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ON THE MULTIPLICITY OF THE EIGENVALUES OF THE VECTORIAL STURM-LIOUVILLE EQUATION

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Abstract. Let $Q(x)$ be a continuous $m \times m$ real symmetric matrix-valued function defined on $[0,1]$, and denote the Sturm-Liouville operator $-\frac{d^2}{dx^2} + Q(x)$ as L_Q with $Q(x)$ as its potential function. In this paper we prove that for each Dirichlet eigenvalue λ_* of L_Q, the geometric multiplicity of λ_* is equal to its algebraic multiplicity. Applying this result, we get a necessary and sufficiently condition such that each Dirichlet eigenvalue of L_Q is of multiplicity m.

1. Introduction

In this paper we shall study some problems related to the multiplicity of the eigenvalue of the following vectorial Sturm-Liouville equation:

$$\begin{cases} y''(x) + (\lambda I_m - Q(x))y(x) = 0, \\ y(0) = y(1) = 0, \end{cases} \tag{1}$$

where I_m is the identity operator on $\mathbb{R}^m$, $Q(x)$ is an $m \times m$ real symmetric matrix-valued continuous function, and $y(x)$ is an $\mathbb{R}^m$-valued function. Denote $L_Q = -\frac{d^2}{dx^2} + Q(x)$ and call it the Sturm-Liouville operator with the *potential function* $Q(x)$. We say that a number λ_* is a Dirichlet eigenvalue of L_Q if and only if the equation (1) has a nontrivial solution. Such a solution is called a *Dirichlet eigenfunction* of L_Q correponding to the eigenvalue λ_*. Let $m_g(\lambda_*)$ denote the *geometric multiplicity* of the eigenvalue λ_*, which is the dimension of the subspace of eigenfunctions corresponding to λ_*. The collection of all Dirichlet eigenvalues of L_Q is called the *Dirichlet spectrum* of L_Q, and denoted by $\sigma_D(L_Q)$. According to the self-adjointness of $Q(x)$, we know that all Dirichlet eigenvalues are real. Counting the geometric multiplicity of the eigenvalues, we arrange the Dirichlet eigenvalues of L_Q in ascending order as:

$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots.$$

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In order to study the Sturm-Liouville eigenvalue problem, we introduce the following initial value problem:

$$\begin{cases} Y''(x) + (\lambda I_m - Q(x))Y(x) = 0_m, \\ Y(0) = 0_m, \quad Y'(0) = I_m, \end{cases}$$

and denote its solution by $Y(x;\lambda)$. Then λ_* is a Dirichlet eigenvalue of L_Q if and only if $\det Y(1;\lambda_*) = 0$, and $m_g(\lambda_*) = \dim(\ker Y(1;\lambda_*))$. Let $m_a(\lambda_*)$ denote the algebraic multiplicity of the eigenvalue λ_*, which is determined by the following equality:

$$m_a(\lambda_*) = \max\{n \in \mathbb{Z}^+ \mid (\lambda - \lambda_*)^{-n} \det Y(1;\lambda) \text{ is analytic at } \lambda_* \}.$$

It is known that for each $\lambda_* \in \sigma_D(L_Q)$, $m_g(\lambda_*) \leqslant m_a(\lambda_*)$. With the help of a homotopy method motivated by the approach of L. Bers in his paper [2], we prove the following theorem:

Theorem 2. *Suppose that $Q(x) \in C([0,1];\mathscr{L}(\mathbb{R}^m))$, and $Q(x) = Q(x)^*$ for all $x \in [0,1]$. Then $m_a(\lambda_*) = m_g(\lambda_*)$ for any $\lambda_* \in \sigma_D(L_Q)$.*

On the other hand, for the Dirichlet eigenvalues of L_Q, it is known that $m_g(\lambda_*) \leqslant m$ for each $\lambda_* \in \sigma_D(L_Q)$. In the works of C-L. Shen [7, 8] about the inverse eigenvalue problems related to equation (1), it was shown that if $Q(x)$ is an even function, and $m_g(\lambda_*) = m$ for each $\lambda_* \in \sigma_D(L_Q)$, then $Q(x)$ is a diagonal matrix-valued function. Note that $m_g(\lambda_*) = m$ if and only if $Y(1;\lambda_*) = 0_m$. Therefore it is interesting to find a spectral condition, only depending on the eigenvalues of equation (1), to tell whether all Dirichlet eigenvalues of L_Q are of multiplicity m. We use a homotopy method to study this problem. Denote $Y(x;\lambda;t)$ as the solution of the following initial value problem:

$$Y''(x) + (\lambda I - tQ(x))Y(x) = 0, \quad Y(0) = 0, \quad Y'(0) = I,$$

where $t \in [0,1]$. Then for sufficiently large $l \in \mathbb{N}$ we know that $Y(1;(l+\frac{1}{2})^2\pi^2;t)$ is an invertible matrix for all $t \in [0,1]$. Thus the following contour integral makes sense:

$$M_l(t) = \frac{1}{2\pi i} \oint_{|\lambda|=(l+\frac{1}{2})^2\pi^2} \frac{\frac{\partial}{\partial\lambda}\det Y(1;\lambda;t)}{\det Y(1;\lambda;t)} d\lambda.$$

Since $M_l(t)$ is a continuous positive integer-valued function on $[0,1]$, then $M_l(1) = M_l(0) = lm$. Thus there are lm Dirichlet eigenvalues (counting multiplicity) smaller than $(l+\frac{1}{2})^2\pi^2$. According to these argument, we obtain a necessary and sufficiently condition which implies all Dirichlet eigenvalues of L_Q are of multiplicity m.

Theorem 4. *Suppose that $Q(x) \in C([0,1];\mathscr{L}(\mathbb{R}^m))$, and $Q(x) = Q(x)^*$ for all $x \in [0,1]$. Then all Dirichlet eigenvalues of L_Q are of multiplicity m if and only if*

$$\sigma_D(L_Q) = \{n^2\pi^2 + \tau_n \mid \{\tau_n\}_{n=1}^{\infty} \textit{ is a bounded sequence}\}.$$

Furthermore, $\{\tau_n\}_{n=1}^{\infty}$ *is a convergent sequence, and*

$$\int_0^1 Q(t)dt = \{\lim_{n\to\infty} \tau_n\} I_m.$$

2. Preliminary

In order to study the Sturm-Liouville eigenvalue problem with the selfadjoint $m \times m$ matrix-valued potential $Q(x)$, we consider the following initial value problems:

$$\begin{cases} Y''(x;\lambda) + (\lambda I_m - Q(x))Y(x;\lambda) = 0_m, \\ Y(0;\lambda) = 0_m, \quad Y'(0;\lambda) = I_m, \end{cases} \tag{2}$$

where 0_m is the $m \times m$ zero matrix. We have that

$$Y(x;\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} I_m + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} Q(t)Y(t;\lambda)dt, \tag{3}$$

furthermore,

$$Y(1;\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} I_m - \frac{\cos\sqrt{\lambda}}{2\lambda}\int_0^1 Q(x)dt + O(\frac{\exp(\|Q\|_\infty|\Im\sqrt{\lambda}|)}{|\lambda|^{\frac{3}{2}}}), \tag{4}$$

where $\|Q\|_\infty = sup\{\|Q(x)\|_{L(\mathbb{R}^m)} \mid x \in [0;1]\}$, and we have that as a function of λ, $Y(x;\lambda)$ is a matrix-valued entire function. It is well known that $\lambda_* \in \sigma_D(L_Q)$, if and only if $Y(1;\lambda_*)$ has a nontrivial kernel, thus $\det Y(1;\lambda_*) = 0$.

To analyze the distribution of the zeros of $\det Y(1;\lambda)$ is helpful for our study of the structure of $\sigma_D(L_Q)$. Now we introduce a Rouché's theorem for analytic matrix valued-functions (see [G2, Ch.XI Thm. 9.2]). Let $\Phi(\lambda)$ be an analytic matrix-valued function defined on an open connected region $\Omega \subset \mathbb{C}$. Define

$$m_g(\lambda_0;\Phi) = \dim \ker\Phi(\lambda_0),$$
$$m_a(\lambda_0;\Phi) = \max\{n \in \mathbb{Z}^+ \mid (\lambda-\lambda_0)^{-n}\det\Phi(\lambda) \text{ is analytic at } \lambda_0\},$$

where $m_g(\lambda_0;\Phi)$ and $m_a(\lambda_0;\Phi)$ are geometric and algebraic multiplicity of λ_0 corresponding to $\Phi(\lambda)$, respectively. Let Γ be a Cauchy contour in Ω with inner domain $\Delta \subset \Omega$. We say that $\Phi(\lambda)$ is *normal* with respect to Γ if $\Phi(\lambda)$ is invertible for all $\lambda \in \Gamma$. Applying the analyticity of $\Phi(\lambda)$, we have that $\det \Phi(\lambda)$ is also analytic on Ω, thus there are only finitely many $\lambda_* \in \Delta$, such that $\Phi(\lambda_*)$ is noninvertible. Then we may define the following quantities:

$$m_g(\Gamma;\Phi) = \sum_{\lambda\in\Delta} m_g(\lambda;\Phi), \quad m_a(\Gamma;\Phi) = \sum_{\lambda\in\Delta} m_a(\lambda;\Phi).$$

If $\Phi(\lambda_*)$ is selfadjoint, then we know $m_a(\lambda_*;\Phi) = m_g(\lambda_*;\Phi)$. But for nonselfadjoint $\Phi(\lambda)$ we only have that $m_a(\lambda_*;\Phi) \geqslant m_g(\lambda_*;\Phi)$. Gohberg and his coworkers proved the following Rouché's theorem for analytic matrix valued-functions (see [G2, Ch.XI Thm. 9.2]).

Theorem 1. *Let $\Phi(\lambda), \Psi(\lambda) : \Omega \in \mathscr{C} \to \mathscr{L}(\mathbb{R}^n)$ be analytic matrix valued-functions, and assume that Φ is normal with respect to the Cauchy contour $\Gamma \in \Omega$. If*

$$\|\Phi(\lambda)^{-1}\Psi(\lambda)\|_{\mathscr{L}(\mathbb{R}^n)} < 1, \quad \textit{for all } \lambda \in \Gamma,$$

then the function $V(\lambda) = \Phi(\lambda) + \Psi(\lambda)$ is also normal with respect to Γ, and $m_a(\Gamma;\Phi) = m_a(\Gamma;V)$.

In the remaining part of this section we shall analyze the distribution of Dirichlet eigenvalues of L_Q. Applying the selfadjointness of $Q(x)$, we have that there exist an $m \times m$ unitary matrix P, such that

$$P^*(\int_0^1 Q(t)dt)P = \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_m \end{pmatrix},$$

where $\{q_j\}_1^m$ are eigenvalues of the constant matrix $\int_0^1 Q(t)dt$, and $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_m$. Applying (4), we have that

$$P^*Y(1;\lambda)P = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} I_m - \frac{\cos\sqrt{\lambda}}{2\lambda} \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_m \end{pmatrix} + O(\frac{\exp(\|Q\|_\infty |\Im\sqrt{\lambda}|)}{|\lambda|^{\frac{3}{2}}}).$$

Denote

$$\Phi(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} I_m - \frac{\cos\sqrt{\lambda}}{2\lambda} \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_m \end{pmatrix},$$
$$\Psi(\lambda) = PY(1;\lambda)P - \Phi(\lambda).$$

Since

$$\det\Phi(\lambda) = \prod_{i=1}^m (\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - \frac{\cos\sqrt{\lambda}}{2\lambda} q_i),$$

Let μ_{lm+i} be the $(lm+i)$-th zero of $\det\Phi(\lambda)$. Applying the Rouché's theorem, we find that μ_{lm+i} satisfies

$$\mu_{lm+i} - (l+1)^2\pi^2 = q_i + O(\frac{1}{l}).$$

On the other hand, for sufficient large l, denote

$$\gamma_l^i = \{\lambda \in \mathbb{C} \mid |\lambda - \mu_{lm+i}| = \varepsilon_l^i\},$$

where ε_l^i is a suitable number, such that $\det\Phi(\lambda) \neq 0$ for all λ inside γ_l^i except μ_{lm+i}, and

$$\lim_{l\to\infty} \varepsilon_l^i = 0,$$

and $\|\Phi(\lambda)^{-1}\Psi(\lambda)\|_{\mathscr{L}(\mathbb{R}^n)} < 1$ for any $\lambda \in \gamma_l^i$. According to Theorem 1, we know that

$$m_a(\gamma_l^i;\Phi) = m_a(\gamma_l^i;\Phi+\Psi) = m_a(\gamma_l^i;Y(1;\lambda)) = m_a(\mu_{lm+i};\Phi). \tag{5}$$

This shows that the eigenvalues of the equation (1) appear near $(l+1)^2\pi^2 + q_i$ for sufficient large l.

3. On the analysis of the structure of $\sigma_D(L_Q)$

The purpose in this section is to prove that $m_a(\lambda_*;Y(1;\lambda)) = m_g(\lambda_*;Y(1;\lambda))$ for all $\lambda_* \in \sigma_D(L_Q)$, where $Q(x) \in \mathscr{L}(\mathbb{R}^n)$. Before we begin to prove our assertion, we need some notation. Consider the following auxiliary eigenvalue problem:

$$\begin{cases} y''(x) + (\lambda I_m - tQ(x))y(x) = 0, \\ y(0) = y(1) = 0, \end{cases} \tag{6}$$

and the following initial value problem:

$$Y''(x) + (\lambda I_m - tQ(x))Y(x) = 0, \quad Y(0) = 0, \quad Y'(0) = I_m, \tag{7}$$

where $t \in [0,1]$. Denote $Y(x;\lambda;t)$ the solution of the equation (7), and let $\lambda_n^g(t)$ denote the n-th eigenvalue (counting geometric multiplicity) of (6). Then $\det Y(1;\lambda_n^g(t);t) = 0$ for all $n \in \mathbb{N}$. Let $\lambda_n^a(t)$ be the n-th zero (counting algebraic multiplicity) of $\det Y(1;\lambda;t)$. Then $\ker Y(1;\lambda_n^a(t)) \neq \{0\}$. Thus for any $\lambda_n^g(t)$, there exists k_n, such that $\lambda_{k_n}^a(t) = \lambda_n^g(t)$, and for any $\lambda_n^a(t)$, there exists j_n, such that $\lambda_{j_n}^g(t) = \lambda_n^a(t)$.

Theorem 2. *Suppose that $Q(x) \in C([0,1];\mathscr{L}(\mathbb{R}^m))$, and $Q(x) = Q(x)^*$ for all $x \in [0,1]$. Then $m_a(\lambda_*) = m_g(\lambda_*)$ for any $\lambda_* \in \sigma_D(L_Q)$.*

Proof. To prove our theorem, we have to prove that $m_g(\lambda_*;Y(1;\lambda)) = m_a(\lambda_*;Y(1;\lambda))$, where $\lambda_* \in \sigma_D(L_Q)$. If we can prove that $\lambda_n^g(t) = \lambda_n^a(t)$ for all $t \in [0,1]$, and $n \in \mathbb{N}$, then $m_g(\lambda_n^a(t); Y(1;\lambda;t)) = m_a(\lambda_n^g(t);Y(1;\lambda;t))$. Since $Y(1;\lambda) = Y(1;\lambda;1)$, then our assertion holds. According to the definition of $\lambda_n^a(t)$, $\lambda_n^g(t)$ and the continuity of $Y(1;\lambda;t)$ corresponding to t-parameter, we know that $\{\lambda_n^a(t)\}_{n=1}^\infty$ and $\{\lambda_n^g(t)\}_{n=1}^\infty$ are two increasing sequences of continuous functions, and $\lambda_n^a(t) \leqslant \lambda_n^g(t)$ for all $t \in [0,1]$, $n \in \mathbb{N}$.

Suppose that there exists $j \in \mathbb{N}$, $t_0 \in [0,1]$, such that $\lambda_j^a(t_0) < \lambda_j^g(t_0)$. Denote

$$\mathcal{N} = \{t \in [0,1] \mid \text{There exist } j_t, \text{ such that } \lambda_{j_t}^a(t) < \lambda_{j_t}^g(t)\}.$$

Since $t_0 \in \mathcal{N}$, then $\mathcal{N}$ is a nonempty subset of $[0,1]$. On the other hand, for any $t_* \in \mathcal{N}$, according to the continuity of $\lambda_{j_{t_*}}^a(t)$ and $\lambda_{j_{t_*}}^g(t)$, there exists an open neighborhood $\mathcal{I}_{t_*}$ of t_*, such that for all $t \in \mathcal{I}_{t_*}$, we have that $\lambda_{j_{t_*}}^a(t) < \lambda_{j_{t_*}}^g(t)$. This implies that $\mathcal{I}_{t_*} \subset \mathcal{N}$, thus $\mathcal{N}$ is an relatively open subset of $[0,1]$. Furthermore, for any $t \in \mathcal{N}$, applying (5) we have that

$$\limsup_{n\to\infty} \frac{\lambda_n^g(t) - \lambda_n^a(t)}{2n\pi} \geqslant 1. \tag{8}$$

Let t_* be an accumulation point of $\mathcal{N}$, but $t_* \notin \mathcal{N}$. Then there exist $\{t_n\}_{n=1}^{\infty} \subset \mathcal{N}$, such that t_n converges to t_* as n tends to infinite. In the previous argument we know that

$$\lambda_{lm+i}^a(t_n) - \lambda_{lm+i}^a(t_*) = (t_n - t_*)q_i + O(\frac{1}{l}). \tag{9}$$

Denote $y_n(x;t)$ the n-th eigenfunction of (6), such that $\int_0^1 \|y_n(x;t)\|^2 dt = 1$. Then applying the variational principle (see [4]), we have that

$$\begin{aligned}
\lambda_n^g(t) &\leqslant \int_0^1 \|y_n'(x;t_*)\|^2 dx + \int_0^1 \langle tQ(x)y_n(x;t_*), y_n(x;t_*)\rangle dt \\
&= \int_0^1 \|y_n'(x;t_*)\|^2 dx + \int_0^1 \langle t_*Q(x)y_n(x;t_*), y_n(x;t_*)\rangle dt \\
&\quad + (t - t_*)\int_0^1 \langle Q(x)y_n(x;t_*), y_n(x;t_*)\rangle dt \\
&= \lambda_n^g(t_*) + (t - t_*)\int_0^1 \langle Q(x)y_n(x;t_*), y_n(x;t_*)\rangle dt,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_n^g(t_*) &\leqslant \int_0^1 \|y_n'(x;t)\|^2 dx + \int_0^1 \langle t_*Q(x)y_n(x;t), y_n(x;t)\rangle dt \\
&= \int_0^1 \|y_n'(x;t)\|^2 dx + \int_0^1 \langle tQ(x)y_n(x;t), y_n(x;t)\rangle dt \\
&\quad + (t_* - t)\int_0^1 \langle Q(x)y_n(x;t), y_n(x;t)\rangle dt \\
&= \lambda_n^g(t) + (t_* - t)\int_0^1 \langle Q(x)y_n(x;t), y_n(x;t)\rangle dt.
\end{aligned}$$

Following the previous two inequalities and applying Hölder inequality, we find that

$$|\lambda_n^g(t) - \lambda_n^g(t_*)| \leqslant |t - t_*| \cdot \|Q\|_\infty. \tag{10}$$

Since $t_* \notin \mathcal{N}$, then $\lambda_n^a(t_*) = \lambda_n^g(t_*)$ for all $n \in \mathbb{N}$. According to (9) and (10), we have that

$$\begin{aligned}\lambda_n^g(t_l) - \lambda_n^a(t_l) &\leqslant |\lambda_n^g(t_l) - \lambda_n^g(t_*)| + |\lambda_n^a(t_l) - \lambda_n^a(t_*)| \\ &\leqslant |t_l - t_*|\{\max_{i=1,\cdots,m} q_i\} + O(|t_l - t_*|) + O(\frac{1}{[\frac{n}{m}]}).\end{aligned}$$

This is a contradiction to the inequality (8). Thus $t_* \in \mathcal{N}$, and we find that $\mathcal{N}$ is a relatively closed subset of $[0,1]$. By the connectedness of $[0,1]$, and the assumption of $\mathcal{N}$ being nonempty, we conclude that $\mathcal{N} = [0,1]$. But $Y(1;\lambda;0) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} I_m$, this shows that $\lambda_n^a(0) = \lambda_n^g(0)$ for all $n \in \mathbb{N}$, $0 \notin \mathcal{N}$, which is absurd. Thus $\mathcal{N}$ is empty. Hence $\lambda_n^g(t) = \lambda_n^a(t)$ for all $n \in \mathbb{N}$. □

Remark. If we consider the Neumann eigenvalue problem of L_Q as follows:

$$\begin{cases} z''(x) + (\lambda I_m - Q(x))z(x) = 0, \\ z'(0) = z'(1) = 0, \end{cases}$$

and denote $Z(x;\lambda)$ the solution of the following initial value problem:

$$\begin{cases} Z''(x;\lambda) + (\lambda I_m - Q(x))Z(x;\lambda) = 0_m, \\ Z(0;\lambda) = I_m, \quad Z'(0;\lambda) = 0_m. \end{cases}$$

Then we know that μ_* is in the Neumann spectrum $\sigma_N(L_Q)$ of L_Q or a Neumann eigenvalue of L_Q, if and only if $\det Z'(1;\mu_*) = 0$. Applying the similar argument in the proof of Theorem 2, we get that $m_g(\mu) = m_a(\mu)$ for all $\mu \in \sigma_N(L_Q)$. Thus for any Neumann eigenvalue of L_Q its geometric multiplicity are equal to its algebraic multiplicity.

Denote $\gamma_l = \{\lambda \in \mathbb{C} : |\lambda - l^2\pi^2| = (l - \frac{1}{4})\pi^2\}$, $l \in \mathbb{N}$. According to the identity (5), there exists a positive integer M_Q, such that

$$\frac{1}{2\pi i}\oint_{\gamma_l} \frac{\frac{\partial}{\partial\lambda}\det Y(1;\lambda)}{\det Y(1;\lambda)} d\lambda = m, \tag{11}$$

for any $l \geqslant M_Q$, $l \in \mathbb{N}$. Applying Theorem 2, we have that there are m Dirichlet eigenvalues of L_Q which are contained inside γ_l. On the other hand, according to (4), we have that

$$Y(1;(l+\frac{1}{2})^2\pi^2;t) = \frac{(-1)^l}{(l+\frac{1}{2})\pi} I_m + tO(\frac{1}{(l+\frac{1}{2})^3\pi^3}),$$

where $t \in [0,1]$. Thus for sufficiently large l we know that $Y(1;(l+\frac{1}{2})^2\pi^2;t)$ is invertible for any $t \in [0,1]$, then $\det Y(1;(l+\frac{1}{2})^2\pi^2;t) \neq 0$ for all $t \in [0,1]$. Thus we can define the following integral

$$M_l(t) = \frac{1}{2\pi i}\oint_{|\lambda|=(l+\frac{1}{2})^2\pi^2} \frac{\frac{\partial}{\partial\lambda}\det Y(1;\lambda;t)}{\det Y(1;\lambda;t)} d\lambda.$$

According to (4), we have that

$$Y(1;\lambda;t)=\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}I_m-\frac{t\cos\sqrt{\lambda}}{2\lambda}\int_0^1 Q(t)dt+tO(\frac{\exp\|Q\|_\infty|\Im\sqrt{\lambda}|}{|\sqrt{\lambda}|^{\frac{3}{2}}}).$$

Then the family $\{\det Y(1;\lambda;t)\}_{t\in[0,1]}$ is a continuous family with parameter t. On the other hand, we know that $\frac{\partial}{\partial\lambda}\det Y(1;\lambda;t)$ are sum of terms with multiplication of entries of $Y(1;\lambda;t)$ and $\frac{\partial}{\partial\lambda}Y(1;\lambda;t)$. Denote $Y_\lambda(x;\lambda;t)=\frac{\partial}{\partial\lambda}Y(x;\lambda;t)$. Then $Y_\lambda(1;\lambda;t)$ satisfies the following equation:

$$\begin{cases} Y''_\lambda(x;\lambda;t)+(\lambda I_m-tQ(x))Y_\lambda(x;\lambda;t)=Q(x)Y(x;\lambda;t),\\ Y_\lambda(0;\lambda;t)=Y'_\lambda(0;\lambda;t)=0_m.\end{cases}$$

According to the continuity of $Y(x;\lambda;t)$ on t variable, we know that $\{Y_\lambda(1;\lambda;t)\}_{t\in[0,1]}$ is a continuous family with parameter t. Then $\{\frac{\partial}{\partial\lambda}\det Y(1;\lambda;t)\}_{t\in[0,1]}$ is also a continuous family with parameter t. Thus $M_l(t)$ is a continuous function on $[0,1]$ with positive integer value, this shows that $M_l(t)$ is a constant function, $M_l(t)\equiv M_l(0)=lm$. From our argument of the distribution of the zeros of $Y(1;\lambda)$, we find that L_Q has exactly lm Dirichlet eigenvalues inside $\{|\lambda|=(l+\frac{1}{2})^2\pi^2\}$. The above argument implies the following corollary:

Corollary 3. *For $Q(x)\in C([0,1];\mathscr{L}(\mathbb{R}^m))$, $Q(x)=Q(x)^*$ for all $x\in[0,1]$. Let λ_n be the n-th Dirichlet eigenvalue of L_Q, then*

$$\lim_{l\to\infty}\sum_{i=1}^m(\lambda_{lm+i}-(l+1)^2\pi^2)=trace\{\int_0^1 Q(t)dt\}.$$

Proof. In the previous argument we find that there are lm Dirichlet eigenvalues of L_Q which are contained in $\{\lambda\in\mathbb{C}:|\lambda|\leqslant(l+\frac{1}{2})^2\pi^2\}$. According to the identities (5) and (11), we find that

$$\lambda_{lm+i}=(l+1)^2\pi^2+q_i+O(\frac{1}{l}),$$

for $i=1,2,\cdots,m$. Thus

$$\sum_{i=1}^m(\lambda_{lm+i}-(l+1)^2\pi^2)=\sum_{i=1}^m q_i+O(\frac{1}{l}).$$

Then our assertion holds. ☐

From the previous argument for the multiplicity and the distribution of the Dirichlet eigenvalues of L_Q, we obtain the following theorem.

Theorem 4. *Suppose that $Q(x)\in C([0,1];\mathscr{L}(\mathbb{R}^m))$, and $Q(x)=Q(x)^*$ for all $x\in[0,1]$. Then all Dirichlet eigenvalues of L_Q are of multiplicity m, if and only if there exists a bounded sequence $\{\tau_n\}_{n=1}^\infty$, such that*

$$\sigma_D(L_Q)=\{n^2\pi^2+\tau_n\mid n\in\mathbb{N}\}.$$

Furthermore, $\{\tau_n\}_{n=1}^\infty$ is a convergent sequence, and

$$\int_0^1 Q(t)dt=\{\lim_{n\to\infty}\tau_n\}I_m.$$

Proof. Suppose that $m_g(\lambda_*; Y(1;\lambda)) = m$ for all $\lambda_* \in \sigma_D(L_Q)$. According to the identities (5) and (11), we find that

$$\lambda_{lm+1} = \lambda_{lm+2} = \cdots = \lambda_{lm+m}.$$

But $\lambda_{lm+i} = (l+1)^2\pi^2 + q_i + O(\frac{1}{l})$, thus $q_1 = q_2 = \cdots = q_m$ and $\tau_l = q_1 + O(\frac{1}{l})$. This implies that $\{\tau_n\}_{n=1}^{\infty}$ is a convergent sequence which converges to q_1, and

$$\int_0^1 Q(t)\,dt = q_1 I_m.$$

Conversely, assume that there exists a bounded sequence $\{\tau_n\}_{n=1}^{\infty}$, such that

$$\sigma_D(L_Q) = \{n^2\pi^2 + \tau_n \mid n \in \mathbb{N}\}.$$

In the identity (24) we find that if $n \geqslant M_Q$, then $m_g(n^2\pi^2 + \tau_n; Y(1;\lambda)) = m$. Denote $\Gamma_l = \{\lambda \in \mathbb{C} : |\lambda| = (l+\frac{1}{2})^2\pi^2\}$, the previous argument implies that $m_a(\Gamma_l; Y(1;\lambda)) = lm = m_g(\Gamma_l; Y(1;\lambda))$ for sufficient large l. From the distribution of $\sigma_D(L_Q)$, we find that there only l eigenvalues inside Γ_l. Thus each eigenvalue inside Γ_l is of multiplicity m. This implies that all Dirichlet eigenvalues of L_Q are of multiplicity m. □

Remark. Denote the n-th Neumann eigenvalue of L_Q as μ_n, then applying the argument similar to those argument in the proof of Corollary 3. and Theorem 4, we have that

$$\mu_{lm+i} = l^2\pi^2 + q_i + o(\frac{1}{l}).$$

Furthermore, we also get that all Neumann eigenvalue of L_Q are of multiplicity m, if and only if

$$\sigma_N(L_Q) = \{n^2\pi^2 + \epsilon_n \mid n \in \mathbb{Z}_+\},$$

where $\{\epsilon_n\}_{n=0}^{\infty}$ is a bounded sequence. Applying these result, we can simplify the vectorial V.A. Ambarzumyan theorem (see [3]) as the followings:

Corollary 5. *Let $Q(x)$ be a continuous $m\times m$ selfadjoint matrix-valued function, then $\sigma_N(L_Q) = \{n^2\pi^2 \mid n \in \mathbb{Z}\}$, if and only if $Q(x) = 0_m$ for all $x \in [0,1]$.*

From [Sh2] we know that if $Q(x) \in C([0,1]; \mathscr{L}(\mathbb{R}^m))$, and $Q(x) = Q(x)^*$, $Q(x) = Q(1-x)$ for all $x \in [0,1]$, and all eigenvalues of the equation (1) are of multiplicity m, then $Q(x) = q(x)I_m$, where $q(x)$ is a scalar continuous function defined in $[0,1]$, and $q(x) = q(1-x)$ for all $x \in [0,1]$. According to Theorem 4, we obtain the following result which is an extension of the previous inverse spectral theorem that we just mentioned.

Corollary 6. *Suppose that* $Q(x) \in C([0,1];\mathscr{L}(\mathbb{R}^m))$*, and* $Q(x) = Q(x)^*$, $Q(x) = Q(1-x)$ *for all* $x \in [0,1]$*. Then the Dirichlet spectrum* $\sigma_D(L_Q)$ *of* L_Q *is of the form*

$$\sigma_D(L_Q) = \{n^2\pi^2 + \tau_n \mid n \in \mathbb{N}\},$$

where $\{\tau_n\}_{n=1}^{\infty}$ *is a bounded sequence, if and only if* $Q(x) = q(x)I_m$ *where* $q(x)$ *is a scalar continuous function defined in* $[0,1]$*, and* $q(x) = q(1-x)$ *for all* $x \in [0,1]$.

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