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SOME PROPERTIES OF WEAK FORM OF γ-SEMI-OPEN SETS

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Abstract. In this paper, we introduce and explore fundamental properties of weak form of γ-semi-open sets namely maximal γ-semi-open sets in topological spaces such as decomposition theorem for maximal γ-semi-open set. Basic properties of intersection of maximal γ-semi-open sets are established, such as the γ-semi-closure law of γ-semi-radical.

1. Introduction

Topology is an important and interesting area of mathematics, the study of which will not only introduce you to new concepts and theorems but also put into context old ones like continuous functions. It is so fundamental that its influence is evident in almost every other branch of mathematics. This makes the study of topology relevant to all who aspire to be mathematicians whether their first love is algebra, analysis, category theory, chaos, continuum mechanics, dynamics, geometry, industrial mathematics, mathematical biology, mathematical economics, mathematical finance, mathematical modelling, mathematical physics, mathematics of communication, number theory, numerical mathematics, operation research or statistics.

S. Kasahara [11] introduced and discussed an operation γ of a topology τ into the power set P(X) of a space X . H. Ogata [13], introduced the concept of γ-open sets and investigated the related topological properties of the associated topology τ_γ and τ by using operation γ. So far several researchers worked on the findings of H. Ogata and a lot of material is available in the literature.

S. Hussain and B. Ahmad [1-10] continued studying the properties of γ-operations on topological spaces and investigated many interesting results. N. Levine [12] introduced the notion of semi-open sets in topological spaces. Several topologist generalized many classical notions of topology using semi open sets introduced by Levine. Recently, S. Hussain, B. Ahmad and T. Noiri[10] introduced γ-semi-open sets in topological spaces as a generalization

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of γ-open sets introduced by H. Ogata [13]. Moreover, they showed [10] that the concepts of g Λ_s-sets; g Λ^s-sets, semi-T_1 space and semi-R_0 space can be generalized by replacing the semi- open sets with γ-semi-open sets. Further B. Ahmad and S. Hussain [2]explored γ-semi-interior (closure) and γ-semi-continuous functions. They introduced and discussed minimal γ-semi-open sets in topological spaces [7] and obtained some properties of pre γ-semi-open sets using properties of minimal γ-semi-open sets. As an application of a theory of minimal γ-semi-open sets, they established a sufficient condition for a γ-semi-locally finite space to be a pre γ-semi-T_2 space.

Although the definition of the maximal γ-semi-open set is obtained by dualizing the definition of the minimal γ-semi-open set, the properties of them are quite different, especially the results in the last two sections. The purpose of this paper is to prove some fundamental properties of maximal γ-semi-open sets and establish a part of the foundation of the theory of maximal γ-semi-open sets in topological spaces.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space in place of a topological space.

2. Preliminaries

Throughout the present paper X denotes the topological space.

Definition([11]). An operation $\gamma : \tau \to P(X)$ is a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ.

Definition.([11]). Let $A \subseteq X$. A point $\in A$ is said to be a γ-interior point of A, if there exists an open nbd N of x such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma(A)$. Thus

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ-open [13] iff A = int_γ(A). A set A is called γ- closed [1] iff X-A is γ-open.

Definition([13]). A point $x \in X$ is called a γ-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $cl_\gamma(A)$. A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ-closed superset of A.

Definition([10]). A subset A of a space (X,τ) is said to be a γ-semi-open set, if there exists a γ-open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ^*-semi-open sets is denoted by $SO_{\gamma(X)}$. A is γ-semi-closed iff X-A is γ-semi-open in X.

Definition.([5]). Let A be a subset of a space X. The intersection of all γ-semi-closed sets containing A is called γ-semi-closure of A and is denoted by $scl_\gamma(A)$.

Note that A is γ-semi-closed if and only if $scl_\gamma(A) = A$.

Definition([5]). Let A be a subset of a space X. The union of γ-semi-open subsets of A is called γ-semi-interior of A and is denoted by $sint_\gamma(A)$.

Definition.([5]) An operation γ on τ is said be semi-regular, if for any semi-open $nbds$ U, V of $x \in X$, there exists an semi-open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition.([5]) An operation γ on τ is said be semi-open, if for every nbd U of $x \in X$, there exists a semi-open set B such that $x \in B$ and $U^\gamma \supseteq B$.

Definition([6]) A space X is said to be γ-semi-T_2, if for each distinct points $x, y \in X$, there exist semi-open sets U, V such that $x \in U$, $y \in V$ and $U^\gamma \cap V^\gamma = \phi$.

3. Maximal γ-semi-open sets

Definition 3.1. A proper nonempty γ-semi-open subset B of X is said to be a maximal γ-semi-open set, if any γ-semi-open set which contains B is X or B.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ be a topology on X. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), \text{ if } b \in A \\ A, \quad \text{if } b \notin A \end{cases}$$

Calculation gives that γ-semi-open sets are $\{a\}$, $\{a, b\}$, $\{a, c\}$, X, ϕ. Clearly $\{a, b\}$, $\{a, c\}$ and X are maximal γ-semi-open sets.

The following Lemma is straightforward from the Definition 3.1:

Lemma 3.3. (a) *Let U be a maximal γ-semi-open set and W a γ-semi-open set. Then $U \cup W = X$ or $W \subseteq U$.*

(b) *Let U and V be maximal γ-semi-open sets. Then $U \cup V = X$ or $U = V$.*

The following is immediate:

Proposition 3.4. *Let U be a maximal γ-semi-open set. If $x \in U$, then $W \cup U = X$ or $W \subseteq U$, for any γ-semi-open nbd W of x.*

Theorem 3.5. *Let U_α, U_β and U_γ be maximal γ-semi-open sets such that $U_\alpha \neq U_\beta$. If $U_\alpha \cap U_\beta \subseteq U_\gamma$, then $U_\alpha = U_\gamma$ or $U_\beta = U_\gamma$.*

Proof. We see that

$$\begin{aligned} U_\alpha \cap U_\gamma &= U_\alpha \cap (U_\gamma \cap X) = U_\alpha \cap (U_\gamma \cap (U_\alpha \cup U_\beta)) \quad \text{(by Lemma 3.3(b))} \\ &= U_\alpha \cap ((U_\gamma \cap U_\alpha) \cup (U_\gamma \cap U_\beta)) = (U_\alpha \cap U_\gamma) \cup (U_\gamma \cap U_\alpha \cap U_\beta) \\ &= (U_\alpha \cap U_\gamma) \cup (U_\alpha \cap U_\beta) \quad \text{(by } U_\alpha \cap U_\beta \subseteq U_\gamma) \end{aligned}$$

$$= U_\alpha \cap (U_\gamma \cup U_\beta).$$

Hence we have $U_\alpha \cap U_\gamma = U_\alpha \cap (U_\gamma \cup U_\beta)$. If $U_\gamma \neq U_\beta$, then $U_\gamma \cup U_\beta = X$, and hence $U_\alpha \cap U_\gamma = U_\alpha$ implies $U_\alpha \subseteq U_\gamma$. Since U_α and U_γ are maximal γ-semi-open sets, we have $U_\alpha = U_\gamma$. Hence the proof. □

The following Theorem gives a relationship among maximal γ-semi-open sets:

Proposition 3.6. *Let U_1, U_2 and U_3 be disjoint maximal γ-semi-open sets. Then, $U_1 \cap U_2 \nsubseteq U_1 \cap U_3$.*

Proof. If $U_1 \cap U_2 \subseteq U_1 \cap U_3$, then we see that $(U_1 \cap U_2) \cup (U_2 \cap U_3) \subseteq (U_1 \cap U_3) \cup (U_2 \cap U_3)$. Therefore $U_2 \cap (U_1 \cup U_3) \subseteq (U_1 \cup U_2) \cap U_3$. Since $U_1 \cup U_3 = X = U_1 \cup U_2$, we have $U_2 \subseteq U_3$. It follows that $U_2 = U_3$. This contradiction completes the proof. □

Proposition 3.7. *Let U be a maximal γ-semi-open set and $x \in U$. Then $U = \cup\{W : W$ is a γ-semi-open nbd of x such that $W \cup U \neq X\}$, where γ is a semi-regular operation.*

Proof. This follows from Proposition 3.4 and the fact that U is a γ-semi-open nbh of x [2]. Hence the proof. □

Recall that the complement of any finite subset is a cofinite subset. The following Proposition shows the existence of maximal γ-semi-open sets for special cases.

Proposition 3.8. *Let V be a proper nonempty cofinite γ-semi-open subset. Then there exists, at least, one (cofinite) maximal γ-semi-open set U such that $V \subseteq U$.*

Proof. If V is a maximal γ-semi-open set, we may set $U = V$. If V is not a maximal γ-semi-open set, then there exists (cofinite) γ-semi-open set V_1 such that $V \subseteq V_1 \neq X$. If V_1 is a maximal γ-semi-open set, we may set $U = V_1$. If V_1 is not a maximal γ-semi-open set, then there exists a (cofinite) γ-semi-open set V_2 such that $V \subseteq V_1 \subseteq V_2 \neq X$ Continuing this process, we have a sequence of γ-semi-open sets

$$V \subseteq V_1 \subseteq V_2 \cdots \subseteq V_k \subseteq \cdots$$

Since V is a cofinite set, this process ends in a finite number of steps. Finally, we get a maximal γ-semi-open set $U = V_n$, for some positive integer n. This completes the proof. □

4. Applications of γ-semi-closure, γ-semi-interior in maximal γ-semi-open sets

Now we study some relationship among γ-semi-closure, γ-semi-interior and a maximal γ-semi-open set. As an application, we prove Theorem 4.12 about pre γ-semi-open sets:

Theorem 4.1. *Let U be a maximal γ-semi-open set and $x \in (X - U)$. Then, $(X - U) \subseteq W$, for any γ-semi-open nbd W of x.*

Proof. Since $x \in (X-U)$, we have $W \nsubseteq U$, for any γ-semi-open nbd W of x. Then $W \cup U = X$, by Lemma 3.3(a). Therefore $(X-U) \subseteq W$. This completes the proof. □

Corollary 4.2. *Let U be a maximal γ-semi-open set. Then, either of the following holds:*

(a) *$W = X$, for each $x \in (X-U)$ and each γ-semi-open nbd W of x.*
(b) *there exists a γ-semi-open set W such that $(X-U) \subseteq W$ and $W \subseteq X$.*

Proof. If (a) does not hold, then there exists an element $x \in (X-U)$ and a γ-semi-open nbd W of x such that $W \subseteq X$. By Theorem 4.1, we have $(X-U) \subseteq W$. Hence the proof. □

Corollary 4.3. *Let U be a maximal γ-semi-open set. Then, either of the following holds:*

(a) *$(X-U) \subseteq W$, for each $x \in (X-U)$ and each γ-semi-open nbd W of x.*
(b) *there exists a γ-semi-open set W such that $(X-U) = W \neq X$.*

Proof. Assume that (b) does not hold. Then by Theorem 4.1, we have $(X-U) \subseteq W$, for each $x \in (X-U)$ and each γ-semi-open nbd W of x. Hence we have $(X-U) \subseteq W$. This completes the proof. □

Corollary 4.4. *Let U be a maximal γ-semi-open set. Then $scl_\gamma(U) = X$ or $scl_\gamma(U) = U$.*

Proof. Since U is a maximal γ-semi-open set. Then by Corollary 4.3, the following two cases hold:

(a) for each $x \in (X-U)$ and each γ-semi-open nbd W of x, we have $(X-U) \subseteq W$. Since $(X-U) \neq W^\gamma$, we have $W \cap U \neq \phi$, for any γ-semi-open nbd and hence open nbd W of x. Hence, $(X-U) \subseteq scl_\gamma(U)$. Since $X = U \cup (X-U) \subseteq U \cup scl_\gamma(U) = scl_\gamma(U) \subseteq X$, we have $scl_\gamma(U) = X$;
(b) there exists a γ-semi-open set W such that $(X-U) = W \neq X$, since $(X-U) = W$ is a γ-semi-open set, U is a γ-semi-closed set. Therefore, $U = scl_\gamma(U)$. This completes the proof. □

The following Corollary follows from Corollary 4.4:

Corollary 4.5. *Let U be a maximal γ-semi-open set. Then $sint_\gamma(X-U) = (X-U)$ or $sint_\gamma(X-U) = \phi$.*

Theorem 4.6. *Let U be a maximal γ-semi-open set and $\phi \neq S \subseteq (X-U)$. Then $scl_\gamma(S) = (X-U)$.*

Proof. Since $\phi \neq S \subseteq (X-U)$, then by Theorem 4.1, we have $W \cap S \neq \phi$, for any $x \in (X-U)$ and any γ-semi-open nbd W of x. Then $(X-U) \subseteq scl_\gamma(S)$. Since $X-U$ is a γ-semi-closed set and $S \subseteq (X-U)$, we see that $scl_\gamma(S) \subseteq scl_\gamma(X-U) = X-U$. Therefore, $X-U = scl_\gamma(S)$. Hence the proof. □

Corollary 4.7. *Let U be a maximal γ-semi-open set and $M \subseteq X$ with $U \subseteq M$. Then $scl_\gamma(M) = X$. Where γ is a semi-regular operation.*

Proof. Since $U \subseteq M \subseteq X$, there exists a $\phi \neq S \subseteq (X-U)$ such that $M = U \cup S$. By semi-regularity of operation γ and Theorem 4.6, we have $scl_\gamma(M) = scl_\gamma(S \cup U) = scl_\gamma(S) \cup scl_\gamma(U) \supseteq (X-U) \cup U = X$. Therefore $scl_\gamma(M) = X$. This completes the proof. □

The following Theorem follows from Corollary 4.7:

Theorem 4.8. *Let U be a maximal γ-semi-open set and $(X-U)$ has at least two elements. Then, for any element $a \in (X-U)$, $scl_\gamma(X-\{a\}) = X$, where γ is a semi-regular operation.*

Theorem 4.9. *Let U be a maximal γ-semi-open set and $U \subseteq N \subseteq X$. Then $sint_\gamma(N) = U$.*

Proof. If $N = U$, then $sint_\gamma(N) = sint_\gamma(U) = U$. Otherwise $N \neq U$, and hence $U \subseteq N$. It follows that $U \subseteq sint_\gamma(N)$. Since U is a maximal γ-semi-open set, we have also $sint_\gamma(N) \subseteq U$. Therefore $sint_\gamma(N) = U$. Hence the proof. □

The following Theorem follows from Theorems 4.6 and 4.9:

Theorem 4.10. *Let U be a maximal γ-semi-open set and $\phi \neq S \subseteq (X-U)$. Then, $X - scl_\gamma(S) = int_\gamma(X-S) = U$.*

Definition 4.11 ([2])**.** A subset M of X is said to be pre-γ-semi-open-set, if $M \subseteq sint_\gamma(scl_\gamma(M))$.

The following result follows from Corollary 4.7.

Theorem 4.12. *Let U be a maximal γ-semi-open set and $U \subseteq M \subseteq X$. Then M is a pre-$\gamma$-semi-open set, where γ is a semi-regular operation.*

Proof. If $M = U$, then M is a γ-semi-open set. Therefore M is a pre-γ-semi-open set [2]. Otherwise $U \subseteq M$, then by Corollary 4.7, $sint_\gamma(scl_\gamma(M)) = sint_\gamma(X) = X \supseteq M$. Therefore M is a pre-γ-semi-open set. Hence the proof. □

The following Corollary directly follows from Theorem 4.12:

Corollary 4.13. *Let U be a maximal γ-semi-open set. Then $X - \{a\}$ is a pre-γ-semi-open set, for any $a \in (X-U)$, where γ is a semi-regular operation.*

5. Basic properties of γ-semi-radical

In this section, we prove fundamental properties of radical of maximal γ-semi-open sets. We establish a very useful decomposition theorem for a maximal γ-semi-open set in Theorem 5.7.

Definition 5.1. Let $U = \{U_\lambda : \lambda \in I\}$ be a class of maximal γ-semi-open sets. Then $\bigcap U = \bigcap_{\lambda \in I} U_\lambda$ is called the γ-semi-radical of U.

Example 5.2. Clearly is Example 3.2, the γ-semi-radical is the set $\{a\}$.

The Symbol $I\backslash\Gamma$ means difference of index sets; namely, $I\backslash\Gamma = I-\Gamma$, and the cardinality of a set I is denoted by $|I|$ in the following arguments:

Theorem 5.3. *Suppose that $|I|\geq 2$. Let U_λ be a maximal γ-semi-open set for any $\lambda\in I$ and $U_\lambda\neq U_\mu$, for any $\lambda,\mu\in I$ with $\lambda\neq\mu$. If $\mu\in I$, then the following hold:*

(a) $X-\bigcap_{\lambda\in I-\{\mu\}}U_\lambda\subseteq U_\mu$.
(b) $\bigcap_{\lambda\in I-\{\mu\}}U_\lambda\neq\phi$.

Proof.

(a) By Lemma 3.3(b), we have $(X-U_\mu)\subseteq U_\lambda$ for any $\lambda\in I$ with $\lambda\neq\mu$. Then $(X-U_\mu)\subseteq\bigcap_{\lambda\in I-\{\mu\}}U_\lambda$. Therefore, we have $X-\bigcap_{\lambda\in I-\{\mu\}}U_\lambda\subseteq U_\mu$.
(b) If $\bigcap_{\lambda\in I-\{\mu\}}U_\lambda=\phi$. By (a), we have $X=U_\mu$. This is contradiction to our supposition that U_λ is a maximal γ-semi-open set. Therefore, we have $\bigcap_{\lambda\in I-\{\mu\}}U_\lambda\neq\phi$. This completes the proof. □

The following Corollary follows from Theorem 5.3(b):

Corollary 5.4. *Let U_λ be a maximal γ-semi-open set, for any $\lambda\in I$ and $U_\lambda\neq U_\mu$, for any $\lambda,\mu\in I$ with $\lambda\neq\mu$. If $|I|\geq 2$, then $U_\lambda\bigcap U_\mu\neq\phi$, for any $\lambda,\mu\in I$ with $\lambda\neq\mu$.*

Theorem 5.5. *Let U_λ be a maximal γ-semi-open set for any $\lambda\in I$ and $U_\lambda\neq U_\mu$, for any $\lambda,\mu\in I$ with $\lambda\neq\mu$. Assume that $|I|\geq 2$. If $\mu\in I$. Then, $\bigcap_{\lambda\in I-\{\mu\}}U_\lambda\nsubseteq U_\mu\nsubseteq\bigcap_{\lambda\in I-\{\mu\}}U_\lambda$.*

Proof. If $\bigcap_{\lambda\in I-\{\mu\}}U_\lambda\subseteq U_\mu$. Then by Theorem 5.3(b), we have
$X=(X-\bigcap_{\lambda\in I-\{\mu\}}U_\lambda)\bigcup(\bigcap_{\lambda\in I-\{\mu\}}U_\lambda)\subseteq U_\lambda$. This is contradiction to our assumption. If $U_\mu\subseteq\bigcap_{\lambda\in I-\{\mu\}}U_\lambda$, then we have $U_\mu\subseteq U_\lambda$ and hence $U_\mu=U_\lambda$ for any element $\lambda\in(I-\{\mu\})$. This contradicts our assumption that $U_\mu\neq U_\lambda$ when $\lambda\neq\mu$. Hence the proof. □

Corollary 5.6. *Let U_λ be a maximal γ-semi-open set, for any $\lambda\in I$ and $U_\lambda\neq U_\mu$, for any $\lambda,\mu\in I$ with $\lambda\neq\mu$. If $\phi\neq\delta\subseteq I$, then $\bigcap_{\lambda\in I-\{\delta\}}U_\lambda\nsubseteq\bigcap_{\iota\in\delta}U_\iota\nsubseteq\bigcap_{\lambda\in I-\{\delta\}}U_\lambda$.*

Proof. Let $\iota\in\delta$. By Theorem 5.5,

$$\bigcap_{\lambda\in I-\{\delta\}}U_\lambda=\bigcap_{\lambda\in(I-(\{\delta\}\bigcup\{\iota\}))}U_\lambda\nsubseteq U_i.$$

Therefore, we see $\bigcap_{\lambda\in I-\{\delta\}}U_\lambda\nsubseteq\bigcap_{\iota\in\delta}U_\iota$. On the other hand, since $\bigcap_{\iota\in\delta}U_\iota=\bigcap_{\iota\in((I-(I-\{\delta\})))}U_\iota\nsubseteq\bigcap_{\lambda\in I-\{\delta\}}U_\lambda$, we have $\bigcap_{\iota\in\delta}U_\iota\nsubseteq\bigcap_{\lambda\in I-\{\delta\}}U_\lambda$. This completes the proof. □

Theorem 5.7. *Let U_λ be a maximal γ-semi-open set for any $\lambda\in I$ and $U_\lambda\neq U_\mu$, for any $\lambda,\mu\in I$ with $\lambda\neq\mu$. If $\phi\neq\delta\subseteq I$, then $\bigcap_{\lambda\in I}U_\lambda\nsubseteq\bigcap_{\iota\in\delta}U_\iota$.*

Proof. By Corollary 5.6, we have $\bigcap_{\lambda\in I}U_\lambda=(\bigcap_{\lambda\in I-\delta}U_\lambda)\bigcap(\bigcap_{\iota\in I}U_\iota)\nsubseteq\bigcap_{\iota\in I}U_\iota$. Hence the proof. □

Theorem 5.8. *[Decomposition Theorem for Maximal γ-semi-Open Set.] Let $|I| \geq 2$. If U_λ be a maximal γ-semi-open set for any $\lambda \in I$ and $U_\lambda \neq U_\mu$, for any $\lambda, \mu \in I$ with $\lambda \neq \mu$. Then for any $\mu \in I$,*

$$U_\mu = (\bigcap_{\lambda \in I} U_\lambda) \bigcup (X - \bigcap_{\lambda \in I - \{\mu\}} U_\lambda).$$

Proof. Let $\mu \in I$. By Theorem 5.3(a), we have

$$\begin{aligned}(\bigcap_{\lambda \in I} U_\lambda) \bigcup (X - \bigcap_{\lambda \in I-\{\mu\}} U_\lambda) &= ((\bigcap_{\lambda \in I-\{\mu\}} U_\lambda) \bigcap U_\mu) \bigcup (X - \bigcap_{\lambda \in I-\{\mu\}} U_\lambda) \\ &= ((\bigcap_{\lambda \in I-\{\mu\}} U_\lambda) \bigcup (X - \bigcap_{\lambda \in I-\{\mu\}} U_\lambda)) \bigcap (U_\mu \bigcup ((X - \bigcap_{\lambda \in I-\{\mu\}} U_\lambda))) \\ &= U_\mu \bigcup (X - \bigcap_{\lambda \in I-\{\mu\}} U_\lambda) = U_\mu.\end{aligned}$$

Therefore, we have $U_\mu = (\bigcap_{\lambda \in I} U_\lambda) \bigcup (X - \bigcap_{\lambda \in I-\{\mu\}} U_\lambda)$. Hence the proof. □

We use Theorem 5.8 to prove the following:

Theorem 5.9. *Let I be a finite set and U_λ be a maximal γ-semi-open set, for any $\lambda \in I$ and $U_\lambda \neq U_\mu$, for any $\lambda, \mu \in I$ with $\lambda \neq \mu$. If $\bigcap_{\lambda \in I} U_\lambda$ is a γ-semi-closed set, then U_λ is a γ-semi-closed set, for any $\lambda \in I$, where γ is a semi-regular operation.*

Proof. By Theorem 5.8, we have $U_\mu = (\bigcap_{\lambda \in I} U_\lambda) \bigcup (X - \bigcap_{\lambda \in I-\{\mu\}} U_\lambda) = (\bigcap_{\lambda \in I} U_\lambda) \bigcup (\bigcup_{\lambda \in I-\{\mu\}} (X - U_\lambda)$. Since I is a finite set and γ is semi-regular so, $\bigcup_{\lambda \in I-\{\mu\}} (X - U_\lambda)$ is a γ-semi-closed set [13]. Hence U_λ is a γ-semi-closed set by our assumption. this completes the proof. □

The following Theorem gives a sufficient condition for the set of all maximal γ-semi-open sets:

Theorem 5.10. *Assume that $|I| \geq 2$. Let U_λ be a maximal γ-semi-open set, for any $\lambda \in I$ and $U_\lambda \neq U_\mu$, for any $\lambda, \mu \in I$ with $\lambda \neq \mu$. If $\bigcap_{\lambda \in I} U_\lambda = \phi$, then $\{U_\lambda : \lambda \in I\}$ is the set of all maximal γ-semi-open sets of X.*

Proof. If there exists another maximal γ-semi-open set U_ν of X, which is not equal to U_λ, for any $\lambda \in I$, then $\phi = \bigcap_{\lambda \in I} U_\lambda = \bigcap_{\lambda \in (I \bigcup \{\nu\}) - \{\nu\}} U_\lambda$. By Theorem 5.3(b), we see that $\bigcap_{\lambda \in (I \bigcup \{\nu\}) - \}\nu\}} U_\lambda \neq \phi$. This contradicts our assumption. Hence the proof. □

Example 5.11. Let X be a γ-semi-T_2 space or a cofinite space or a cocountable space. If each point $\{x\}$ is γ-semi-closed , then $X - \{a\}$ is a maximal γ-semi-open set for any $a \in X$. Moreover, we see that $\{X - \{a\} : a \in X\}$ is the set of all maximal γ-semi-open sets of X by Theorem 5.10, since $\bigcap_{a \in X} (X - \{a\}) = \phi$.

6. More about γ-semi-radical of maximal γ-semi-open sets

In this section, we study the γ-semi-closure of γ-semi-radical of maximal γ-semi-open sets, we begin with a proposition.

Proposition 6.1. *Let U_λ be a set, for any $\lambda \in I$. If $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) = X$, then $scl_\gamma(U_\lambda) = X$, for any $\lambda \in I$.*

Proof. We see that $X = scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) \subseteq scl_\gamma(U_\lambda)$. It follows that $scl_\gamma(U_\lambda) = X$, for any $\lambda \in I$. Hence the proof. □

Theorem 6.2. *Let I be a finite set and U_λ be a maximal γ-semi-open set for any $\lambda \in I$. If $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) \neq X$, then there exists $\lambda \in I$ such that $scl_\gamma(U_\lambda) = U_\lambda$, where γ is a semi-regular and an semi-open operation.*

Proof. Suppose that $scl_\gamma(U_\lambda) = X$ for any $\lambda \in I$. Let $\mu \in I$. Since γ is semi-open, so $\bigcap_{\lambda\in I-\{\mu\}} U_\lambda$ is a γ-semi-open set. Also semi-regularity of operation γ implies that

$$\begin{aligned} scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) &= scl_\gamma((\bigcap_{\lambda\in I-\{\{\mu\}} U_\lambda)\bigcap U_\mu) = scl_\gamma((\bigcap_{\lambda\in I-\{\{\mu\}} U_\lambda))\bigcap scl_\gamma(U_\mu) \\ &\supseteq (\bigcap_{\lambda\in I-\{\{\mu\}} U_\lambda)\bigcap scl_\gamma(U_\mu) = (\bigcap_{\lambda\in I-\{\{\mu\}} U_\lambda)\bigcap X = (\bigcap_{\lambda\in I-\{\{\mu\}} U_\lambda). \end{aligned}$$

Therefore, $scl_\gamma(\bigcap_{\lambda\in I-\{\{\mu\}} U_\lambda) \subseteq scl_\gamma(scl_\gamma(\bigcap_{\lambda\in I} U_\lambda)) = scl_\gamma(\bigcap_{\lambda\in I} U_\lambda)$, since γ is semi-open. On the other hand, we see that $\bigcap_{\lambda\in I} U_\lambda \subseteq \bigcap_{\lambda\in I-\{\mu\}} U_\lambda$. It follows that $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) = scl_\gamma(\bigcap_{\lambda\in I-\{\mu\}} U_\lambda)$. Then by induction on the element of I, we see that $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) = scl_\gamma(U_\lambda) = X$, for any $\lambda \in I$. This contradicts our assumption that $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) \neq X$. Therefore, we see that there exists $\lambda \in I$ such that $scl_\gamma(U_\lambda) = U_\lambda$. This completes the proof. □

The following example shows that the condition that I is finite is necessary in Theorem 6.2.

Example 6.3. Let $X = \Re^n$, the n-dimensional Euclidean space. Let $U_x = X - \{x\}$, for any $x \in X$. Then U_x is a maximal γ-semi-open set and hence we have

$$scl_\gamma(\bigcap_{x\in X} U_x) = scl_\gamma(\phi) = \phi \neq X$$

However, $scl_\gamma(U_x) = X$ for any $x \in X$.

The γ-semi-radical of maximal γ-semi-open sets have the following outstanding property:

Theorem 6.4. *[The γ-Semi-Closure Law of γ-Semi-Radical] Let I be finite and U_λ is a maximal γ-semi-open set for each $\lambda \in I$. Let $\Gamma \subseteq I$ such that*

$$scl_\gamma(U_\lambda) = \begin{cases} U_\lambda, & \text{for any } \lambda \in \Gamma \\ X, & \text{for any } \lambda \in I - \Gamma \end{cases}$$

Then, $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) = \bigcap_{\lambda\in I} U_\lambda (= X$, if $\Gamma = \phi)$, where γ is a semi-regular and an semi-open operation.

Proof. If $I = \phi$, then we have the result by Theorem 6.2. Otherwise $I \neq \phi$, and hence we see that

$$scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) = scl_\gamma((\bigcap_{\lambda\in\Gamma} U_\lambda)\bigcap(\bigcap_{\lambda\in I-\Gamma} U_\lambda)) = scl_\gamma((\bigcap_{\lambda\in\Gamma} U_\lambda))\bigcap scl_\gamma((\bigcap_{\lambda\in I-\Gamma} U_\lambda))$$
$$\supseteq (\bigcap_{\lambda\in\Gamma} U_\lambda)\bigcap scl_\gamma(\bigcap_{\lambda\in I-\Gamma} U_\lambda) = (\bigcap_{\lambda\in\Gamma} U_\lambda)\bigcap X = \bigcap_{\lambda\in\Gamma} U_\lambda.$$

By Theorem 6.2 and the fact that $(\bigcap_{\lambda\in\Gamma} U_\lambda)$ is a γ-semi-open set. Since γ is semi-open [2], it follows that

$$scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) = scl_\gamma(scl_\gamma(\bigcap_{\lambda\in I} U_\lambda)) \supseteq scl_\gamma(\bigcap_{\lambda\in\Gamma} U_\lambda).$$

On the other hand, we see that $\bigcap_{\lambda\in I} U_\lambda \subseteq \bigcap_{\lambda\in\Gamma} U_\lambda$, and hence $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) \subseteq scl_\gamma(\bigcap_{\lambda\in\Gamma} U_\lambda)$. It follows that $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) = scl_\gamma(\bigcap_{\lambda\in\Gamma} U_\lambda)$. The γ-semi-radical $\bigcap_{\lambda\in\lambda} U_\lambda$ is a γ-semi-closed set since U_λ is a γ-semi-closed set for any $\lambda \in \Gamma$ by our assumption. Therefore, we see that $scl_\gamma(\bigcap_{\lambda\in I} U_\lambda) = \bigcap_{\lambda\in\Gamma} U_\lambda$. This completes the proof. □

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