TAMKANG JOURNAL OF MATHEMATICS
Volume 41, Number 4, 303-312, Winter 2010

SOME NEW INTEGRAL REPRESENTATIONS OF GENERALIZED MATHIEU SERIES AND ALTERNATING MATHIEU SERIES

ŽIVORAD TOMOVSKI

Dedicated to Prof. Hari M. Srivastava on the occasion of his 70^{th} *birthday*

Abstract. The main purpose of this paper is to present a number of new integral representations for the familiar Mathieu series $S_\mu^{(\alpha,\beta)}(r;\{a_k\}_{k=1}^\infty)(r \in R,\ \alpha,\ \beta,\ \mu,\ \{a_k\}_{k=1}^\infty \in R^+)$ [12] as well as for its alternating version [8, 16] when $a_k = \{k^p\}_{k=1}^\infty$, $a_k = \{(k!)^p\}_{k=1}^\infty$, $a_k = \{(\ln k!)^p\}_{k=1}^\infty$ with $p = \gamma$, $\gamma(\mu\alpha - \beta) > 1$ and $p = \frac{q}{\alpha}$, $\mu - \frac{\beta}{\alpha} > q^{-1}$, $q \in N$.

1. Introduction

The following familiar infinite series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} \qquad (r \in R) \tag{1.1}$$

is named after Emile Leonard Mathieu (1835-1890), who investigated it in his 1890 work [7] on elasticity of solid bodies. A remarkably useful integral representation for $S(r)$ in the elegant form

$$S(r) = \frac{1}{r}\int_0^\infty \frac{t\sin(rt)}{e^t-1}dt \tag{1.2}$$

was given by Emersleben [5]. An alternating version of (1.1)

$$\widetilde{S}(r) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{(n^2+r^2)^2} \qquad (r \in R) \tag{1.3}$$

was recently introduced by Pogany et al. in [8].

In [8] it was given the following relationship between $S(r)$ and $\widetilde{S}(r)$:

$$\widetilde{S}(r) = S(r) - \frac{1}{4}S\Big(\frac{r}{2}\Big). \tag{1.4}$$

Received September 16, 2009; revised June 3, 2010.
2000 *Mathematics Subject Classification*. Primary 33E20; Secondary 33C20.
Key words and phrases. Mathieu series, alternating Mathieu series, integral representations, Laplace transform.

Let F be a Laplace transform of f, i.e.

$$F(p) = \mathcal{L}_p(f(x)). \tag{1.5}$$

Using the relations (see [9]), p.651)

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} \frac{f(x)}{e^x - 1} dx \tag{1.6}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} F(n) = \int_0^{\infty} \frac{f(x)}{e^x + 1} dx \tag{1.7}$$

Pogany et al. [8] gave an integral representation for $\widetilde{S}(r)$:

$$\widetilde{S}(r) = \frac{1}{r} \int_0^{\infty} \frac{t \sin(rt)}{e^t + 1} dt. \tag{1.8}$$

Choi and Srivastava (see [2], Theorem 1) presented a relationship between the Mathieu series $S(r)$ and certain series involving the Riemann Zeta function. Pogany et al. (see [8], Proposition 1) gave a relationship between the alternating Mathieu series $\widetilde{S}(r)$ and certain series involving the Dirichlet Eta function. By means of these relationships Choi and Srivastava (see [2]) presented various integral representations of $S(r)$ and $\widetilde{S}(r)$, in terms of the Trigamma function $\psi'(z)$ or (equivalently) the Hurwitz (or generalized) Zeta function $\zeta(s, a)$.

Several interesting problems and solutions dealing with integral representations and bounds for the following slight generalization of the Mathieu series with a fractional power

$$S_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^\mu} \qquad (r \in R; \mu > 1) \tag{1.9}$$

can be found in the works by Diananda [3], Tomovski and Trencevski [14], Cerone and Lenard [1] and Choi and Srivastava [2]. Namely, Tomovski and Trencevski [14] presented a relationship (integral representation) of (1.9) in terms of the Polygamma function $\psi^{(\mu)}(z)$, $\mu = 2, 3, 4, \ldots$:

$$S_{\mu+1}(r) = \frac{2}{\mu!(2r)^\mu} \mathrm{Re}[e^{i\mu\pi/2} \psi^{(\mu)}(1 + ir)] + \sum_{k=1}^{\mu} \frac{2(1-K)}{(2r)^{2\mu-k+1} k!(\mu-k+1)} \binom{-(\mu+1)}{\mu-k} \mathrm{Re}[e^{i(2\mu-k+1)\pi/2} \psi^{(\mu)}(1+ir)] \; (\mu \in N). \tag{1.10}$$

Choi and Srivastava [2] presented two relationships (integral representations) for the series $S_3(r)$ when $0 < |r| < 1$ and $\mathrm{Re}(r) > 0$:

$$S_3(r) = \frac{i}{8r^3} [\psi'(1 + ir) - \psi'(1 - ir)] + \frac{1}{8r^3} [\psi''(1 + ir) + \psi''(1 - ir)] = \frac{i}{8r^3} [\zeta'(2, 1+ir) - \zeta'(2, 1-ir)] - \frac{1}{4r^2} [\zeta(3, 1+ir) + \zeta(3, 1-ir)] \; (0 < |r| < 1), \tag{1.11}$$

$$S_3(r) = \frac{3 + r^2}{2(1 + r^2)^3} + \frac{1}{4r^3} \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t e^{-t} [\sin(rt) - rt \cos(rt)] \; (\mathrm{Re}(r) > 0).$$

Motivated essentially by the works of Cerone and Lenard [1] (and Qi [10]) the *five-parameter* family of generalized Mathieu series

$$S_{\mu}^{(\alpha,\beta)}(r;a) = S_{\mu}^{(\alpha,\beta)}(r;\{a_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} \frac{2a_k^{\beta}}{(a_k^{\alpha}+r^2)^{\mu}} \quad (r \in R, \alpha, \beta, \mu \in R^{+}) \tag{1.12}$$

was defined in [12], where it is tacitly assumed that the positive sequence

$$a = \{a_n\}_{k=1}^{\infty} = \{a_1, a_2, a_3, \ldots\} \quad (\lim_{k\to\infty} a_k = \infty)$$

is so chosen that the infinite series in definition (1.12) converges, that is, that the following auxiliary series

$$\sum_{k=1}^{\infty} \frac{1}{a_k^{\mu\alpha-\beta}}$$

is convergent. Comparing the definitions (1.1), (1.9) and (1.12), we see that $S_2(r) = S(r)$ and $S_{\mu}(r) = S_{\mu}^{(2,1)}(r, \{k\}_{k=1}^{\infty})$. Furthermore, the special cases $S_2^{(2,1)}(r;\{a_k\}_{k=1}^{\infty})$, $S_{\mu}(r) = S_{\mu}^{(2,1)}(r;\{k\}_{k=1}^{\infty})$, $S_{\mu}^{(2,1)}(r;\{k^{\gamma}\}_{k=1}^{\infty})$ and $S_{\mu}^{(\alpha,\alpha/2)}(r;\{k\}_{k=1}^{\infty})$ were investigated by Qi [10]; Diananda [3]; Tomovski [15] and Cerone-Lenard [1]. Let (see [8, 16])

$$\widetilde{S}_{\mu}^{(\alpha,\beta)}(r;a) = \widetilde{S}_{\mu}^{(\alpha,\beta)}(r;\{a_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2a_k^{\beta}}{(a_k^{\alpha}+r^2)^{\mu}} \quad (r \in R, \alpha, \beta, \mu \in R^{+}) \tag{1.13}$$

be an alternating variant of (1.12), where the positive sequence $\{a_k\}_{k=1}^{\infty}$ satisfies the same conditions of the definition (1.12). In [4, 8, 12, 16] several integral representations for (1.12) and their alternating variants (1.13) were presented in terms of the generalized hypergeometric functions, the Bessel function of first kind and the generalized Mittag-Leffler functions.

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by

$${}_pF_q[(a_l)_{1,p};(b_j)_{1,q};x] = \sum_{m=0}^{\infty} \frac{\prod_{l=1}^{p}(a_l)_m}{\prod_{j=1}^{q}(b_j)_m} \frac{x^m}{m!} \tag{1.14}$$

where $(\delta)_m$ is the Pochhammer symbol, defined by

$$(\delta)_0 = 1, \ (\delta_m) = \delta(\delta+1)\cdots(\delta+m-1) = \frac{\Gamma(\delta+m)}{\Gamma(\delta)} \quad (m \in N).$$

The Fox-Wright generalization ${}_p\Psi_q$ of the hypergeometric ${}_pF_q$ function is defined by

$$ {}_p\Psi_q(x) = {}_p\Psi_q[(a_l,\alpha_l)_{1,p};(b_j,\beta_j)_{1,q};x] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}(a_l+\alpha_l k)}{\prod_{j=1}^{q}(b_j+\beta_j k)} \frac{x^k}{k!} $$

$$ \Big(a_l,b_j,\alpha_l,\beta_j \in R; l=1,2,\ldots,p, j=1,2,\ldots,q; 1+\sum_{j=1}^{q}\beta_j - \sum_{l=1}^{p}\alpha_l > 0\Big) \tag{1.15} $$

so that, obviously

$$ {}_p\Psi_q[(a_l,1)_{1,p};(b_j,1)_{1,q};x] = \frac{\prod_{l=1}^{p}\Gamma(a_l)}{\prod_{j=1}^{q}\Gamma(b_j)} \, {}_pF_q[(a_l)_{1,p};(b_j)_{1,q};x] \quad (a_l>0, b_j \notin Z_0^-). \tag{1.16} $$

2. Integral Representations for $\widetilde{S}_\mu^{(\alpha,\beta)}(r;a)$

In this section we present some new integral representations for generalized alternating Mathieu series (1.13) by using some recent investigated integral representations for (1.12).

Theorem 2.1. *For the alternating Mathieu series $\widetilde{S}(r)$ and its generalizations $\widetilde{S}_\mu^{(\alpha,\beta)}(r;\{k^\gamma\})$, $\widetilde{S}_\mu^{(\alpha,\beta)}(r;\{k^{q/\alpha}\})$ the following integral representations hold:*

$$ \widetilde{S}(r) = \widetilde{S}_2^{(2,1)}(r;\{k\}) = \frac{1}{r}\int_0^\infty \frac{t}{e^t-1}\Big[\sin(tr) - \frac{1}{2}\sin(\frac{tr}{2})\Big]dt \ (r\in R), \tag{2.1} $$

$$ \begin{aligned} \widetilde{S}_\mu^{(\alpha,\beta)}(r;\{k^\gamma\}) = \frac{2}{\Gamma(\mu)}\int_0^\infty \frac{t^{\gamma(\mu\alpha-\beta)-1}}{e^t-1}\Big\{ & {}_1\Psi_1\Big[(\mu,1),(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2t^{\gamma\alpha}\Big] \\ & -\Big(\frac{1}{2}\Big)^{\gamma(\mu\alpha-\beta)-1} {}_1\Psi_1\Big[(\mu,1),(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2\Big(\frac{t}{2}\Big)^{\gamma\alpha}\Big]\Big\}dt \\ & (r\in R,\alpha,\beta,\gamma\in R^+,\gamma(\mu\alpha-\beta)>1), \end{aligned} \tag{2.2} $$

$$ \begin{aligned} \widetilde{S}_\mu^{(\alpha,\beta)}(r;\{k^{q/\alpha}\}) = \frac{2}{\Gamma\Big(q\Big(\mu-\frac{\beta}{\alpha}\Big)\Big)}\int_0^\infty \frac{t^{q(\mu-\beta/\alpha)-1}}{e^t-1}\Big\{ & {}_1F_q\Big[\mu,\Delta\Big(q;q\Big(\mu-\frac{\beta}{\alpha}\Big)\Big);-r^2\Big(\frac{t}{q}\Big)^q\Big] \\ & -\Big(\frac{1}{2}\Big)^{q(\mu-\beta/(\alpha))} {}_1F_1\Big[\mu,\Delta\Big(q;q\Big(\mu-\frac{\beta}{\alpha}\Big)\Big);-r^2\Big(\frac{t}{2q}\Big)^q\Big]\Big\}dt \\ & (r\in R,\alpha,\beta,\gamma\in R^+,\mu-\frac{\beta}{\alpha}>q^{-1},\ q \textit{ belong to } N), \end{aligned} \tag{2.3} $$

where $\Delta(q;\lambda)$ *is the* q*-tuple* $\Big(\frac{\lambda}{q},\frac{\lambda+1}{q},\ldots,\frac{\lambda+q-1}{q}\Big)$.

Proof. Using the relation (1.4) and integral representation (1.2) we get

$$\widetilde{S}(r) = S(r) - \frac{1}{4}S\Big(\frac{r}{2}\Big) = \frac{1}{r}\int_0^\infty \frac{t\sin(tr)}{e^t-1}dt - \frac{1}{2r}\int_0^\infty \frac{t\sin(\frac{tr}{2})}{e^t-1}dt$$
$$= \frac{1}{r}\int_0^\infty \frac{t}{e^t-1}\Big[\sin(tr) - \frac{1}{2}\sin\Big(\frac{tr}{2}\Big)\Big]dt.$$

In [12] the following two integral representations were proved:

$$S_\mu^{(\alpha,\beta)}(r;\{n^\gamma\}) = \frac{2}{\Gamma(\mu)}\int_0^\infty \frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^x-1}\,{}_1\Psi_1[(\mu,1),(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2x^{\gamma\alpha}]dt$$
$$(r\in R, \alpha,\beta,\gamma\in R^+, \gamma(\mu\alpha-\beta)>1) \tag{2.4}$$
$$S_\mu^{(\alpha,\beta)}(r;\{n^{q/\alpha}\}) = \frac{2}{\Gamma\Big(q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big)}\int_0^\infty \frac{x^{q[\mu-\frac{\beta}{\alpha}]-1}}{e^t-1}\,{}_1F_q\Big[\mu;\Delta[q;q[\mu-\beta/\alpha]);-r^2\Big(\frac{x}{q}\Big)^q\Big]dx$$
$$(r\in R, \alpha,\beta\in R^+, \mu-\frac{\beta}{\alpha}>q^{-1}; q\in N). \tag{2.5}$$

Substituting the last two integral representations in the following two relationships

$$\widetilde{S}_\mu^{(\alpha,\beta)}(r;\{n^\gamma\}) = S_\mu^{(\alpha,\beta)}(r;\{n^\gamma\}) - \frac{1}{2^{\gamma(\mu\alpha-\beta)-1}}S_\mu^{(\alpha,\beta)}\bigg(\frac{r}{2^{\frac{\gamma\alpha}{2}}};\{n^\gamma\}\bigg), \tag{2.6}$$
$$\widetilde{S}_\mu^{(\alpha,\beta)}(r;\{n^{q/\alpha}\}) = S_\mu^{(\alpha,\beta)}(r;\{n^{q/\alpha}\}) - \frac{1}{2^{q(\mu-\beta/\alpha)-1}}S_\mu^{(\alpha,\beta)}\bigg(\frac{r}{2^{\frac{q}{2}}};\{n^{q/\alpha}\}\bigg), \tag{2.7}$$

we obtain (2.2) and (2.3) respectively. □

3. Integral representations for $S_\mu^{(\alpha,\beta)}(r;a)$ and $\widetilde{S}_\mu^{(\alpha,\beta)}(r;a)$ with $a_k = \{(k!)^p\}_{k=1}^\infty$ and $a_k = \{(\ln k!)^p\}_{k=1}^\infty$

The positive sequences $\{(k!)_{k=1}^\infty\}$ and $\{(\ln k!)_{k=1}^\infty\}$ tends to infinity and the auxiliary series $\sum_{k=1}^\infty \frac{1}{(k!)^p}$ and $\sum_{k=2}^\infty \frac{1}{(\ln k!)^p}$ are convergent for any $p>1$. Hence we can consider the following special cases of (1.12): $S_\mu^{(\alpha,\beta)}(r;\{(k!)^\gamma\}_{k=1}^\infty)$, $S_\mu^{(\alpha,\beta)}(r;\{(k!)^{q/\alpha}\}_{k=1}^\infty)$, $S_\mu^{(\alpha,\beta)}(r;\{(\ln k!)^\gamma\}_{k=2}^\infty)$, $S_\mu^{(\alpha,\beta)}(r;\{(\ln k!)^{q/\alpha}\}_{k=2}^\infty)$, as well as their alternating variants as special cases of (1.13). To express their integral representations we need the following Lemmas.

Lemma 3.1. *The following two integral representations are valid*

$$\sum_{k=1}^\infty F(k!) = \int_0^\infty f(y)D_{k!}(y)dy \tag{3.1}$$
$$\sum_{k=1}^\infty (-1)^{k-1}F(k!) = \int_0^\infty f(y)\widetilde{D}_{k!}(y)dy \tag{3.2}$$

where $D_{k!}(y) = \sum_{k=1}^\infty \frac{1}{e^{yk!}}$ *and* $\widetilde{D}_{\ln k!}(y) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{e^{yk!}}$ *are Dirichlet series.*

Proof. It is sufficient to proof the first equality since the second can be proved analogously.

$$\sum_{k=1}^{\infty} F(k!) = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-yk!} f(y)dy = \int_0^{\infty} f(y)\Big(\sum_{k=1}^{\infty} \frac{1}{e^{yk!}}\Big)dy = \int_0^{\infty} f(y) D_{k!}(y)dy. \quad \square$$

Lemma 3.2. *The following two integral representations are valid*

$$\sum_{k=1}^{\infty} F(\ln k!) = \int_0^{\infty} f(y) D_{\ln k!}(y)dy \tag{3.3}$$

$$\sum_{k=1}^{\infty} (-1)^{k-1} F(\ln k!) = \int_0^{\infty} f(y) \widetilde{D}_{\ln k!}(y)dy \tag{3.4}$$

where $D_{\ln k!}(y) = \sum_{k=1}^{\infty} \frac{1}{(k!)^y}$ *and* $\widetilde{D}_{\ln k!}(y) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k!)^y}$ *are Dirichlet series.*

Proof. It is sufficient to proof the first equality, since the second can be proved analogously. Indeed we have

$$\sum_{k=1}^{\infty} F(\ln k!) = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-y\ln k!} f(y)dy = \int_0^{\infty} f(y)\Big(\sum_{k=1}^{\infty} \frac{1}{(k!)^y}\Big)dy = \int_0^{\infty} f(y) D_{\ln k!}(y)dy.$$

Throughout next Lemmas, we also find it to be useful to consider the function $a_x = a(x)$ such that

$$a_x = a(x)\Big|_{x\in N} = \boldsymbol{a}$$

a and $\lfloor a^{-1}(x) \rfloor$ denotes the integer part of the inverse of the function $a(x)$. $\square$

Lemma 3.3. *For Dirichlet series* $D_{k!}(y)$ *and* $\widetilde{D}_{k!}(y)$ *the following integral representations hold:*

$$D_{k!}(y) = y \int_0^{\infty} e^{-yt}([\Gamma^{-1}(t)] - 1)dt \tag{3.5}$$

$$\widetilde{D}_{k!}(y) = y \int_0^{\infty} e^{-yt} \cos^2\Big(\frac{\pi}{2}[\Gamma^{-1}(t)]\Big)dt \tag{3.6}$$

where $\Gamma^{-1}(x)$ *is the inverse function of Gamma function.*

Proof. When $\boldsymbol{a} = \{a_n\}_{n=1}^{\infty}$ is monotone increasing to infinity then (see [8], [16])

$$D_{\boldsymbol{a}}(y) = \sum_{k=1}^{\infty} e^{-a_k y} = y \int_0^{\infty} e^{-yt} A(t)dt \tag{3.7}$$

$$\widetilde{D}_{\boldsymbol{a}}(y) = \sum_{k=1}^{\infty} (-1)^{k-1} e^{-a_k y} = y \int_0^{\infty} e^{-yt} \widetilde{A}(t)dt \tag{3.8}$$

where the so-called countining functions $A(t)$ and $\widetilde{A}(t)$ has been found easily in the following manners:

$$A(t) = \sum_{k:a_k \le t} 1 = [a^{-1}(t)] \tag{3.9}$$

$$\widetilde{A}(t) = \sum_{k:a_k \le t} (-1)^{k-1} = \frac{1-(-1)^{[a^{-1}(t)]}}{2} = \sin^2\Big(\frac{\pi}{2}[a^{-1}(t)]\Big). \tag{3.10}$$

Since

$$A(t) = \sum_{k:k! \le t} 1 = \sum_{k=1}^{k! \le t} 1 = \sum_{k=1}^{\Gamma(k+1)\le t} 1 = \sum_{k=1}^{[\Gamma^{-1}(t)]-1} 1 = [\Gamma^{-1}(t)] - 1, \tag{3.11}$$

$$\widetilde{A}(t) = \sum_{k:k! \le t} (-1)^{k-1} = \sin^2\Big(\frac{\pi}{2}[\Gamma^{-1}(t)] - \frac{\pi}{2}\Big) = \cos^2\Big(\frac{\pi}{2}[\Gamma^{-1}(t)]\Big) \tag{3.12}$$

applying the integral representations (3.7) and (3.8) with (3.11) and (3.12) respectively to $D_{k!}(y)$ and $\widetilde{D}_{k!}(y)$ we get (3.5) and (3.6). □

Lemma 3.4. *For Dirichlet series $D_{\ln k!}(y)$ and $\widetilde{D}_{\ln k!}(y)$ the following integral representations hold:*

$$D_{\ln k!}(y) = y\int_1^\infty x^{-y-1}([\Gamma^{-1}(t)] - 1)dt \tag{3.13}$$

$$\widetilde{D}_{\ln k!}(y) = y\int_1^\infty x^{-y-1}\cos^2\Big(\frac{\pi}{2}[\Gamma^{-1}(x)]\Big)dx \tag{3.14}$$

where $\Gamma^{-1}(x)$ is the inverse function of Gamma function.

Proof. Since

$$A(t) = \sum_{k:\ln k! \le t} 1 = \sum_{k=1}^{\ln k! \le t} 1 = \sum_{k=1}^{[\Gamma^{-1}(e^t)]-1} 1 = [\Gamma^{-1}(e^t)] - 1 \tag{3.15}$$

$$\widetilde{A}(t) = \sum_{k:\ln k! \le t} (-1)^{k-1} = \sin^2\Big(\frac{\pi}{2}[\Gamma^{-1}(e^t)] - \frac{\pi}{2}\Big) = \cos^2\Big(\frac{\pi}{2}[\Gamma^{-1}(e^t)]\Big) \tag{3.16}$$

applying the integral representations (3.7) and (3.8) respectively to $D_{\ln k!}(y)$ and $\widetilde{D}_{\ln k!}(y)$ after substitution $x = e^t$ we get (3.13) and (3.14). □

Substituting (3.5) and (3.6) in (3.1) and (3.2) respectively, we obtain the following Theorem.

Theorem 3.5. *The following two integral representations are valid*

$$\sum_{k=1}^\infty F(k!) = \int_0^\infty\int_0^\infty yf(y)e^{-yt}([\Gamma^{-1}(t)] - 1)dtdy, \tag{3.17}$$

$$\sum_{k=1}^\infty (-1)^{k-1}F(k!) = \int_0^\infty\int_0^\infty yf(y)e^{-yt}\cos^2\Big(\frac{\pi}{2}[\Gamma^{-1}(t)]\Big)dtdy. \tag{3.18}$$

Comparing (1.6) and (1.7) with integral representations (2.4) and (2.5) and applying the last two equations with

$$f(y) = \frac{2}{\Gamma(\mu)} y^{\gamma(\mu\alpha-\beta)-1} \, {}_1\Psi_1[(\mu,1);(\gamma(\mu\alpha-\beta,\gamma\alpha)); -r^2 y^{\gamma\alpha}] \tag{3.19}$$

and

$$f(y) = \frac{2}{\Gamma\Big(q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big)} y^{q[\mu-\frac{\beta}{\alpha}]-1} \, {}_1F_q\Big[\mu; \Delta\Big(q; q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big); -r^2\Big(\frac{y}{q}\Big)^q\Big] \tag{3.20}$$

we get the following results.

Theorem 3.6. *For the generalized Mathieu series the following integral representations hold:*

$$S_\mu^{(\alpha,\beta)}\Big(r;\{(k!)^\gamma\}_{k=1}^\infty\Big) = \frac{2}{\Gamma(\mu)}\int_0^\infty\!\!\int_0^\infty y^{\gamma(\mu\alpha-\beta)} e^{-yx} \, {}_1\Psi_1[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha)); -r^2 y^{\gamma\alpha}] \times\Big(\Big[\Gamma^{-1}(x)\Big]-1\Big)dxdy, \tag{3.21}$$

$$\widetilde{S}_\mu^{(\alpha,\beta)}\Big(r;\{(k!)^\gamma\}_{k=1}^\infty\Big) = \frac{2}{\Gamma(\mu)}\int_0^\infty\!\!\int_0^\infty y^{\gamma(\mu\alpha-\beta)} e^{-yx} \, {}_1\Psi_1[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha)); -r^2 y^{\gamma\alpha}] \times\cos^2\Big(\frac{\pi}{2}\Big[\Gamma^{-1}(x)\Big]\Big)dxdy,$$

$$(r\in R, \alpha,\beta,\gamma\in R^+, \gamma(\mu\alpha-\beta)>1); \tag{3.22}$$

$$S_\mu^{(\alpha,\beta)}\Big(r;\{(k!)^{q/\alpha}\}_{k=1}^\infty\Big) = \frac{2}{\Gamma\Big(q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big)}\int_0^\infty\!\!\int_0^\infty y^{q[\mu-\frac{\beta}{\alpha}]} e^{-yx} \, {}_1F_q\Big[\mu;\Delta\Big(q;q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big); -r^2\Big(\frac{y}{q}\Big)^q\Big]\Big(\Big[\Gamma^{-1}(x)\Big]-1\Big)dxdy, \tag{3.23}$$

$$\widetilde{S}_\mu^{(\alpha,\beta)}\Big(r;\{(k!)^{q/\alpha}\}_{k=1}^\infty\Big) = \frac{2}{\Gamma\Big(q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big)}\int_0^\infty\!\!\int_0^\infty y^{q[\mu-\frac{\beta}{\alpha}]} e^{-yx} \, {}_1F_q\Big[\mu;\Delta\Big(q;q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big); -r^2\Big(\frac{y}{q}\Big)^q\Big]\cos^2\Big(\frac{\pi}{2}\Big[\Gamma^{-1}(x)\Big]\Big)dxdy,$$

$$(r\in R, \alpha,\beta\in R^+, \mu-\frac{\beta}{\alpha}>q^{-1}; q\in N). \tag{3.24}$$

Now substituting (3.13) and (3.14) in (3.1) and (3.2) respectively, we obtain the following Theorem.

Theorem 3.7. *The following two integral representations are valid*

$$\sum_{k=1}^\infty F(\ln k!) = \int_0^\infty\!\!\int_1^\infty y f(y) x^{-y-1}([\Gamma^{-1}(x)]-1)dxdy, \tag{3.25}$$

$$\sum_{k=1}^\infty (-1)^{k-1} F(\ln k!) = \int_0^\infty\!\!\int_1^\infty y f(y) x^{-y-1}\cos^2\Big(\frac{\pi}{2}[\Gamma^{-1}(x)]\Big)dxdy. \tag{3.26}$$

Comparing (1.6) and (1.7) with integral representations (2.4) and (2.5) and applying (3.25) and (3.26) with (3.19) and (3.20) we get the following results.

Theorem 3.8. *For the generalized Mathieu series the following integral representations hold:*

$$S_\mu^{(\alpha,\beta)}\Big(r;\{(\ln k!)^\gamma\}_{k=2}^\infty\Big) = \frac{2}{\Gamma(\mu)}\int_0^\infty\int_1^\infty y^{\gamma(\mu\alpha-\beta)}\,{}_1\Psi_1[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha));-r^2y^{\gamma\alpha}] \times x^{-y-1}\Big(\Big[\Gamma^{-1}(x)\Big]-1\Big)dxdy, \tag{3.27}$$

$$\widetilde{S}_\mu^{(\alpha,\beta)}\Big(r;\{(\ln k!)^\gamma\}_{k=2}^\infty\Big) = \frac{2}{\Gamma(\mu)}\int_0^\infty\int_1^\infty y^{\gamma(\mu\alpha-\beta)}\,{}_1\Psi_1[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha));-r^2y^{\gamma\alpha}] \times x^{-y-1}\cos^2\Big(\frac{\pi}{2}\Big[\Gamma^{-1}(x)\Big]\Big)dxdy,$$

$$(r\in R,\alpha,\beta,\gamma\in R^+,\gamma(\mu\alpha-\beta)>1); \tag{3.28}$$

$$S_\mu^{(\alpha,\beta)}\Big(r;\{(\ln k!)^{q/\alpha}\}_{k=2}^\infty\Big) = \frac{2}{\Gamma\Big(q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big)}\int_0^\infty\int_1^\infty y^{q[\mu-\frac{\beta}{\alpha}]}\,{}_1F_q\Big[\mu;\Delta\Big(q;q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big); -r^2\Big(\frac{y}{q}\Big)^q\Big]x^{-y-1}\Big(\Big[\Gamma^{-1}(x)\Big]-1\Big)dxdy, \tag{3.29}$$

$$\widetilde{S}_\mu^{(\alpha,\beta)}\Big(r;\{(\ln k!)^{q/\alpha}\}_{k=2}^\infty\Big) = \frac{2}{\Gamma\Big(q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big)}\int_0^\infty\int_1^\infty y^{q[\mu-\frac{\beta}{\alpha}]}\,{}_1F_q\Big[\mu;\Delta\Big(q;q\Big[\mu-\frac{\beta}{\alpha}\Big]\Big); -r^2\Big(\frac{y}{q}\Big)^q\Big]x^{-y-1}\cos^2\Big(\frac{\pi}{2}\Big[\Gamma^{-1}(x)\Big]\Big)dxdy$$

$$(r\in R,\alpha,\beta\in R^+,\mu-\frac{\beta}{\alpha}>q^{-1};q\in N). \tag{3.30}$$

References

[1] P. Cerone and C. T. Lenard, *On integral forms of generalized Mathieu series*, JIPAM **4**(2003), Art.100, 1–11(electronic).

[2] J. Choi and H. M. Srivastava, *Mathieu series and associated sums involving the Zeta function*, Comput. Math. Appl., **59** (2010), 861–867.

[3] P. H. Diananda, *Some inequalities related to an inequality of Mathieu*, Math. Ann., **250** (1980), 95–98.

[4] N. Elezovic, H. M. Srivastava, Ž. Tomovski, *Integral representations and integral transforms of some families of Mathieu type series*, Integral Transforms and Special Functions, **19** (2008), 481–495.

[5] O. Emersleben, *Uber die Reihe* $\sum_{k=1}^\infty k/(k^2+c^2)^2$, Math. Ann., **125** (1952), 165–171.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Tables of Integral Transforms, Vol. I, Mc. Graw-Hill Book Company, New York, Toronto and London, 1954.

[7] E. L. Mathieu, Traite' de Physique Mathematique. VI-VII: Theory de l'Elasticite' des Corps Solides (Part 2), Gauthier-Villars, Paris, 1890.

[8] T. K. Pogany, H. M. Srivastava, Ž. Tomovski, *Some families of Mathieu a-series and alternating Mathieu a-series*, Appl. Math. Computation, **173** (2006), 69–108.

[9] A. P. Prudnikov, Yu. A. Bryckov and O. I. Maricev, Integrals and Series (Elementary Functions), "Nauka", Moscow, 1981 (Russian); English translation: Integrals and Series, Vol.1: Elementary Functions, Gordon and Breach Science Publishers, New York, 1990.

[10] F. Qi, *An integral expression and some inequalities of Mathieu type series*, Rostock. Math. Kolloq., **58** (2004), 37–46.

[11] H. M. Srivastava and H. L. Manocha, A Treatise on generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1984.

[12] H. M. Srivastava, Ž. Tomovski, *Some problems and solutions involving Mathieu's series and its generalizations*, JIPAM, **5**(2004), Article 45, 1-13 (electronic).

[13] H. M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[14] Ž. Tomovski and K. Trencevski, *On an open problem of Bai-Ni Guo and Feng Qi*, J. Inequal. Pure Appl. Math., **4**(2003), Article 29, 1–7 (electronic).

[15] Ž. Tomovski, *New double inequality for Mathieu series*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat., **15** (2004), 79–83.

[16] Ž. Tomovski, *Integral representations of generalized Mathieu series via Mittag-Leffler type functions*, Fract. Calc. & Appl. Anal., **10**(2007), 127–138.

Faculty of Mathematics and Natural Sciences, Institute of Mathematics, PO.Box.162, Gazi Baba b.b. 1000 Skopje, Macedonia.

E-mail: tomovski@pmf.ukim.mk; zivoradt@yahoo.com