

ON WEAKLY PRIME SUBMODULES

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Abstract. Let R be a commutative ring with non-zero identity. We define a proper submodule N of an R -module M to be weakly prime if $0 \neq rm \in N$ ($r \in R, m \in M$) implies $m \in N$ or $rM \subseteq N$. A number of results concerning weakly prime submodules are given. For example, we give three other characterizations of weakly prime submodules.

1. Introduction

Throughout this paper R will denote a commutative ring with non-zero identity and M a unital module. Several authors have extended the notion of prime ideal to modules, see, for example [5, 6]. A proper ideal P of R is said to be weakly prime ideal if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D. D. Anderson and E. Smith in [1]. Also, weakly primary ideals in a commutative ring with non-zero identity have been introduced and studied in [5]. Here we study weakly prime submodules of a module over a commutative ring. Various properties of weakly prime submodules are considered (see Sec. 2). In fact, the aim of this paper is to prove for weakly prime submodules some of the results given in [1] for weakly prime ideals. For example, we show that if P is a submodule of a finitely generated multiplication R -module M , then P is weakly prime if and only if for submodules N and K of M with $0 \neq NK \subseteq P$, either $N \subseteq P$ or $K \subseteq P$.

Now we define the concepts that we will use. If R is a ring and N is a submodule of an R -module M , the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M . An R -module M is called a multiplication module if for each submodule N of M , $N = IM$ for some ideal I of R . In this case we can take $I = (N : M)$ (see [3]).

Let M be a module over a ring R . A proper submodule N of M with $(N : M) = P$ is said to be prime (or P -prime) if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r \in P$. We say that M is a prime module if the zero submodule of M is prime submodule of M (see [5]).

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2. Weakly prime submodules

Recall that a proper submodule N of a module M over a commutative ring R is said to be weakly prime submodule if whenever $0 \neq rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$. Clearly, every prime submodule of a module is a weakly prime submodule. However, since 0 is always weakly prime (by definition), a weakly prime submodule need not be prime.

Remark. We know that if N is a prime submodule of an R -module M , then $(N : M)$ is a prime ideal of R . Suppose that N is weakly prime which is not prime. Contrary to what happens for a prime submodules, the ideal $(N : M)$ is not, in general, a weakly prime ideal of R . For example, let M denote the cyclic Z -module $Z/8Z$. Take $N = \{0\}$. Certainly N is a weakly prime submodule of M , but $(N : M) = 8Z$ is not a weakly prime ideal of R , but we have the following results:

Proposition 2.1. *Let R be a commutative ring, M a faithful cyclic R -module, and N a weakly prime submodule of M . Then $(N : M)$ is a weakly prime ideal of R .*

Proof. Assume that $M = Rx$ and let $0 \neq ab \in (N : M)$ with $a \notin (N : M)$. Then there exists $r \in R$ such that $a(rx) \notin N$, so $ax \notin N$. As $0 \neq abM \subseteq N$, it follows that $0 \neq abx \in N$ (for if $abx = 0$, then $ab \in (0 : x) = (0 : M) = 0$, a contradiction), so $b \in (N : M)$ since N is a weakly prime submodule of M , as needed.

Proposition 2.2. *Let R be a commutative ring, M a P -prime R -module, and N a weakly prime submodule of M . Then $(N : M)$ is a weakly prime ideal of R .*

Proof. Let $0 \neq ab \in (N : M)$ with $a \notin (N : M)$. Then there exists $m \in M - N$ such that $am \notin N$. As $0 \neq abM \subseteq N$ we have $abm \in N$. If $abm = 0$, then $ab \in (0 : m) = (0 : M) = P$, so $b \in P \subseteq (N : M)$. If $abm \neq 0$, then $b \in (N : M)$ since $am \notin N$ and N is a weakly prime submodule of M , as required.

If we adapt the proof of the [4, Proposition 2.14 and Theorem 2.15], then we get the following corollary:

Corollary 2.3. *Let R a commutative ring, M an R -module, and N a weakly prime submodule of M that is not prime. Then the following hold:*

- (i) *If I is an ideal of R such that $I \subseteq (N : M)$, then $IN = 0$. In particular, $(N : M)N = 0$.*
- (ii) *If M is a multiplication R -module, then $N^2 = 0$.*

We next give three other characterizations of weakly prime submodules.

Theorem 2.4. *Let R be a commutative ring, M an R -module, and N a proper submodule of M . Then the following statements are equivalent.*

- (i) *For ideal I of R and submodule D of M with $0 \neq ID \subseteq N$, either $IM \subseteq N$ or $D \subseteq N$.*

- (ii) N is a weakly prime submodule of M .
 (iii) For $m \in M - N$, $(N : Rm) = (N : M) \cup (0 : Rm)$.
 (iv) For $m \in M - N$, $(N : Rm) = (N : M)$ or $(0 : Rm) = (N : Rm)$.

Proof. (i) \Rightarrow (ii) Suppose that $0 \neq sm \in N$ where $s \in R$ and $m \in M$. Take $I = Rs$ and $D = Rm$. Then $0 \neq ID \subseteq N$, so either $I \subseteq (N : M)$ or $D \subseteq N$; hence either $r \in (N : M)$ or $m \in N$. Thus N is weakly prime.

(ii) \Rightarrow (i) Suppose that N is a weakly prime submodule of M . If N is prime, then the result is clear. So we can assume that N is weakly prime that is not prime. Let $0 \neq ID \subseteq N$ with $x \in D - N$. We show that $I \subseteq (N : M)$. Let $r \in I$. If $0 \neq rx$, then N weakly prime gives $r \in (N : M)$. So assume that $rx = 0$. First suppose that $rD \neq 0$, say $rd \neq 0$ where $d \in D$. If $d \notin N$, then $r \in (N : M)$. If $d \in N$, then $r(d+x) = rd \in N$, so $r \in (N : M)$ or $d+x \in N$. Thus, $r \in (N : M)$; hence $I \subseteq (N : M)$. So we can assume that $rD = 0$. Suppose that $Ix \neq 0$, say $ax \neq 0$ where $a \in I$. Then N weakly prime gives $a \in (N : M)$. As $(r+a)x = ax \in N$, we get $r \in (N : M)$, so $I \subseteq (N : M)$. Therefore, we can assume that $Ix = 0$.

Since $ID \neq 0$, there exist $b \in I$ and $d_1 \in D$ such that $bd_1 \neq 0$. As $(N : M)N = 0$ (by Corollary 2.3.) and $0 \neq b(d_1+x) = bd_1 \in N$ we can divide the proof into the following two cases:

Case 1. $b \in (N : M)$ and $d_1+x \notin N$.

Since $0 \neq (r+b)(d_1+x) = bd_1 \in N$, we obtain $r+b \in (N : M)$, so $r \in (N : M)$. Hence $I \subseteq (N : M)$.

Case 2. $b \notin (N : M)$ and $d_1+x \in N$.

As $0 \neq bd_1 \in N$ we have $d_1 \in N$, so $x \in N$ which is a contradiction. Thus $I \subseteq (N : M)$.

(ii) \Rightarrow (iii) Clearly, if $m \in M - N$, then $H = (N : M) \cup (0 : Rm) \subseteq (N : Rm)$. Let $a \in (N : Rm)$ where $m \in M - N$. Then $am \in N$. If $am \neq 0$, then $a \in (N : M)$ since N is weakly prime, so $a \in H$. If $am = 0$, then $a \in (0 : Rm)$, so $a \in H$, and hence we have equality.

(iii) \Rightarrow (iv) Is obvious.

(iv) \Rightarrow (ii) Suppose that $0 \neq rm \in N$ with $r \in R$ and $m \in M - N$. Then $r \in (N : Rm)$ and $r \notin (0 : Rm)$. It follows from (iv) that $r \in (N : Rm) = (N : M)$, as required.

Let M be an R -module and N be a submodule of M such that $N = IM$ for some ideal I of R . Then we say that I is a presentation ideal of N . Note that it is possible that for a submodule N , no such presentation ideal exist. For example, assume that M is a vector space over an arbitrary field F with $\dim_F M \geq 2$ and let N be a proper subspace of M such that $N \neq 0$. Then M is finite length (so M is noetherian, artinian and injective), but M is not multiplication and N has not any presentation. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module.

Let N and K be submodules of a multiplication R -module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R . The product N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [2, Theorem 3.4], the product of N and K is

independent of presentations of N and K . Moreover, for $a, b \in M$, by ab , we mean the product of Ra and Rb . Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [2]).

Lemma 2.5. *Let M be a multiplication module over a commutative ring R , and let N and K be submodules of M .*

- (i) *If for every $a \in N$, $aK = 0$, then $NK = 0$.*
- (ii) *If for every $b \in K$, $Nb = 0$, then $NK = 0$.*
- (iii) *If $ab = 0$ for every $a \in N$, $b \in K$, then $NK = 0$.*

Proof. (i) Assume that $a \in N$ and let I_a and J be the presentation ideals Ra and K respectively. Then $aK = (Ra)K = I_aJM = 0$, so $I_aJ \subseteq (0 : M)$. By [2, Proposition 3.8] we have

$$NK = \left(\sum_{a \in N} Ra \right) K = \left(\sum_{a \in N} I_a M \right) (JM) = \left(\sum_{a \in N} I_a J \right) M \subseteq (0 : M) M = 0.$$

(ii) This proof is similar to that in case (i) and we omit it.

(iii) Assume that $a \in N$, $b \in K$. There are ideals I_a and J_b of R such that $ab = (Ra)(Rb) = I_a J_b M = 0$, so $I_a J_b \subseteq (0 : M)$. By [2, Proposition 3.8] we have

$$NK = \left(\sum_{a \in N} Ra \right) \left(\sum_{b \in K} Rb \right) = \left(\sum_{a \in N} \sum_{b \in K} I_a J_b \right) M \subseteq (0 : M) M = 0.$$

Compare the next result with [1, Theorem 3(4)].

Theorem 2.6. *Let R be a commutative ring, M a finitely generated multiplication R -module, and P a proper submodule of M . Then the following statements are equivalent.*

- (i) *P is a weakly prime submodule of M .*
- (ii) *For submodules N and K of M with $0 \neq NK \subseteq P$, either $N \subseteq P$ or $K \subseteq P$.*

Proof. (i) \Rightarrow (ii) If P is prime, then (ii) follows from [2, Theorem 3.16], so we can assume that P is not prime. Suppose that N and K are submodules of M with $0 \neq NK \subseteq P$, but $N \not\subseteq P$ and $K \not\subseteq P$. We show that $NK = 0$ which is a contradiction. Let $a \in N$. First, suppose that $a \in N - P$. Let I_1, I_2 and I_3 be the presentation ideals Ra , N and K , respectively. Since $Ra \subseteq N$ and $NK \subseteq P$, we have $I_1 \subseteq I_2 + (0 : M) = I_2$ and $I_2 I_3 \subseteq (P : M)$ by [7, p.231 Corollary], so $aK = (Ra)K = I_1 I_3 M \subseteq I_2 I_3 M \subseteq (P : M) M = P$; hence $I_3 \subseteq (P : Ra)$. It follows from Theorem 2.4 that either $I_3 \subseteq (P : N)$ or $I_3 \subseteq (0 : Ra)$. As $K \not\subseteq P$ we have $I_3 \subseteq (0 : Ra)$. Therefore, $aK = (Ra)I_3 M = 0$, so $NK = 0$ by Lemma 2.5. Next suppose that $a \in N \cap P$. Let $b \in K$. If $b \in P$, then $ab = (Ra)(Rb) \subseteq P^2 = 0$ by Corollary 2.3, so $NK = 0$. If $b \in K - P$, then as previously noted, $Nb = 0$, hence $ab \subseteq Nb = 0$, so $NK = 0$ by Lemma 2.5.

(ii) \Rightarrow (i) Suppose that $0 \neq ID \subseteq N$ where I is an ideal of R and D is a submodule of M . Take $K = IM$. Then $0 \neq KD = I(D : M)M = ID \subseteq N$, so either $K = IM \subseteq (N : M)M = N$ or $D \subseteq N$ by (ii). It follows from [8, p.231 Corollary] that either $I \subseteq (N : M)$ or $D \subseteq N$. Thus N is weakly prime by Theorem 2.5.

Let N be a proper submodule M . Then, the radical of N denoted by \sqrt{N} is defined in [3] to be the intersection of all prime submodules of M containing N . If N is a submodule of a multiplication R -module, then by [2, Theorem 3.13] we have

$$\sqrt{N} = \{m \in M : m^k \subseteq N \text{ for some non-zero integer } k\}$$

Compare the following theorem with [1, Theorem 4].

Theorem 2.7. *Let R be a commutative ring, M a finitely generated faithful multiplication R -module, and N a weakly prime submodule of M that is not prime. Then $N\sqrt{0} = 0$.*

Proof. Let $x \in \sqrt{0}$. Then $Rx = JM$ and $N = IM$ for some ideals J and I of R . If $x \in N$, then $J \subseteq I$ by [7, p.231 Corollary] and $N(Rx) = IJM \subseteq N^2 = 0$ by Corollary 2.3, so $Nx = 0$. It follows from Lemma 2.5 that $N\sqrt{0} = 0$. So suppose that $x \notin N$. By Theorem 2.4, either $(N : Rx) = (N : M)$ or $(N : Rx) = (0 : Rx)$. If $(N : Rx) = (0 : Rx)$, then $N \subseteq (N : Rx)M = (0 : Rx)M$ (since $(N : M) \subseteq (N : Rx)$). Therefore, $N_x \subseteq J(0 : Rx)M = 0$, and hence $N\sqrt{0} = 0$ by Lemma 2.5. Suppose that $(N : M) = (N : Rx)$. Let $x^n = J^n M = 0$, but $x^{n-1} = J^{n-1} M \neq 0$, so $J^n = 0$ by [7, p.231 Corollary]. As $J^{n-1} \subseteq (I : J)$ we have $0 \neq x^{n-1} \subseteq (IM, JM)M = (N : Rx)M = (N : M)M = N$, so $x \in N$ by Theorem 2.6 which is a contradiction. Thus $N\sqrt{0} = 0$.

Corollary 2.8. *Let M be a finitely generated faithful multiplication module over a commutative ring R . Suppose that N and K are weakly prime submodules of M that are not prime. Then $NK = 0$.*

Proof. Assume that N and K are weakly prime submodule of M that are not prime. If $b \in N$, then $b^2 \in N^2 = 0$ by Corollary 2.3, so $N \subseteq \sqrt{0}$. Similarly, $K \subseteq \sqrt{0}$. Then $NK \subseteq N\sqrt{0} = 0$ by Theorem 2.7, as required.

Corollary 2.9. *Let M be a finitely generated faithful multiplication module over a commutative ring R with unique maximal submodule N , and let every prime submodule of M is maximal. If P is a weakly prime submodule of M , then either $P = N$ or $PN = 0$.*

Proof. If P is prime, then $P = N$. If P is not prime, then $\sqrt{0} = N$ and $PN = 0$ by Theorem 2.7.

Corollary 2.10. *Let M be a finitely generated faithful module over a local ring R with unique maximal submodule N , and let every prime submodule of M is maximal. If P is a weakly prime submodule M , then either $P = N$ or $PN = 0$.*

Proof. By [6, p.3748 Corollary 1], M is cyclic, and hence M is a multiplication R -module. Now the assertion follows from Corollary 2.9.

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