TAMKANG JOURNAL OF MATHEMATICS
Volume 42, Number 3, 329-342, Autumn 2011
doi:10.5556/j.tkjm.42.2011.329-342

Available online at http://journals.math.tku.edu.tw/

RECONSTRUCTION OF THE STURM-LIOUVILLE OPERATORS ON A GRAPH WITH δ'_s COUPLINGS

CHUAN-FU YANG

Abstract. Inverse nodal problems consist in constructing operators from the given zeros of their eigenfunctions. In this work, we deal with the inverse nodal problems of reconstructing the Sturm-Liouville operator on a star graph with δ'_s couplings at the central vertex. The uniqueness theorem is proved and a constructive procedure for the solution is provided from a dense subset of zeros of the eigenfunctions for the problem as a data.

1. Introduction

In 1988, the inverse nodal problem was posed and solved for Sturm-Liouville problems by J. R. McLaughlin [30], who showed that knowledge of a dense subset of nodal points of the eigenfunctions alone can determine the potential function of the Sturm-Liouville problem up to a constant. Some numerical schemes were provided by O. H. Hald and J. R. McLaughlin for the reconstruction of the potential [14]. From the physical point of view this corresponds to finding, e.g., the density of a string or a beam from the zero-amplitude positions of their eigenvibrations. Recently, some authors have reconstructed the potential function for generalizations of the Sturm-Liouville problem from the nodal points (for example, refer to [3, 5, 7, 10, 11, 14, 19, 24, 25, 26, 30, 33, 34, 35, 37, 41, 42]).

Quantum graphs became in the last decade a useful and versatile tool to describe several classes of physical systems, in particular, various combinations of quantum wires. There are numerous papers devoted to the subject and we restrict ourselves to mentioning the bibliography given in [17, 20], where also basic concepts of theory are discussed. In [17, 20], all symmetrical vertex matching conditions are described (something more general than standard boundary conditions). For example, so-called Kirchhoff boundary conditions as the most common case of the standard δ couplings; a kind of δ'_s couplings similar to δ couplings, just with roles of the function and its derivative exchanged. For Kirchhoff boundary conditions, its important applications are clear, i.e., in electrical circuits, it expresses Kirchhoff's law; in

2000 *Mathematics Subject Classification*. 34A55, 34B24, 47E05.
Key words and phrases. Sturm-Liouville operator, Star-graph with δ'_s couplings, Inverse nodal problem, Reconstruction formula.

elastic string network, it expresses the balance of tension, and so on. A graph with δ'_s couplings was introduced and investigated by Peter Exner, and Peter Kuchment, and so on. A graph with δ'_s couplings has important applications in lattice Kronig-Penney models, and the δ'_s couplings at a d adge vertex can be approximated by means of $d+1$ couplings of the δ-type [9]. The question of physical meaning of such a coupling on graphs was addressed and a pair of simple nontrivial examples of the so-called δ'_s couplings was presented in [12, 13].

Recently, the spectral problems of quantum graphs have become a rapidly-developing field of mathematics and mathematical physics, and spectral properties of quantum graphs and different inverse problems have been studied in both forward [20, 21, 22, 31, 39] and inverse [4, 23, 32, 38, 40, 41, 42], etc. Nowadays there are only a number of papers devoted to inverse nodal problems for differential operators on graphs (for example, refer to [8, 11, 41, 42]).

In this work we concern ourselves with reconstructing Sturm-Liouville operators on a star graph with δ'_s couplings from nodal data. We prove the corresponding uniqueness theorem and provide a constructive procedure for the solution. For the Sturm-Liouville operators on a graph with δ'_s couplings, the uniqueness theorem and recovery algorithm for the potential obtained in this work are new. We also show connections of these problems with inverse spectral problems of Sturm-Liouville operators on a star graph with δ'_s couplings at the central vertex.

2. Preliminary

In this work, we consider the following boundary value problem for the Sturm-Liouville operator on a star-shaped graph consisting of d segments of equal length:

$$-y''_j(x)+q_j(x)y_j(x)=\lambda y_j(x),\ x\in(0,\pi),\ j=\overline{1,d};d\geq 2, \tag{2.1}$$

which are subject to the boundary conditions

$$y_j(0)=0,\ j=\overline{1,d} \tag{2.2}$$

or

$$y'_j(0)=0,\ j=\overline{1,d}, \tag{2.3}$$

at the pendant vertices 0, and

$$y'_1(\pi)=y'_2(\pi)=\cdots=y'_d(\pi), \tag{2.4}$$

$$\sum_{j=1}^{d} y_j(\pi)=0, \tag{2.5}$$

at the central vertex π. In the equation (2.1), $q_j \in C^1[0,\pi]$, $j = 1,2,\ldots,d$, are real-valued functions. The boundary conditions (2.4) and (2.5) are the so-called a δ'_s couplings.

For convenience, we denote by A_1, A_2 the operator acting in Hilbert space $L^2_d[0,\pi] =: \bigoplus_{i=1}^d L^2[0,\pi]$ for the problem (2.1), (2.2), (2.4) and (2.5) or (2.1), (2.3), (2.4) and (2.5), respectively.

In [39], regularized trace formulae for the operators A_1 and A_2 are calculated with some techniques in classical analysis; next, these trace formulae are used to obtain a result of inverse problem in the spirit of Ambarzumyan; finally we give the asymptotic expressions of eigenvalues for the operators A_1 and A_2, and show that there are d sequences of eigenvalues which one sequence is simple while the others might not be (see Lemmas 2.1 and 2.2 in this paper). Let $\{\lambda^D_{n,j}, j = \overline{1,d}\}_{n=1}^\infty$ be the sequence of the eigenvalues for the operator A_1 and $\{\lambda^N_{n,j}, j = \overline{1,d}\}_{n=0}^\infty$ be the sequence of eigenvalues for the operator A_2, and denote

$$\bar{q}_j = \frac{1}{2\pi}\int_0^\pi q_j(x)dx,\ \ \bar{q} = \frac{2}{d}\sum_{j=1}^d \bar{q}_j. \tag{2.6}$$

Lemma 2.1 (see [39])**.** *For sufficiently large n, the eigenvalues of the operator A_1 possess the following asymptotic expression*

$$\sqrt{\lambda^D_{n,d}} = n + \frac{\bar{q}/2}{n} + o\left(\frac{1}{n}\right), \tag{2.7}$$

and

$$\sqrt{\lambda^D_{n,j}} = n - \frac{1}{2} + \frac{c_{j,0}}{n-\frac{1}{2}} + o\left(\frac{1}{n}\right),\ j = \overline{1,d-1}, \tag{2.8}$$

where $c_{j,0}$, $1 \le j \le d-1$, are the solutions of the equation for c

$$\sum_{j=1}^d \prod_{j\neq l\in\{1,2,\ldots,d\}} (c - \bar{q}_j) = 0. \tag{2.9}$$

Remark 1. *Define $f(x) = \prod_{j=1}^d (x-\bar{q}_j)$, then $f'(x) = \sum_{j=1}^d \prod_{j\neq l=\overline{1,d}}(x-\bar{q}_j)$ is a polynomial with order $d-1$ and its zeros are identified with all solutions to the equation (2.9). By the Rolle Theorem, it follows that all solutions to the equation (2.9) are real.*

Lemma 2.2 (see [39])**.** *For sufficiently large n, the eigenvalues of the operator A_2 possess the following asymptotic expression*

$$\sqrt{\lambda^N_{n,d}} = (n - \frac{1}{2}) + \frac{\bar{q}/2}{n-\frac{1}{2}} + o\left(\frac{1}{n}\right), \tag{2.10}$$

and

$$\sqrt{\lambda^N_{n,j}} = n + \frac{c_{j,0}}{n} + o\left(\frac{1}{n}\right),\ j = \overline{1,d-1}, \tag{2.11}$$

where $c_{j,0}$, $1 \le j \le d-1$, are the solutions of the equation (2.9).

3. Inverse nodal problems

Denote by $\varphi_j(\lambda,x)$, $j=\overline{1,d}$, the solutions of (2.1) satisfying the initial conditions

$$\varphi_j(\lambda,0)=0,\ \varphi_j'(\lambda,0)=1, \tag{3.1}$$

then, we have [28]

$$\varphi_j(\lambda,x)=\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}-\frac{\cos(\sqrt{\lambda}x)}{\lambda}K_j(x,x)+\frac{1}{\lambda}\int_0^x K'_{j,t}(x,t)\cos(\sqrt{\lambda}t)\,dt, \tag{3.2}$$

where both of the first partial derivatives $K'_{j,x}(x,t)$ and $K'_{j,t}(x,t)$ of $K_j(x,t)$, $j=1,2,\dots,d$, exist and $K'_{j,x}(x,\cdot)\in L^2[0,\pi]$ and $K'_{j,t}(x,\cdot)\in L^2[0,\pi]$.

Similarly, denote by $\psi_j(\lambda,x)$, $j=\overline{1,d}$, the solutions of (2.1) satisfying the initial conditions

$$\psi_j(\lambda,0)=1,\ \psi_j'(\lambda,0)=0, \tag{3.3}$$

then, we get [28]

$$\psi_j(\lambda,x)=\cos(\sqrt{\lambda}x)+\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\widetilde{K}_j(x,x)-\frac{1}{\sqrt{\lambda}}\int_0^x \widetilde{K}'_{j,t}(x,t)\sin(\sqrt{\lambda}t)\,dt, \tag{3.4}$$

where $\widetilde{K}_j(x,t)$ has the same properties as $K_j(x,t)$ and

$$K_j(x,x)=\widetilde{K}_j(x,x)=\frac{1}{2}\int_0^x q_j(t)\,dt. \tag{3.5}$$

For definiteness, we take $\lambda_n=\lambda^D_{n,d}$ and study the zeros of eigenfunctions corresponding to the eigenvalue λ_n for the operator A_1 in more details.

Using (3.2), we get the asymptotics for the components $\varphi_j(\lambda_n,x)$ of eigenfunctions, for $n\to\infty$ uniformly in $x\in[0,\pi]$:

$$\begin{aligned}\rho_n\varphi_j(\lambda_n,x)&=\sin(\rho_n x)-\frac{\cos(\rho_n x)K_j(x,x)}{\rho_n}+o\left(\frac{1}{n}\right)\\&=\sin(nx)-\frac{\int_0^x q_j(x)-\bar{q}x}{2n}\cos(nx)+o\left(\frac{1}{n}\right),\ \rho_n:=\sqrt{\lambda_n}.\end{aligned}$$

For a fixed n and j, we estimate the nodal point $x^k_{n,j}$ of the eigenfunction $\varphi_j(\lambda_n,x)$. From

$$0=\sin(nx)-\frac{\int_0^x q_j(x)-\bar{q}x}{2n}\cos(nx)+o\left(\frac{1}{n}\right),$$

we obtain

$$\tan(nx)=\frac{\int_0^x q_j(x)-\bar{q}x}{2n}+o\left(\frac{1}{n}\right). \tag{3.6}$$

Using Taylor's expansion for the arctangent, we obtain the following asymptotic formulae for nodal points, as $n \to \infty$ uniformly in $k \in \mathbf{N}$:

$$nx_{n,j}^k = k\pi + \frac{1}{2n}\left[\int_0^{x_{n,j}^k} q_j(t)\,dt - \bar{q}x_{n,j}^k\right] + o\left(\frac{1}{n}\right),$$

which implies

$$\begin{aligned} x_{n,j}^k &= \frac{k\pi}{n} + \frac{1}{2n^2}\left[\int_0^{x_{n,j}^k} q_j(t)\,dt - \bar{q}x_{n,j}^k\right] + o\left(\frac{1}{n^2}\right) \\ &= \frac{k\pi}{n} + \frac{1}{2n^2}\left[\int_0^{\frac{k\pi}{n}} q_j(t)\,dt - \bar{q}\frac{k\pi}{n}\right] + o\left(\frac{1}{n^2}\right). \end{aligned} \tag{3.7}$$

The equality (3.7) gives

$$x_{n,j}^{k+1} - x_{n,j}^k := l_{n,j}^k = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right),\ n \to \infty,$$

uniformly with respect to k. For $k = 0, 1, \ldots, n$, the formula (3.7) gives

$$\begin{aligned} x_{n,j}^0 &= O\left(\frac{1}{n^2}\right), \quad x_{n,j}^1 = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right), \ldots, \\ x_{n,j}^{n-1} &= \frac{(n-1)\pi}{n} + O\left(\frac{1}{n^2}\right),\ x_{n,j}^n = \pi + O\left(\frac{1}{n^2}\right). \end{aligned}$$

For a fixed $j \in \{1, 2, \ldots, d\}$, there exists N_0 such that for all $n > N_0$ the function $\varphi_j(\lambda_n, x)$ has exactly $n-1$ simple zeros inside the interval $(0, \pi)$, namely: $0 < x_{n,j}^1 < \cdots < x_{n,j}^{n-1} < \pi$. The points $X_j^1 := \{x_{n,j}^k\}$ are called nodal points on the edge e_j related to the eigenvalues $\{\lambda_n\}$.

Thus, according to the order of $x_{n,j}^k$, for large n, the components $\varphi_j(\lambda_n, x)$ of eigenfunctions has exactly $n-1$ nodes in the interval $(0, \pi)$, i.e., $x_{n,j}^k$, $k = \overline{1, n-1}$.

In the above results, the order estimate is independent of k. As a result,

$$l_{n,j}^k = \frac{\pi}{n} + o\left(\frac{1}{n}\right). \tag{3.8}$$

Corollary 1. *The sets $X_j^1 = \{x_{n,j}^k\}$ is dense in $[0, \pi]$, where $x_{n,j}^k$ is defined by (3.7).*

We consider the following inverse problem.

Problem. Given nodal points set X_j^1 or its subset $X_j^{1,0}$ which is dense in $(0, \pi)$, how to find the potential $q_j(x)$ on the edge e_j.

Using (3.7) we arrive at the following assertions.

Theorem 1. *Fix $j \in \{1,2,\dots,d\}$ and $x \in [0,\pi]$. Let $\{x_{n,j}^k\} \subset X_j^1$ be dense in $(0,\pi)$ so that there exists $k = k(n)$ such that $\lim_{n\to\infty} x_{n,j}^{k(n)} = x$. Then the following finite limit exists and the corresponding equality holds.*

$$\lim_{n\to\infty} 2n\left[nx_{n,j}^k - k\pi\right] := g_j(x), \tag{3.9}$$

and

$$g_j(x) = \int_0^x q_j(t)dt - \bar{q}x. \tag{3.10}$$

Remark 2. *In the proof of Theorem 1, we use the nodal data corresponding to the eigenvalue of the form as in (2.7) to derive the reconstruction formula (3.9). If the eigenvalues of the form as in (2.8) are chosen, a similar formula still holds. We take $\sqrt{\lambda_{n,j_0}^D} = n - \frac{1}{2} + \frac{c_{j_0,0}}{n-\frac{1}{2}} + o\left(\frac{1}{n}\right)$, $j_0 \in \{1,2,\dots,d-1\}$ and obtain the zeros of eigenfunctions corresponding to the eigenvalue λ_{n,j_0}^D for the operator A_1:*

$$x_{n,j}^k = \frac{k\pi}{n-\frac{1}{2}} + \frac{1}{2\left(n-\frac{1}{2}\right)^2}\left[\int_0^{\frac{k\pi}{n}} q_j(t)dt - 2c_{j_0,0}\frac{k\pi}{n-\frac{1}{2}}\right] + o\left(\frac{1}{n^2}\right).$$

Thus, the formula (3.9) has the following form

$$\lim_{n\to\infty} 2\left(n-\frac{1}{2}\right)\left[\left(n-\frac{1}{2}\right)x_{n,j}^k - k\pi\right] := h_j(x)$$

and

$$h_j(x) = \int_0^x q_j(t)dt - 2c_{j_0,0}x.$$

And the reconstruction formula (3.11) has the following form

$$q_j(x) - \frac{1}{\pi}\int_0^x q_j(t)dt = h_j'(x) - \frac{h_j(\pi)}{\pi}.$$

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

Theorem 2. *Fix $j \in \{1,2,\dots,d\}$ and $x \in [0,\pi]$. Let $X_j^{1,0} \subset X_j^1$ be a subset of nodal points which is dense in $(0,\pi)$. Let $\{x_{n,j}^k\} \subset X_j^{1,0}$ be dense in $(0,\pi)$ so that there exists $k = k(n)$ such that $\lim_{n\to\infty} x_{n,j}^{k(n)} = x$. Then, the specification of $X_j^{1,0}$ uniquely determines the potential $q_j(x) - \bar{q}$ in $(0,\pi)$. The potential $q_j(x) - \bar{q}$ can be constructed via the following algorithm:*

(1) *for each $x \in [0,\pi]$ choose a sequence $\{x_{n,j}^k\} \subset X_j^{1,0}$ such that $x_{n,j}^k \to x$ as $n \to \infty$;*

(2) *find the function $g_j(x)$ via (3.9) and from (3.10) calculate*

$$q_j(x) - \bar{q} = g_j'(x). \tag{3.11}$$

Similarly, we take $\lambda_n = \lambda^N_{n,d}$ and study the operator A_2.

Using (3.4), we get the asymptotics for the components $\psi_j(\lambda_n, x)$ of eigenfunctions, for $n \to \infty$ uniformly in $x \in [0,\pi]$:

$$\begin{aligned}\psi_j(\lambda_n, x) &= \cos(\rho_n x) + \frac{\sin(\rho_n x)\widetilde{K}_j(x,x)}{\rho_n} + o\left(\frac{1}{n}\right)\\ &= \cos(n-\tfrac{1}{2})x + \frac{\int_0^x q_j(x) - \bar{q}x}{2n}\sin(n-\tfrac{1}{2})x + o\left(\tfrac{1}{n}\right),\ \rho_n := \sqrt{\lambda_n}.\end{aligned}$$

For a fixed n and j, we estimate the nodal point $x^k_{n,j}$ of the eigenfunction $\psi_j(\lambda_n, x)$. From

$$0 = \cos(n-\frac{1}{2})x + \frac{\int_0^x q_j(x) - \bar{q}x}{2n}\sin(n-\frac{1}{2})x + o\left(\frac{1}{n}\right),$$

we obtain

$$\cot(n-\frac{1}{2})x = \frac{\bar{q}x - \int_0^x q_j(x)}{2n} + o\left(\frac{1}{n}\right). \tag{3.12}$$

Using Taylor's expansions, we obtain the following asymptotic formulae for nodal points, as $n \to \infty$ uniformly in $k \in \mathbf{N}$:

$$\begin{aligned}x^k_{n,j} &= \frac{(k+\frac{1}{2})\pi}{n-\frac{1}{2}} + \frac{1}{2n^2}\left[\int_0^{x^k_{n,j}} q_j(t)dt - \bar{q}x^k_{n,j}\right] + o\left(\frac{1}{n^2}\right)\\ &= \frac{(k+\frac{1}{2})\pi}{n-\frac{1}{2}} + \frac{1}{2n^2}\left[\int_0^{\frac{(k+\frac{1}{2})\pi}{n-\frac{1}{2}}} q_j(t)dt - \bar{q}\frac{(k+\frac{1}{2})\pi}{n-\frac{1}{2}}\right] + o\left(\frac{1}{n^2}\right).\end{aligned} \tag{3.13}$$

The equality (3.13) gives

$$x^{k+1}_{n,j} - x^k_{n,j} := l^k_{n,j} = \frac{\pi}{n} + o(1),\ n \to \infty,$$

uniformly with respect to k. For $k = 0,1,\ldots,n$, the formula (3.7) gives

$$\begin{aligned}x^0_{n,j} &= \frac{\frac{1}{2}\pi}{n-\frac{1}{2}} + O\left(\frac{1}{n^2}\right),\ x^1_{n,j} = \frac{\frac{3}{2}\pi}{n-\frac{1}{2}} + O\left(\frac{1}{n^2}\right),\ \ldots,\\ x^{n-1}_{n,j} &= \frac{(n-\frac{1}{2})\pi}{n} + O\left(\frac{1}{n^2}\right),\ x^n_{n,j} = \frac{(n+\frac{1}{2})\pi}{n} + O\left(\frac{1}{n^2}\right).\end{aligned}$$

For a fixed $j \in \{1,2,\ldots,d\}$. The points $X^2_j := \{x^k_{n,j}\}$ are called nodal points on the edge e_j related to the eigenvalues $\{\lambda_n\}$. Thus, according to the order of $x^k_{n,j}$, for large n, the components $\varphi_j(\lambda_n, x)$ of eigenfunctions has exactly n nodes in the interval $(0,\pi)$, i.e., $x^k_{n,j}$, $k = \overline{1,n}$.

Corollary 2. *The sets $X^2_j = \{x^k_{n,j}\}$ is dense in $[0,\pi]$, where $x^k_{n,j}$ is defined by (3.13).*

Using (3.13) we arrive at the following assertions.

Theorem 3. *Fix $j \in \{1,2,\dots,d\}$ and $x \in [0,\pi]$. Let $\{x_{n,j}^k\} \subset X_j^2$ be dense in $(0,\pi)$ so that there exists $k = k(n)$ such that $\lim_{n\to\infty} x_{n,j}^{k(n)} = x$. Then the following finite limit exists and the corresponding equality holds.*

$$\lim_{n\to\infty} 2n\left[(n-\frac{1}{2})x_{n,j}^k - (k+\frac{1}{2})\pi\right] := f_j(x), \tag{3.14}$$

and

$$f_j(x) = \int_0^x q_j(t)\,dt - \bar{q}x. \tag{3.15}$$

Theorem 4. *Fix $j \in \{1,2,\dots,d\}$ and $x \in [0,\pi]$. Let $X_j^{2,0} \subset X_j^2$ be a subset of nodal points which is dense in $(0,\pi)$. Let $\{x_{n,j}^k\} \subset X_j^{2,0}$ be dense in $(0,\pi)$ so that there exists $k = k(n)$ such that $\lim_{n\to\infty} x_{n,j}^{k(n)} = x$. Then, the specification of $X_j^{2,0}$ uniquely determines the potential $q_j(x) - \bar{q}$ in $(0,\pi)$. The potential $q_j(x) - \bar{q}$ can be constructed via the following algorithm:*

(1) *for each $x \in [0,\pi]$ choose a sequence $\{x_{n,j}^k\} \subset X_j^{2,0}$ such that $x_{n,j}^k \to x$ as $n \to \infty$;*

(2) *find the function $f_j(x)$ via (3.14) and from (3.15) calculate*

$$q_j(x) - \bar{q} = f_j'(x). \tag{3.16}$$

4. Incomplete inverse problems

Together with A_i we consider a boundary value problem $\widetilde{A}_i$ of the same form (2.1)–(2.5) with the potential functions $\widetilde{q}_1(x),\dots,\widetilde{q}_d(x)$. We agree that if a certain symbol δ denotes an object related to A_i, then $\widetilde{\delta}$ will denote an analogous object related to $\widetilde{A}_i$.

Now we give the following incomplete inverse spectral problem. Suppose that $q_k(x)$ are known a priori for $k \in \{1,2,\dots,d\} \setminus \{j\}$, $x \in (0,\pi)$. Moreover, suppose that $q_j(x)$ is known on a part of the interval, namely, for $x \in (b,\pi)$. The inverse problem is to construct $q_j(x)$ for $x \in (0,b)$ from a part of the spectrum for the operator A_i. Denote by σ_k^i $(i = 1,2)$ the spectrum of the boundary value problem

$$-y_k''(x) + q_k(x)y_k(x) = \lambda y_k(x),\ y_k^{(i-1)}(0) = 0 = y_k'(\pi).$$

Theorem 5. *Fix $j \in \{1,2,\dots,d\}$ and $b \in (0,\frac{\pi}{2})$. Let $q_k(x)$ are known a priori for $k \in \{1,2,\dots,d\} \setminus \{j\}$, $x \in (0,\pi)$ and $q_j(x) = \widetilde{q}_j(x)$ on (b,π). Let $\Lambda \subset \mathbf{N}$ be a subset of positive integer numbers, and suppose $\Omega := \{\lambda_n\}_{n\in\Lambda}$ is a part of the spectrum for the operator A_i such that $\sigma_k^i \cap \Omega = \emptyset$, $k \in \{1,2,\dots,d\} \setminus \{j\}$ and the system of the functions $\{\cos 2\sqrt{\lambda_n}x\}_{n\in\Lambda}$ is complete in $L^2(0,b)$. If $\Omega = \widetilde{\Omega}$, then $q_j(x) = \widetilde{q}_j(x)$ on $(0,\pi)$.*

Proof. Here we only need prove that Theorem 5 is true for the operator A_1. Since

$$
\begin{aligned}
-\varphi_j''(\lambda,x)+q_j(x)\varphi_j(\lambda,x) &= \lambda\varphi_j(\lambda,x),\\
-\widetilde{\varphi}_j''(\lambda,x)+\widetilde{q}_j(x)\widetilde{\varphi}_j(\lambda,x) &= \lambda\widetilde{\varphi}_j(\lambda,x),\\
\varphi_j(\lambda,0)=\widetilde{\varphi}_j(\lambda,0)=0,\ \varphi_j'(\lambda,0) &= \widetilde{\varphi}_j'(\lambda,0)=1,
\end{aligned}
$$

from the boundary conditions (2.2) it follows that

$$\int_0^\pi Q_j(x)\varphi_j(\lambda,x)\widetilde{\varphi}_j(\lambda,x)dx=\varphi_j'(\lambda,\pi)\widetilde{\varphi}_j(\lambda,\pi)-\varphi_j(\lambda,\pi)\widetilde{\varphi}_j'(\lambda,\pi), \tag{4.1}$$

where $Q_j(x)=q_j(x)-\widetilde{q}_j(x)$. Moreover,

$$\varphi_k(\lambda,x)=\widetilde{\varphi}_k(\lambda,x),\ \varphi_k'(\lambda,x)=\widetilde{\varphi}_k'(\lambda,x), k\in\{1,2,\dots,d\}\setminus\{j\}, \tag{4.2}$$

since for $k\in\{1,2,\dots,d\}\setminus\{j\}$, $q_k(x)=\widetilde{q}_k(x)$ on $(0,\pi)$. Clearly, $\varphi_k'(\lambda_n,\pi)=\widetilde{\varphi}_k'(\lambda_n,\pi)\neq 0$ for $k\in\{1,2,\dots,d\}\setminus\{j\}$ since $\sigma_k^1\cap\Omega=\emptyset$. Using the boundary conditions (2.4) and (2.5) we obtain

$$\sum_{k=1}^d\frac{\varphi_k(\lambda_n,\pi)}{\varphi_k'(\lambda_n,\pi)}=0,\quad \sum_{k=1}^d\frac{\widetilde{\varphi}_k(\lambda_n,\pi)}{\widetilde{\varphi}_k'(\lambda_n,\pi)}=0. \tag{4.3}$$

From (4.2) and (4.3) it follows that(in details, refer (3.12)–(3.15) in [38])

$$\frac{\varphi_j(\lambda_n,\pi)}{\varphi_j'(\lambda_n,\pi)}=\frac{\widetilde{\varphi}_j(\lambda_n,\pi)}{\widetilde{\varphi}_j'(\lambda_n,\pi)},\ \lambda_n\in\Omega. \tag{4.4}$$

Using (4.1) and $q_j(x)=\widetilde{q}_j(x)$ on (b,π) we obtain

$$\int_0^b Q_j(x)\varphi_j(\lambda_n,x)\widetilde{\varphi}_j(\lambda_n,x)dx=0,\quad \lambda_n\in\Omega. \tag{4.5}$$

Since $\varphi_j(\lambda,x)$ is the solution of the equation (2.1) satisfying the initial conditions $\varphi_j(\lambda,0)=0$ and $\varphi_j'(\lambda,0)=1$, there exists a bounded function $K_j(x,t)$ (independent of λ) such that [15]

$$\rho^2\varphi_j(\lambda,x)\widetilde{\varphi}_j(\lambda,x)=\frac{1}{2}-\frac{1}{2}\cos(2\rho x)-\frac{1}{2}\int_0^x K_j(x,t)\cos(2\rho t)dt, \tag{4.6}$$

where $\rho^2=\lambda$. Substituting (4.6) into (4.5) and using the Riemann-Lebesgue Lemma, we obtain

$$\int_0^b Q_j(x)dx=0$$

and

$$\int_0^b [Q_j(x)+\int_x^b K_j(x,t)Q_j(t)dt]\cos 2\sqrt{\lambda_n}x dx=0,\quad \lambda_n\in\Omega, \tag{4.7}$$

and consequently,

$$Q_j(x) + \int_x^b K_j(x,t)Q_j(t)\,dt = 0 \text{ a.e. on } (0,b). \tag{4.8}$$

But this homogeneous Volterra integral equation has only the trivial solution it follows that $Q_j(x) = 0$ a.e. on $(0,b)$, i.e., $q_j(x) = \widetilde{q}_j(x)$ on $x \in [0,b]$. The proof is finished. $\square$

For $X_j^{i,0} \subset X_j^i$, the set $X_j^{i,0}$ is called **twin** if together with each of its point $x_{n,j}^k$, the set $X_j^{i,0}$ contains at least one of adjacent nodal x_{nj}^{k-1} or/and $x_{n,j}^{k+1}$. For $X_j^{i,0} \subset X_j^i$ we denote $\Lambda_{X_j^{i,0}} := \{m(n) : \exists k, x_{m(n),j}^k \in X_j^{i,0}\}$, where $m(n)$ is a sequence of natural numbers.

Theorem 6. *Fix $j \in \{1,2,\ldots,d\}$ and $b \in (0,\frac{\pi}{2})$. Let $q_k(x) = \widetilde{q}_k(x)$ on $x \in (0,\pi)$ for $k \in \{1,2,\ldots,d\} \setminus \{j\}$. Let $X_j^{i,0} \subset X_j^i \bigcap (b,\pi)$ be a dense on (b,π) twin subset of nodal points such that*

$$\sigma_k^i \cap \{\lambda_{m(n)}\}_{m(n)\in\Lambda_{X_j^{i,0}}} = \emptyset,\ k \in \{1,2,\ldots,d\} \setminus \{j\}$$

and the system of the functions $\{\cos(2\sqrt{\lambda_{m(n)}}x)\}_{m(n)\in\Lambda_{X_j^{i,0}}}$ is complete in $L^2(0,b)$. If $X_j^{i,0} = \widetilde{X}_j^{i,0}$ and $\overline{q} = \overline{\widetilde{q}}$, then $q_j(x) = \widetilde{q}_j(x)$ on $(0,\pi)$.

Proof. To prove this theorem we need a result [10, 37, 41]: Fix n, j, k, let $x_{n,j}^k = \widetilde{x}_{nj}^k$, $x_{n,j}^{k+1} = \widetilde{x}_{nj}^{k+1}$, and $q_j(x) = \widetilde{q}_j(x)$ on $(x_{n,j}^k, x_{n,j}^{k+1})$. Then $\lambda_n = \widetilde{\lambda}_n$.

Since $X_j^{i,0} = \widetilde{X}_j^{i,0}$, it follows from Theorems 2 and 4 that $g_j(x) = \widetilde{g}_j(x)$ or $f_j(x) = \widetilde{f}_j(x)$ for $x \in (b,\pi)$. Using (3.11) and (3.16) we obtain $q_j(x) = \widetilde{q}_j(x)$ on (b,π). Thus, we have $\lambda_{m(n)} = \widetilde{\lambda}_{m(n)}$ for $m(n) \in \Lambda_{X_j^{i,0}}$. Applying Theorem 5 we conclude that $q_j(x) = \widetilde{q}_j(x)$ on $(0,\pi)$. $\square$

Theorem 7. *Fix $j \in \{1,2,\ldots,d\}$ and $b \in (0,\frac{\pi}{2})$. Let $q_k(x) = \widetilde{q}_k(x)$ on $x \in (0,\pi)$ for $k \in \{1,2,\ldots,d\} \setminus \{j\}$. Let $X_j^{i,0} \subset X_j^i \bigcap (b,\pi)$ be a dense on (b,π) twin subset of nodal points such that*

$$\sigma_k^i \cap \{\lambda_{m(n)}\}_{m(n)\in\Lambda_{X_j^{i,0}}} = \emptyset,\ k \in \{1,2,\ldots,d\} \setminus \{j\}$$

and $m(n)$ be a sequence of natural numbers such that

$$m(n) = \frac{n}{\sigma}(1+\epsilon_n),\ 0 < \sigma \le 1,\ \epsilon_n \to 0. \tag{4.9}$$

If $X_j^{i,0} = \widetilde{X}_j^{i,0}$, $\overline{q} = \overline{\widetilde{q}}$ and $\sigma > \frac{2b}{\pi}$, then $q_j(x) = \widetilde{q}_j(x)$ on $(0,\pi)$.

Proof. Here we only need prove that Theorem 5 is true for the operator A_1. First, by the assumption $X_j^{i,0} = \widetilde{X}_j^{i,0}$ and Theorem 2 we obtain

$$q_j(x) = \widetilde{q}_j(x) \text{ on } (b,\pi).$$

Together with a result [10, 37, 41](see the proof of Theorem 6), we have $\lambda_{m(n)} = \widetilde{\lambda}_{m(n)}$ for $m(n) \in \Lambda_{X_j^{i,0}}$.

Since

$$\begin{aligned} -\varphi_j''(\lambda, x) + q_j(x)\varphi_j(\lambda, x) &= \lambda\varphi_j(\lambda, x), \\ -\widetilde{\varphi}_j''(\lambda, x) + \widetilde{q}_j(x)\widetilde{\varphi}_j(\lambda, x) &= \lambda\widetilde{\varphi}_j(\lambda, x), \\ \varphi_j(\lambda, 0) = \widetilde{\varphi}_j(\lambda, 0) = 0,\ \varphi_j'(\lambda, 0) &= \widetilde{\varphi}_j'(\lambda, 0) = 1, \end{aligned}$$

it follows that

$$\begin{aligned} G(\rho) &:= \int_0^b Q_j(x) 2\varphi_j(\lambda, x)\widetilde{\varphi}_j(\lambda, x)\,dx \\ &= \varphi_j'(\lambda, \pi)\widetilde{\varphi}_j(\lambda, \pi) - \varphi_j(\lambda, \pi)\widetilde{\varphi}_j'(\lambda, \pi),\ \rho = \sqrt{\lambda}, \end{aligned} \tag{4.10}$$

where $Q_j(x) = q_j(x) - \widetilde{q}_j(x)$. Using the method in the proof of Theorem 5, from (4.10) we obtain

$$G(s_{m(n)}) = 0,\ s_{m(n)} = \sqrt{\lambda_{m(n)}}. \tag{4.11}$$

Next, we will show that $G(\rho) = 0$ on the whole ρ-plane.

From (4.6) we see that the entire function $G(\rho)$ is a function of exponential type $\le 2b$. One has

$$|G(\rho)| \le Ce^{2br|\sin\theta|} \tag{4.12}$$

for some positive constant C, $\rho = \sqrt{\lambda} = re^{i\theta}$.

Define an indicator of the function $G(\rho)$ by

$$h(\theta) = \limsup_{r\to\infty} \frac{\ln|G(re^{i\theta})|}{r}. \tag{4.13}$$

Since $|\mathrm{Im}\sqrt{\lambda}| = r|\sin\theta|$, $\theta = \arg\sqrt{\lambda}$, from (4.12) and (4.13) one obtains the following estimate

$$h(\theta) \le 2b|\sin\theta|. \tag{4.14}$$

It is known [27] that for any entire function $G(\rho)$ of exponential type, not identically zero, one has

$$\liminf_{r\to\infty} \frac{n(r)}{r} \le \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta, \tag{4.15}$$

where $n(r)$ is the number of zeros of $G(\rho)$ in the disk $|\rho| \le r$. By (4.14),

$$\frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta \le \frac{b}{\pi}\int_0^{2\pi} |\sin\theta|\,d\theta = \frac{4b}{\pi}. \tag{4.16}$$

From the known asymptotic expression (2.7) of the eigenvalues λ_n, for the number of zeros of $G(\rho)$ in the disk $|\rho| \le r$ we have the estimate

$$n(r) \ge 2 \sum_{\frac{n}{\sigma}(1+o(1))<r} 1 = 2\sigma r[1+o(1)],\ r \to \infty. \tag{4.17}$$

Since $\sigma > \dfrac{2b}{\pi}$, we get

$$\lim_{n\to\infty} \frac{n(r)}{r} \ge 2\sigma > \frac{4b}{\pi} \ge \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{4.18}$$

Thus, inequalities (4.15) and (4.18) imply that $G(\rho) \equiv 0$ on the whole ρ-plane.

Repeating the proof of Theorem 5, we have

$$Q_j(x) = 0,\ x \in [0,b],$$

i.e.,

$$q_j(x) = \tilde{q}_j(x) \text{ on } x \in [0,b].$$

The proof is completed. □

Acknowledgements

The author would like to thank the referees for their careful reading and valuable comments. This work was supported by Natural Science Foundation of Jiangsu Province of China(BK 2010489) and the Outstanding Plan-Zijin Star Foundation of Nanjing University of Science and Technology(AB 41366), and NUST Research Funding(No. AE 88787). This work was supported by the National Natural Science Foundation of China (11100000/A010602).

References

[1] G. Borg, *Uniqueness theorems in the spectral theory of* $y'' + (\lambda - q(x))y = 0$, Proc. 11th Scandinavian Congress of Mathematicians (Oslo: Johan Grundt Tanums Forlag) (1952), 276–287.

[2] J. Borman and P. Kurasov, *Symmetries of quantum graphs and the inverse scattering problem*, Adv. in Appl. Math., **35** (2005), 58–70.

[3] P. J. Browne and B. D. Sleeman, *Inverse nodal problem for Sturm-Liouville equation with eigenparameter dependent boundary conditions*, Inverse Problems, **12** (1996), 377–381.

[4] B. M. Brown and R. Weikard, *A Borg-Levinson theorem for trees*, Proc. Royal Soc. London Ser. A, **461** (2005), 3231–3243.

[5] S. A. Buterin and C. T. Shieh, *Inverse nodal problem for differential pencils*, Applied Mathematics Letters, **22**(2009), 1240–1247.

[6] R. Carlson, *Large eigenvalues and trace formulas for matrix Sturm-Liouville problems*, SIAM J. Math. Anal., **30** (1999), 949–962.

[7] Y. T. Chen, Y. H. Cheng, C. K. Law and J. Tsay, L^1 *convergence of the reconstruction formula for the potential function*, Proc. Amer. Math. Soc., **130** (2002), 2319–2324.

[8] Cheng Y. H. , Reconstruction of the Sturm-Liouville operator on a p-star graph with nodal data, Rocky Mountain Journal of Mathematics, to appear.

[9] T. Cheon and P. Exner, *An approximation to δ' couplings on graphs*, J. Phys. A: Math. Gen., **37**(2004), L329–L335.

[10] Y. H. Cheng, C. K. Law and J. Tsay, *Remarks on a new inverse nodal problem*, J. Math. Anal. Appl., **248**(2000), 145–155.

[11] S. Currie and B. A. Watson, *Inverse nodal problems for Sturm-Liouville equations on graphs*, Inverse Problems, **23** (2007), 2029–2040.

[12] P. Exner, *Lattice Kronig-Penney models*, Phys. Rev. Lett., **74** (1995), 3503–3506.

[13] P. Exner, *Contact interactions on graph superlattices*, J. Phys. A: Math. Gen., **29**(1996), 87–102.

[14] O. H. Hald and J. R. McLaughlin, *Solutions of inverse nodal problems*, Inverse Problems, **5** (1989), 307–347.

[15] H. Hochstadt and B. Lieberman, *An inverse Sturm-Liouville problem with mixed given data*, SIAM J. Appl. Math., **34** (1978), 676–680.

[16] M. Horváth, *Inverse spectral problems and closed exponential systems*, Ann. Math., **162** (2005), 885–918.

[17] V. Kostrykin and R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A: Math. Gen., **32** (1999), 595–630.

[18] V. Kostrykin and R. Schrader, *Quantum wires with magnetic fluxes*, Commun. Math. Phys., **237** (2003), 161–179.

[19] H. Koyunbakan, *A new inverse problem for the diffusion operator*, Applied Mathematics Letters, **19** (2006), 995–999.

[20] P. Kuchment, *Quantum graphs I. Some basic structures*, Waves Random Media, **14** (2004), 107–128.

[21] P. Kuchment, *Quantum graphs II. Some spectral properties of quantum and combinatorial graphs*, J. Phys. A: Math. Gen., **38** (2005), 4887–4900.

[22] P. Kurasov, *Schrödinger operators on graphs and geometry I. Essentially bounded potentials*, Report N9, Dept. of Math., Lund Univ. (2007).

[23] P. Kurasov and M. Nowaszyk, *Inverse spectral problem for quantum graphs*, J. Phys. A: Math. Gen., **3** (2005), 4901–4915. Corrigendum: J. Phys. A: Math. Gen., *39* (2006), 993.

[24] Law C. K., Shen C. L. and Yang C. F.: The inverse nodal problem on the smoothness of the potential function, Inverse Problems 15, 253–263 (1999); Errata, 17, 361–364 (2001).

[25] C. K. Law and J. Tsay, *On the well-posedness of the inverse nodal problem*, Inverse Problems, **17** (2001), 1493–1512.

[26] C. K. Law and C. F. Yang, *Reconstructing the potential function and its derivatives using nodal data*, Inverse Problems, **14**(1998), 299–312.

[27] B. Ja. Levin, *Distribution of Zeros of entire functions*, AMS Transl. Vol.5, Providence, 1964.

[28] B. M. Levitan and M. G. Gasymov, *Determination of a differential equation by two of its spectra*, Usp. Mat. Nauk, **19** (1964), 3–63.

[29] V. A. Marchenko, Sturm-Liouville Operators and Applications (Operator Theory: Advances and Applications, 22), Birkhäuser, Basel, 1986.

[30] J. R. McLaughlin, *Inverse spectral theory using nodal points as data–a uniqueness result*, J. Differential Equations, **73** (1988), 354–362.

[31] K. Naimark and M. Solomyak, *Eigenvalue estimates for the weighted Laplacian on metric trees*, Proc. London Math. Soc., **80** (2000), 690–724.

[32] V. N. Pivovarchik, *Inverse problem for the Sturm-Liouville equation on a simple graph*, SIAM J. Math. Anal., **32** (2000), 801–819.

[33] C. L. Shen, *On the nodal sets of the eigenfunctions of the string equations*, SIAM J. Math. Anal., **19** (1988), 1419–1424.

[34] C. L. Shen and C. T. Shieh, *An inverse nodal problem for vectorial Sturm-Liouville equation*, Inverse Problems, **16** (2000), 349–356.

[35] C. T. Shieh and V. A. Yurko, *Inverse nodal and inverse spectral problems for discontinuous boundary value problems*, J. Math. Anal. Appl., **347** (2008), 266–272.

[36] B. Simon, *A new approach to inverse spectral theory I. Fundamental formalism*, Ann. Math., **150** (1999), 1029–1057.

[37] X. F. Yang, *A solution of the inverse nodal problem*, Inverse Problems, **13** (1997), 203–213.

[38] C. F. Yang, *Inverse spectral problems for the Sturm-Liouville operator on a d-star graph*, J. Math. Anal. Appl., **365** (2010), 742–749.

[39] C. F. Yang, Z. Y. Huang and X. P. Yang, *Trace formulae for Schrödinger systems on graphs*, Turk. J. Math., **34** (2010), 181–196.

[40] V. A. Yurko, *Inverse spectral problems for Sturm-Liouville operators on graphs*, Inverse Problems, **21** (2005), 1075–1086.

[41] V. A. Yurko, *Inverse nodal problems for Sturm-Liouville operators on star-type graphs*, J. Inv. Ill-Posed Problems, **16** (2008), 715–722.

[42] V. A. Yurko, *Inverse nodal problems for Sturm-Liouville operators on a star-type graph*, Siberian Mathematical Journal, **50** (2009), 373–378.

Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, People's Republic of China.

E-mail: chuanfuyang@tom.com