Fixed Point and Coincidence Point Theorems

Arjamand Bano[1] and Saima Naheed [2]

Abstract

In this paper, we present a generalization of some fixed point and coincidence point theorems using the notion of a $w-$distance on a complete metric space. Consequently, we improve and generalize various results existing in the literature.

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1. Introduction and Preliminaries

In 1969, Nadler [5] extended the well known Banach contraction principle [1], which asserts that "each single-valued contraction self map on a complete metric space has a unique fixed point.", to multi-valued mapping using the concept of Hausdorff metric. In [3], Kada et al. introduced a notion of w-distance on a metric space and improved several results replacing the involved metric by a generalized distance. In this paper, using the concept of w-distance, we give a generalization of Nadler's fixed point theorem and coincidence point results in complete metric space.

Let (E,d) be a complete metric space and let $CB(E)$ be the collection of all nonempty closed bounded subsets of E. For $A,B \in CB(E)$, and $p \in E$, define

$D(p,A) = \inf\{d(p,a) : a \in A\}$ and

$$H(A,B)=\max\{\sup_{a\in A} D(a,B),\sup_{b\in B} D(b,A)\}$$

where H is Hausdorff metric induced by d.

One can easly prove that $(CB(E),H)$ is a complete metric space, whenever (E,d) is a complete metric space (see e.g. Lemma 8.1.4, of [8]).

Definition 1. An element $p\in E$ is said to be a fixed point of a multi-valued mapping $T:E\to CB(E)$, if $p\in T(p)$.

Definition 2. Let $C(f,T)$ be the set of coincidence points of f and T

i.e. $C(f,T)=\{p\in E: fp\in Tp\}$.

Definition 3. A map $\varphi:E\to R$ is called lower semi-continuous if for any sequence $\{p_n\}\subset E$ with $p_n\to p\in E$ implies $\varphi(p)\leq \liminf_{n\to\infty}\varphi(p_n)$.

Definition 4. A function $w:E\times E\to[0,\infty)$ is called $w-$distance on E if it satisfies the following for any $p_1,p_2,p_3\in E$:

(w_1) $w(p_1,p_3)\leq w(p_1,p_2)+w(p_2,p_3)$;

(w_2) a map $w(p,.):E\to[0,\infty)$ is lower semi-continuous;

(w_3) for any $\varepsilon>0,$ there exists $\delta>0$ such that $w(p_3,p_1)\leq\delta$ and $w(p_3,p_2)\leq\delta$ imply $d(p_1,p_2)\leq\varepsilon.$

The metric d is a w-distance on E. Note that, in general for $p_1,p_2\in E, w(p_1,p_2)\neq w(p_2,p_1)$. Define $w(p,A)=\inf\{w(p,a):a\in A\},$ and

$$H_w(A,B)=\max\{\sup_{a\in A} w(a,B),\sup_{b\in B} w(b,A)\}.$$

It is easy to see that H_w is a metric on $CB(E)$. H_w is called the Hausdorff metric induced by w.

Lemma 1. Let E be a metric space and $A,B \in CB(E)$. Then for each $a \in A$ and $\varepsilon > 0$ there exists an $b \in B$ such that

$$w(a,b) = H_w(A,B) + \varepsilon \ .$$

2. MAIN RESULTS

Theorem 1. *Let (E,d) be a complete metric space and let $T: E \to CB(E)$ be a mapping such that*

$$H_w(Tp_1,Tp_2) \le \alpha w(p_1,p_2) + \beta[w(p_1,Tp_1) + w(p_2,Tp_2)] + \gamma[w(p_1,Tp_2) + w(p_2,Tp_1)]$$

for all $p_1, p_2 \in E$,where $\alpha,\beta,\gamma \ge 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

Proof: Let $p_0 \in E, p_1 \in Tp_0$ and define $h = \dfrac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}$. If $h = 0$ then proof is trivial. Now, assume that $h > 0$. Then it follows from Lemma 1 that

$$\exists p_2 \in Tp_1; \qquad w(p_1,p_2) \le H_w(Tp_0,Tp_1) + h$$

$$\exists p_3 \in Tp_2; \qquad w(p_2,p_3) \le H_w(Tp_1,Tp_2) + h^2,$$

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$$\exists p_{n+1} \in Tp_n; \qquad w(p_n,p_{n+1}) \le H_w(Tp_{n-1},Tp_n) + h^n.$$

Hence, we have

$$\begin{aligned} w(p_n,p_{n+1}) &\le H_w(Tp_{n-1},Tp_n) + h^n \\ &\le \alpha w(p_{n-1},p_n) + \beta[w(p_n,Tp_n) + w(p_{n-1},Tp_{n-1})] \\ &\quad + \gamma[w(p_n,Tp_{n-1}) + w(p_{n-1},Tp_n)] + h^n \end{aligned}$$

$$\leq \alpha w(p_{n-1}, p_n) + \beta[w(p_n, p_{n+1}) + w(p_{n-1}, p_n)]$$

$$+\gamma[w(p_{n-1}, p_n) + w(p_n, p_{n+1})] + h^n$$

for all $n \in \mathrm{N}$. It follows that

$$w(p_n, p_{n+1}) \leq hw(p_{n-1}, p_n) + \frac{h^n}{1-(\beta+\gamma)}$$

for all $n \in \mathrm{N}$. It can be conclude that

$$w(p_n, p_{n+1}) \leq h^n w(p_0, p_1) + \frac{nh^n}{1-(\beta+\gamma)}$$

for all $n \in \mathrm{N}$. Now, since $h < 1$, then $\sum_{n=1}^{\infty} w(p_n, p_{n+1}) < \infty$. It follows that $\{p_n\}$ is a Cauchy sequence in E. By completeness of E, there exists $p^* \in E$ such that $\lim_{n\to\infty} p_n = p^*$. Now we shall show that p^* is a fixed point of T. We have

$$w(p^*, Tp^*) \leq w(p^*, p_{n+1}) + w(p_{n+1}, Tp^*) \leq w(p^*, p_{n+1}) + H_w(Tp_n, Tp^*)$$

$$\leq w(p^*, p_{n+1}) + \alpha w(p_n, p^*) + \beta[w(p_n, Tp_n) + w(p^*, Tp^*)]$$

$$+\gamma[w(p_n, Tp^*) + w(p^*, Tp_n)]$$

for all $n \in \mathrm{N}$. Therefore,

$$w(p^*, Tp^*) \leq w(p^*, p_{n+1}) + \alpha w(p_n, p^*) + \beta[w(p_n, p_{n+1}) + w(p^*, Tp^*)]$$

$$+\gamma[w(p_n, Tp^*) + w(p_{n+1}, p^*)]$$

for all $n \in \mathrm{N}$. As $n \to \infty$, we have

$$w(p^*, Tp^*) \leq (\beta+\gamma) w(p^*, Tp^*).$$

On the other hand $\beta + \gamma < 1$, then $w(p^*, Tp^*) = 0$. It follows that $p^* \in Tp^*$.

Theorem 2. *Let* (E,d) *be a complete metric space and let* $T:E\to CB(E)$ *and* $f:E\to E$, f *is continuous mapping which commute with* T *and* $T(E)\subseteq f(E)$. *Suppose there exists* $h\in(0,1)$ *such that*

$$w(Tp_1,Tp_2)\leq \alpha w(fp_1,fp_2)+\beta[w(fp_1,Tp_1)+w(fp_2,Tp_2)]+\gamma[w(fp_1,Tp_2)+w(fp_2,Tp_1)]$$

for all $p_1,p_2\in E$, *where* $\alpha,\beta,\gamma\geq 0$ *and* $\alpha+2\beta+2\gamma<1$. *Then* f *and* T *have a coincidence point.*

Proof: Let p_0 be an arbitrary but fixed element of E and let $p_1\in E$ be a point such that $fp_1\in Tp_0$. Using the definition of Hausdorff metric and the fact that $f(E)\subseteq T(E)$.

We may choose $p_2\in E$ such that $fp_2\in Tp_1$ and

$$w(fp_2,fp_1)\leq H_w(Tp_1,Tp_0)+h^2$$

Inductively after p_n has been determined, we choose $p_{n+1}\in E$, so that $fp_{n+1}\in Tp_n$ and

$$w(fp_n,fp_{n+1})\leq H_w(Tp_{n-1},Tp_n)+h^n$$

Then

$$w(fp_n,fp_{n+1})\leq H_w(Tp_{n-1},Tp_n)+h^n$$

$$\leq \alpha w(fp_{n-1},fp_n)+\beta[w(fp_n,Tp_n)+w(fp_{n-1},Tp_{n-1})]+\gamma[w(fp_n,Tp_{n-1})+w(fp_{n-1},Tp_n)]+h^n$$

$$\leq \alpha w(fp_{n-1},fp_n)+\beta[w(fp_n,fp_{n+1})+w(fp_{n-1},fp_n)]+\gamma[w(fp_n,fp_{n+1})+w(fp_{n-1},fp_n)]+h^n$$

for all $n\in\mathrm{N}$. It follows that

$$w(fp_n,fp_{n+1})\leq hw(fp_{n-1},fp_n)+\frac{h^n}{1-(\beta+\gamma)}$$

for all $n \in \mathrm{N}$. It can be conclude that

$$w(fp_n, fp_{n+1}) \le h^n w(fp_0, fp_1) + \frac{nh^n}{1-(\beta+\gamma)}$$

for all $n \in \mathrm{N}$. Now , since $h<1$, then $\sum_{n=1}^{\infty} w(fp_n, fp_{n+1}) < \infty$. It follows that $\{fp_n\}$ is a Cauchy sequence in E. Since E is complete metric space, there exists a $u \in E$ and $fu = u$. By (w_1) of definition 4 we have

$$\begin{aligned} w(Tu, fu) &\le w(Tu, fp_{n+1}) + w(fp_{n+1}, fu) \\ &\le H_w(Tu, Tp_n) + w(fp_{n+1}, fu) \\ &\le \alpha w(fu, fp_n) + \beta[w(fu, Tu) + w(fp_n, Tp_n)] \\ &\quad +\gamma[w(fu, Tp_n) + w(fp_n, Tu)] + w(fp_{n+1}, fu) \end{aligned}$$

For all $n \in \mathrm{N}$. Therefore,

$$\begin{aligned} w(Tu, fu) &\le \alpha w(fu, fp_n) + \beta[w(fu, Tu) + w(fp_n, fp_{n+1})] \\ &\quad +\gamma[w(fu, fp_{n+1}) + w(fp_n, Tu)] + w(fp_{n+1}, fu) \end{aligned}$$

For all $n \in \mathrm{N}$. Taking limit $n \to \infty$, then we have

$$w(Tu, fu) \le (\beta+\gamma) w(fu, Tu).$$

On the other hand $\beta+\gamma<1$, then $w(Tu, fu) = 0.$ It follows that $fu \in Tu.$

Corollary 1. *Let E be complete metric space and let $T: E \to E$ such that*

$$w(Tp_1, Tp_2) \le \alpha w(p_1, p_2) + \beta[w(p_1, Tp_1) + w(p_2, Tp_2)] + \gamma[w(p_1, Tp_2) + w(p_2, Tp_1)]$$

for all $p_1, p_2 \in E$ where $\alpha, \beta, \gamma \ge 0$ and $\alpha + 2\beta + \gamma < 1$. Then T has a fixed point.

Corollary 2. *Let E be complete metric space and let $T: E \to CB(E)$ satisfies*

$$H_w(Tp_1, Tp_2) \le \alpha w(p_1, p_2)$$

for all $p_1, p_2 \in E$, *where* $0 \le \alpha < 1$. *Then* T *has a fixed point.*

Corollary 3. *Let* E *be a complete metric space and let* $T : E \to CB(E)$ *satisfies*

$$H_w(Tp_1, Tp_2) \le \beta[w(p_1, Tp_1) + w(p_2, Tp_2)]$$

for all $p_1, p_2 \in E$, *where* $\beta \in [0, \frac{1}{2})$. *Then* T *has a fixed point.*

Corollary 4. *Let* E *be complete metric space and let* $T : E \to CB(E)$ *satisfies*

$$H_w(Tp_1, Tp_2) \le \gamma[w(p_1, Tp_2) + w(p_2, Tp_1)]$$

for all $p_1, p_2 \in E$, *where* $\gamma \in [0, \frac{1}{2})$. *Then* T *has a fixed point.*

Corollary 5. *Let* E *be complete metric space and let* $T : E \to CB(E)$ *satisfies*

$$H_w(Tp_1, Tp_2) \le \alpha w(p_1, p_2) + \beta[w(p_1, Tp_1) + w(p_2, Tp_2)]$$

for all $p_1, p_2 \in E$, *where* $\alpha + 2\beta < 1$. *Then* T *has a fixed point.*

Corollary 6. *Let* E *be complete metric space and let* $T : E \to CB(E)$ *satisfies*

$$H_w(Tp_1, Tp_2) \le a_1 w(p_1, p_2) + a_2 w(p_1, Tp_1) + a_3 w(p_2, Tp_2) + a_4 w(p_1, Tp_2) + a_5 w(p_2, Tp_1)$$

for all $p_1, p_2 \in E$, *where* $a_i \ge 0$ *for each* $i \in \{1, 2, ..., 5\}$ *and* $\sum_{i=1}^{5} a_i < 1$. *Then* T *has a fixed point.*

References:

[1] S. Banach, Sure operations dans les ensembles abstraits et leur application aux equations integrals, Fund .Math. 3 (1922) 133-181.

[2] G.E. Hardy and T. D. Rogers, A generalization of a fixed point theorem of Reich, Canad. Math. Bull. 16 (1973), 201-206.

[3] O. Kada, T. Suzuki, W. Takahashi, Non convex minimization theorems and fixed point theorems in complete metric space, Math. Japon. 44, (1996), 381-391.

[4] J.R. Morales, Generalization of Rakotch's fixed point theorem, Revista de matematica: teoreay aplicaciones. Enero 2002. V. 9(1) p. 25-33.

[5] N. B. Nadler J., Multi-valued contraction mappings, Pacific J. Math. 30 (1969) 475-488.

[6] S. Reich, Fixed points of contractive functions, Boll. Un. Mat. Ital. 5 (1972), 26-42.

[7] S. Reich, Kannan's fixed point theorem, Boll. Un. Mat. Ital. 4 (1971) ,1-11.

[8] I. A. Rus, Generalized contraction and Applications, Cluj University Press, Cluj-Nappa, 2001.

Department of Mathematics, Gomal University Dera Ismail Khan, Pakistan.

Email: arjamandbano2002@yahoo.com

Email: saima_naheed86@yahoo.com