right)^{\frac{1}{2}}\left(\sum_{s=M}^{n-1}\frac{(\Delta x_{s+1})^2}{f^2(x_{s+2})}\right)^{\frac{1}{2}}+\left(\sum_{s=M}^{n-1}a_{s+2}^2\right)^{\frac{1}{2}}\left(\sum_{s=M}^{n-1}\frac{(s+1)^2g^2(x_{s+2},x_{s+1})(\Delta x_{s+1})^2(\Delta x_{s+2})^2}{f^2(x_{s+1})f^2(x_{s+2})}\right)^{\frac{1}{2}}$$

$$-\left(\sum_{s=M}^{n-1}a_{s+1}^2\right)^{\frac{1}{2}}\left(\sum_{s=M}^{n-1}\frac{(s+1)^2g^2(x_{s+2},x_{s+1})(\Delta x_{s+1})^4}{f^2(x_{s+1})f^2(x_{s+2})}\right)^{\frac{1}{2}}-\left(\sum_{s=M}^{n-1}(s+1)p_s^2\right)^{\frac{1}{2}}\left(\sum_{s=M}^{n-1}\frac{(s+1)(\Delta x_s)^2}{f^2(x_{s+1})}\right)^{\frac{1}{2}}$$

$$\leq\left(\frac{(M+1)a_{M+1}\Delta x_{M+1}}{f(x_{M+1})}-\frac{(M+1)a_M\Delta x_M}{f(x_{M+1})}\right)-\sum_{s=M}^{n-1}(s+1)q_s \tag{2.13}$$

In view of (H5), (H6) and (H8),we get from (2.13) that

$$\frac{(n+1)(a_{n+1}\Delta x_{n+1}-a_n\Delta x_n)}{f(x_{n+1})}\to-\infty \text{ as } n\to\infty$$

$$\frac{(n+1)\Delta(a_n\Delta x_n)}{f(x_{n+1})}\to-\infty \text{ as } n\to\infty.$$

Hence there exists $M_1\geq M$ such that $\Delta(a_n\Delta x_n)<0$ for $n\geq M$
which implies $\Delta(a_n\Delta x_n)<-k, k>0$
Summing the last inequality from m to $(n-1)$,we obtain

$$\sum_{s=m}^{n-1}\Delta(a_s\Delta x_s)<\sum_{s=m}^{n-1}(-k)$$

$$(a_s\Delta x_s)_{s=m}^{n}<(-k)(n-m)$$

(i.e) $a_n\,\Delta x_n<-k(n-m)+a_m\,\Delta x_m$
Therefore $a_n\Delta x_n\to-\infty$ as $n\to\infty$
Hence there exists

$$M_2\geq M_1 \text{ such that } \Delta x_n<0 \text{ for } n\geq M_2$$

Rewriting (2.7),we have

$$\frac{(n+1)a_{n+1}\Delta x_{n+1}}{f(x_{n+1})}+\sum_{s=M_2}^{n-1}\frac{(s+1)a_{s+2}g(x_{s+2},x_{s+1})\Delta x_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})}$$

$$= \frac{(n+1)a_n\Delta x_n}{f(x_{n+1})} + \frac{(M+1)a_{M+1}\Delta x_{M+1}}{f(x_{M+1})} - \frac{(M+1)a_M\Delta x_M}{f(x_{M+1})}$$
$$- \sum_{s=M}^{n-1}(s+1)q_s + \sum_{s=M_2}^{n-1}\frac{(s+1)a_{s+1}g(x_{s+2},x_{s+1})(\Delta x_{s+1})^2}{f(x_{s+1})f(x_{s+2})}$$
$$- \sum_{s=M}^{M_2-1}\left(\frac{(s+1)a_{s+2}g(x_{s+2},x_{s+1})\Delta x_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})}\right) + \sum_{s=M}^{M_2-1}\left(\frac{(s+1)a_{s+1}g(x_{s+2},x_{s+1})(\Delta x_{s+1})^2}{f(x_{s+1})f(x_{s+2})}\right)$$
$$+ \sum_{s=M}^{M_2-1}\left(\frac{a_{s+2}\Delta x_{s+2}}{f(x_{s+2})} - \frac{a_{s+1}\Delta x_{s+1}}{f(x_{s+2})}\right) + \sum_{s=M_2}^{n-1}\left(\frac{a_{s+2}\Delta x_{s+2}}{f(x_{s+2})} - \frac{a_{s+1}\Delta x_{s+1}}{f(x_{s+2})}\right)$$
$$+ \sum_{s=M_2}^{n-1}\left(\frac{(s+1)p_s\Delta x_s}{f(x_{s+1})}\right) + \sum_{s=M}^{M_2-1}\left(\frac{(s+1)p_s\Delta x_s}{f(x_{s+1})}\right) \tag{2.14}$$

From (H1), (H8), (**??**) and (2.14), there exists an integer $M_3 \geq M_2$, such that

$$\frac{(n+1)a_{n+1}\Delta x_{n+1}}{f(x_{n+1})} + \sum_{s=M_2}^{n-1}\frac{(s+1)a_{s+2}g(x_{s+2},x_{s+1})\Delta x_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})} \leq -k, k \geq M_3$$

where k is a positive constant.

$$\frac{-(n+1)a_{n+1}\Delta x_{n+1}}{f(x_{n+1})} - \sum_{s=M_2}^{n-1}\frac{(s+1)a_{s+2}g(x_{s+2},x_{s+1})\Delta x_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})} \geq k \tag{2.15}$$

let $u_{n+1} = -(n+1)\Delta x_{n+1}$, (2.15) becomes
$$\frac{u_{n+1}a_{n+1}}{f(x_{n+1})} \geq k + \sum_{s=M_3}^{n-1}\frac{(s+1)a_{s+2}g(x_{s+2},x_{s+1})\Delta x_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})}; n \geq M_3$$

$$\Rightarrow u_{n+1} \geq k\frac{f(x_{n+1})}{a_{n+1}} + \sum_{s=M_3}^{n-1}\frac{a_{s+2}f(x_{n+1})g(x_{s+2},x_{s+1})(-\Delta x_{s+2})u_{s+1}}{a_{n+1}f(x_{s+1})f(x_{s+2})} \tag{2.16}$$

Also,

$$\text{Let } v_{n+1} = k\frac{f(x_{n+1})}{a_{n+1}} + \sum_{s=M_3}^{n-1}\frac{a_{s+2}f(x_{n+1})g(x_{s+2},x_{s+1})(-\Delta x_{s+2})v_{s+1}}{a_{n+1}f(x_{s+1})f(x_{s+2})} \tag{2.17}$$

Using Lemma 1, we have, from (2.16) and (2.17)

$$\Rightarrow u_{n+1} \geq v_{n+1} \tag{2.18}$$

(2.16) implies

$$v_{n+1} = \frac{f(x_{n+1})}{a_{n+1}}\left(k + \sum_{s=M_3}^{n-1}\frac{a_{s+2}g(x_{s+2},x_{s+1})(-\Delta x_{s+2})v_{s+1}}{f(x_{s+1})f(x_{s+2})}\right)$$

This implies

$$v_{n+1} \geq \frac{kf(x_{M_3})}{a_{n+1}} \text{ for } n \geq M_3 \tag{2.19}$$

From (2.19) and (2.20),we have

$$-(n+1)\Delta x_{n+1} \geq \frac{kf(x_{M_3})}{a_{n+1}}$$
$$\Rightarrow \Delta x_{n+1} \leq \frac{-kf(x_{M_3})}{(n+1)a_{n+1}} \tag{2.20}$$

Summing (2.20) from M_3 to $(n-1)$

$$\sum_{s=M_3}^{n-1} \Delta x_{s+1} \leq -kf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(s+1)a_{s+1}}$$
$$(x_{s+1})_{s=M_3}^{n} \leq -kf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(s+1)a_{s+1}}$$
$$\Rightarrow x_{n+1} \leq x_{M_3+1} - kf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(s+1)a_{s+1}}$$
$$\Rightarrow x_n \leq 0 \text{ for sufficiently large } n,$$

which yields a contradiction to the fact that x_n is eventually positive.The proof is similar for the case when x_n is eventually negative. Hence the theorem is completely proved. □

Example 1. Consider the difference equation

$$\Delta^2\left(\frac{1}{n}\Delta x_n\right) - \frac{9n^2+18n+5}{2n(n+1)(n+2)}\Delta x_n + \frac{n^2+2n+1}{n(n+1)(n+2)}x_{n+1} = 0 \tag{2.21}$$

All the conditions of theorem 1 are satisfied .Hence every solution of equation(E1) is oscillatory.
One such solution of (2.21) is $x_n = (-1)^n$.

Example 2. Consider the difference equation

$$\Delta^2\left(\frac{1}{n+1}\Delta x_n\right) - \frac{n^4+7n^3+35n^2+92n+45}{(n+1)(n+2)(n+3)}2^{n+1}\Delta x_n + (n+1)8^{n+1}x_{n+1}^3 = 0 \tag{2.22}$$

All the conditions of Theorem 1 are satisfied .Hence every solution of equation (2.22) is oscillatory.
One such solution of (2.22) is $x_n = \frac{(-1)^n}{2^n}$.

Corollary 1. *In addition to* (H1), (H2), (H3) *and* (H4), *assume that* (H5), (H6) *and* (H7) *hold. Then, every solution of* (1.2) *is oscillatory.*

Example 3. Consider the difference equation

$$\Delta^2\left(\frac{1}{2n}(x_n^4)\Delta x_n\right) - \frac{3n^4+9n^3+14n^2+16n+4}{4n(n+1)(n+2)}\Delta x_n + n(x_{n+1}^3 + \frac{x_{n+1}^5}{2}) = 0 \tag{2.23}$$

All the conditions of Corollary 1 are satisfied .Hence every solution of equation (2.23) is oscillatory.
One such solution of (2.23) is $x_n = (-1)^{n+1}$.

Remark 1. In Corollary 1, assume $\phi(x_n) = c > 0$, a constant, in (1.2) yields (1.1).

Corollary 2. *In addition to* (H1), (H2) *and* (H3), *assume that* (H6) *and* (H7) *hold. Then, every solution of* (1.3) *is oscillatory.*

Example 4. Consider the difference equation

$$\Delta^2\left(\frac{1}{n}\Delta x_n\right) + \frac{2(3n+2)}{n(n+1)(n+2)}x_{n+1}^5 = 0 \tag{2.24}$$

All the conditions of Corollary 2 are satisfied. Hence every solution of equation (2.24) is oscillatory.
One such solution of (2.24) is $x_n = (-1)^{n+1}$.

Corollary 3. *In addition to* (H1), (H2), (H4) *and* (H4), *assume that* (H6) *and* (H7) *hold. Then, every solution of* (1.4) *is oscillatory.*

Example 5. Consider the difference equation

$$\Delta^2\left(\frac{1}{2n+1}(x_n^2)\Delta x_n\right) + \frac{8(n+1)}{(n+1)^3(n^2+2n+3)}(x_{n+1}^3 + \frac{x_{n+1}^5}{2}) = 0 \tag{2.25}$$

All the conditions of Corollary 3 are satisfied .Hence every solution of equation (2.25) is oscillatory.
One such solution of (2.25) is $x_n = n(-1)^n$.

Proofs of Corollary 1, Corollary 2 and Corollary 3 are simillar to the proof of Theorem 1 and hence the details are omitted.

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