



FIXED POINT AND COINCIDENCE POINT THEOREMS

ARJAMAND BANO AND SAIMA NAHEED

Abstract. In this paper, we present a generalization of some fixed point and coincidence point theorems using the notion of a w -distance on a complete metric space. Consequently, we improve and generalize various results existing in the literature.

1. Introduction and Preliminaries

In 1969, Nadler [4] extended the well known Banach contraction principle [1], which asserts that “each single-valued contraction self map on a complete metric space has a unique fixed point.”, to multi-valued mapping using the concept of Hausdorff metric. In [3], Kada et al. introduced a notion of w -distance on a metric space and improved several results replacing the involved metric by a generalized distance. In this paper, using the concept of a w -distance, we give a generalization of Nadler’s fixed point theorem and coincidence point results in complete metric space.

Let (E, d) be a complete metric space and let $CB(E)$ be the collection of all nonempty closed bounded subsets of E . For $A, B \in CB(E)$ and $p \in E$, define

$$D(p, A) = \inf\{d(p, a) : a \in A\}$$

and

$$H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}$$

where H is the Hausdorff metric induced by d .

One can easily prove that $(CB(E), H)$ is a complete metric space, whenever (E, d) is a complete metric space (see e.g. Lemma 8.1.4, of [8]).

Definition 1. An element $p \in E$ is said to be a fixed point of a multi-valued mapping $T : E \rightarrow CB(E)$, if $p \in T(p)$.

Definition 2. Let $C(f, T)$ be the set of coincidence points of f and T
i.e. $C(f, T) = \{p \in E : fp \in Tp\}$.

Corresponding author: Saima Naheed.

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Definition 3. A map $\varphi : E \rightarrow R$ is called lower semi-continuous if for any sequence $\{p_n\} \subset E$ with $p_n \rightarrow p \in E$ implies $\varphi(p) \leq \liminf_{n \rightarrow \infty} \varphi(p_n)$.

Definition 4. A function $w : E \times E \rightarrow [0, \infty)$ is called a w -distance on E if it satisfies the following for any $p_1, p_2, p_3 \in E$ such that

1. $w(p_1, p_3) \leq w(p_1, p_2) + w(p_2, p_3)$;
2. a map $w(p, \cdot) : E \rightarrow [0, \infty)$ is lower semi-continuous;
3. for any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(p_3, p_1) \leq \delta$ and $w(p_3, p_2) \leq \delta$ imply $d(p_1, p_2) \leq \varepsilon$.

The metric d is a w -distance on E . Note that, in general for $p_1, p_2 \in E$, $w(p_1, p_2) \neq w(p_2, p_1)$. Define $w(p, A) = \inf\{w(p, a) : a \in A\}$ and

$$H_w(A, B) = \max\{\sup_{a \in A} w(a, B), \sup_{b \in B} w(b, A)\}.$$

It is easy to see that H_w is a metric on $CB(E)$. H_w is called the Hausdorff metric induced by w .

Lemma 1. Let E be a metric space and $A, B \in CB(E)$. Then for each $a \in A$ and $\varepsilon > 0$ there exists an $b \in B$ such that

$$w(a, b) = H_w(A, B) + \varepsilon.$$

2. Main Results

Theorem 1. Let (E, d) be a complete metric space and let $T : E \rightarrow CB(E)$ be a mapping such that $H_w(Tp_1, Tp_2) \leq \alpha w(p_1, p_2) + \beta[w(p_1, Tp_1) + w(p_2, Tp_2)] + \gamma[w(p_1, Tp_2) + w(p_2, Tp_1)]$ for all $p_1, p_2 \in E$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

Proof. Let $p_0 \in E$, $p_1 \in Tp_0$ and define $h = \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}$. If $h = 0$ then proof is trivial.

Now, assume that $h > 0$. Then it follows from Lemma 1 that

$$\begin{aligned} \exists p_2 \in Tp_1; \quad & w(p_1, p_2) \leq H_w(Tp_0, Tp_1) + h \\ \exists p_3 \in Tp_2; \quad & w(p_2, p_3) \leq H_w(Tp_1, Tp_2) + h^2, \\ & \vdots \\ \exists p_{n+1} \in Tp_n; \quad & w(p_n, p_{n+1}) \leq H_w(Tp_{n-1}, Tp_n) + h^n. \end{aligned}$$

Hence, we have

$$\begin{aligned}
w(p_n, p_{n+1}) &\leq H_w(Tp_{n-1}, Tp_n) + h^n \\
&\leq \alpha w(p_{n-1}, p_n) + \beta[w(p_n, Tp_n) + w(p_{n-1}, Tp_{n-1})] \\
&\quad + \gamma[w(p_n, Tp_{n-1}) + w(p_{n-1}, Tp_n)] + h^n \\
&\leq \alpha w(p_{n-1}, p_n) + \beta[w(p_n, p_{n+1}) + w(p_{n-1}, p_n)] \\
&\quad + \gamma[w(p_{n-1}, p_n) + w(p_n, p_{n+1})] + h^n
\end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$w(p_n, p_{n+1}) \leq h w(p_{n-1}, p_n) + \frac{h^n}{1 - (\beta + \gamma)}$$

for all $n \in \mathbb{N}$. It can be conclude that

$$w(p_n, p_{n+1}) \leq h^n w(p_0, p_1) + \frac{nh^n}{1 - (\beta + \gamma)}$$

for all $n \in \mathbb{N}$. Now, since $h < 1$, then $\sum_{n=1}^{\infty} w(p_n, p_{n+1}) < \infty$. It follows that $\{p_n\}$ is a Cauchy sequence in E . By completeness of E , there exists $p^* \in E$ such that $\lim_{n \rightarrow \infty} p_n = p^*$. Now we shall show that p^* is a fixed point of T . We have

$$\begin{aligned}
w(p^*, Tp^*) &\leq w(p^*, p_{n+1}) + w(p_{n+1}, Tp^*) \leq w(p^*, p_{n+1}) + H_w(Tp_n, Tp^*) \\
&\leq w(p^*, p_{n+1}) + \alpha w(p_n, p^*) + \beta[w(p_n, Tp_n) + w(p^*, Tp^*)] \\
&\quad + \gamma[w(p_n, Tp^*) + w(p^*, Tp_n)]
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
w(p^*, Tp^*) &\leq w(p^*, p_{n+1}) + \alpha w(p_n, p^*) + \beta[w(p_n, p_{n+1}) + w(p^*, Tp^*)] \\
&\quad + \gamma[w(p_n, Tp^*) + w(p_{n+1}, p^*)]
\end{aligned}$$

for all $n \in \mathbb{N}$. As $n \rightarrow \infty$, we have

$$w(p^*, Tp^*) \leq (\beta + \gamma)w(p^*, Tp^*).$$

Since $\beta + \gamma < 1$, then $w(p^*, Tp^*) = 0$. It follows that $p^* \in Tp^*$.

Theorem 2. *Let (E, d) be a complete metric space and let $T : E \rightarrow CB(E)$ and $f : E \rightarrow E$, f is continuous mapping which commute with T and $T(E) \subseteq f(E)$. Suppose there exists $h \in (0, 1)$ such that*

$$w(Tp_1, Tp_2) \leq \alpha w(fp_1, fp_2) + \beta[w(fp_1, Tp_1) + w(fp_2, Tp_2)] + \gamma[w(fp_1, Tp_2) + w(fp_2, Tp_1)]$$

for all $p_1, p_2 \in E$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then f and T have a coincidence point.

Proof. Let p_0 be an arbitrary but fixed element of E and let $p_1 \in E$ be a point such that $f p_1 \in T p_0$. Using the definition of the Hausdorff metric and the fact that $f(E) \subseteq T(E)$, we may choose $p_2 \in E$ such that $f p_2 \in T p_1$ and

$$w(f p_2, f p_1) \leq H_w(T p_1, T p_0) + h^2.$$

Inductively after p_n has been determined, we choose $p_{n+1} \in E$, so that $f p_{n+1} \in T p_n$ and

$$w(f p_n, f p_{n+1}) \leq H_w(T p_{n-1}, T p_n) + h^n$$

Then

$$\begin{aligned} w(f p_n, f p_{n+1}) &\leq H_w(T p_{n-1}, T p_n) + h^n \\ &\leq \alpha w(f p_{n-1}, f p_n) + \beta [w(f p_n, T p_n) + w(f p_{n-1}, T p_{n-1})] \\ &\quad + \gamma [w(f p_n, T p_{n-1}) + w(f p_{n-1}, T p_n)] + h^n \\ &\leq \alpha w(f p_{n-1}, f p_n) + \beta [w(f p_n, f p_{n+1}) + w(f p_{n-1}, f p_n)] \\ &\quad + \gamma [w(f p_n, f p_{n+1}) + w(f p_{n-1}, f p_n)] + h^n \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$w(f p_n, f p_{n+1}) \leq h w(f p_{n-1}, f p_n) + \frac{h^n}{1 - (\beta + \gamma)}$$

for all $n \in \mathbb{N}$. It can be conclude that

$$w(f p_n, f p_{n+1}) \leq h^n w(f p_0, f p_1) + \frac{nh^n}{1 - (\beta + \gamma)}$$

for all $n \in \mathbb{N}$. Now, since $h < 1$, then $\sum_{n=1}^{\infty} w(f p_n, f p_{n+1}) < \infty$. It follows that $\{f p_n\}$ is a Cauchy sequence in E . Since E is complete metric space, there exists a $u \in E$ and $f u = u$. By definition 4 we have

$$\begin{aligned} w(Tu, f u) &\leq w(Tu, f p_{n+1}) + w(f p_{n+1}, f u) \\ &\leq H_w(Tu, T p_n) + w(f p_{n+1}, f u) \\ &\leq \alpha w(f u, f p_n) + \beta [w(f u, Tu) + w(f p_n, T p_n)] \\ &\quad + \gamma [w(f u, T p_n) + w(f p_n, Tu)] + w(f p_{n+1}, f u). \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} w(Tu, f u) &\leq \alpha w(f u, f p_n) + \beta [w(f u, Tu) + w(f p_n, f p_{n+1})] \\ &\quad + \gamma [w(f u, f p_{n+1}) + w(f p_n, Tu)] + w(f p_{n+1}, f u) \end{aligned}$$

For all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$, we have

$$w(Tu, f u) \leq (\beta + \gamma) w(f u, Tu).$$

Since $\beta + \gamma < 1$, then $w(Tu, f u) = 0$. It follows that $f u \in Tu$.

Corollary 1 ([2]). *Let E be complete metric space and let $T : E \rightarrow E$ such that*

$$w(Tp_1, Tp_2) \leq \alpha w(p_1, p_2) + \beta[w(p_1, Tp_1) + w(p_2, Tp_2)] + \gamma[w(p_1, Tp_2) + w(p_2, Tp_1)]$$

for all $p_1, p_2 \in E$ where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + \gamma < 1$. Then T has a fixed point.

Corollary 2. *Let E be complete metric space and let $T : E \rightarrow CB(E)$ satisfies*

$$H_w(Tp_1, Tp_2) \leq \alpha w(p_1, p_2)$$

for all $p_1, p_2 \in E$, where $0 \leq \alpha < 1$. Then T has a fixed point.

Corollary 3 ([5, 6, 7]). *Let E be a complete metric space and let $T : E \rightarrow CB(E)$ satisfy*

$$H_w(Tp_1, Tp_2) \leq \beta[w(p_1, Tp_1) + w(p_2, Tp_2)]$$

for all $p_1, p_2 \in E$, where $\beta \in [0, \frac{1}{2})$. Then T has a fixed point.

Corollary 4. *Let E be complete metric space and let $T : E \rightarrow CB(E)$ satisfy*

$$H_w(Tp_1, Tp_2) \leq \gamma[w(p_1, Tp_2) + w(p_2, Tp_1)]$$

for all $p_1, p_2 \in E$, where $\gamma \in [0, \frac{1}{2})$. Then T has a fixed point.

Corollary 5. *Let E be complete metric space and let $T : E \rightarrow CB(E)$ satisfy*

$$H_w(Tp_1, Tp_2) \leq \alpha w(p_1, p_2) + \beta[w(p_1, Tp_1) + w(p_2, Tp_2)]$$

for all $p_1, p_2 \in E$, where $\alpha + 2\beta < 1$. Then T has a fixed point.

Corollary 6. *Let E be complete metric space and let $T : E \rightarrow CB(E)$ satisfy*

$$H_w(Tp_1, Tp_2) \leq a_1 w(p_1, p_2) + a_2 w(p_1, Tp_1) + a_3 w(p_2, Tp_2) + a_4 w(p_1, Tp_2) + a_5 w(p_2, Tp_1)$$

for all $p_1, p_2 \in E$, where $a_i \geq 0$ for each $i \in \{1, 2, \dots, 5\}$ and $\sum_{i=1}^5 a_i < 1$. Then T has a fixed point.

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Department of Mathematics, Gomal University Dera Ismail Khan, Pakistan.

E-mail: arjamandbano2002@yahoo.com

E-mail: saima_naheed86@yahoo.com