INTEGRAL PROPERTIES OF CERTAIN CLASSES OF MULTIVALENT ANALYTIC FUNCTIONS WITH COMPLEX ORDER

TARIQ O. SALIM

Abstract. In the present paper, the generalized Komatu integral operator and the generalized Jung-Kim-Srivastava integral operator are applied to study the integral properties of two subclasses of analytic and $p$-valent functions of complex order. Some special cases (known or new) of the main theorems are indicated.

1. Introduction

Let $A_p(n)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad (a_k \geq 0; n, p \in \mathbb{N})$$

that are analytic and $p$-valent in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$.

Differentiating both sides of (1.1) $m$-times with respect to $z$, we get

$$f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m}$$

where $n, p \in \mathbb{N}$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $p > m$.

Now, let us recall the subclasses $R_{n,m}^p(\lambda, b)$ and $L_{n,m}^p(\lambda, b)$ introduced and studied recently by Srivastava and Orhan [7] and Güney and Breaz [1], which consists of functions $f(z)$ belonging to $A_p(n)$ and satisfying the inequality, respectively:

$$\left| \frac{1}{b} \left( z f^{(1+m)}(z) + \lambda z^2 f^{(2+m)}(z) \right) - \lambda z f^{(1+m)}(z) \right| < 1$$

and

$$\left| \frac{1}{b} \left( f^{(1+m)}(z) + \lambda z f^{(2+m)}(z) - (p-m) \right) \right| < p-m.$$
Srivastava and Orhan [7] (see, e.g. Güney and Breaz [1]) proved the following lemmas, in which coefficient inequalities are given as necessary and sufficient conditions for functions \( f(z) \in A_p(n) \) to belong to the classes \( R^p_{n,m}(\lambda, b) \) and \( L^p_{n,m}(\lambda, b) \).

**Lemma 1.1.** Let \( f(z) \in A_p(n) \) be given by (1.1). Then \( f(z) \in R^p_{n,m}(\lambda, b) \) if and only if

\[
\sum_{k=n+p}^{\infty} \frac{(k+|b|-p)k!\lambda(k-m-1)+1}{(k-m)!} a_k \leq \frac{|b|p!(\lambda(p-m)+1)}{(p-m)!}.
\] (1.5)

**Lemma 1.2.** Let \( f(z) \in A_p(n) \) be given by (1.1). Then \( f(z) \in L^p_{n,m}(\lambda, b) \) if and only if

\[
\sum_{k=n+p}^{\infty} \binom{k}{m}(k-m)\lambda(k-m-1)+1 a_k \leq (p-m) \frac{|b|!}{m!} \lambda(p-m) + 1.
\] (1.6)

The generalized Komatu integral operator [4], \( P^\delta_{c,p} : A_p(n) \to A_p(n) \) and the generalized Jung-Kim-Srivastava integral operator [2], \( Q^\delta_{c,p} : A_p(n) \to A_p(n) \) (see, e.g. [3] and [6]) are defined for \( \delta > 0 \) and \( c > -p \) as

\[
P^\delta_{c,p} f(z) = \frac{(c+p)^\delta}{\Gamma(\delta)} \int_0^z t^{c-1} \left( \log \frac{z}{t} \right)^{\delta-1} f(t) dt
\] (1.7)

and

\[
Q^\delta_{c,p} f(z) = \frac{\delta + c - p - 1}{c + p - 1} \frac{\delta}{z^\delta} \int_0^z t^{c-1} \left( 1 - \frac{t}{z} \right)^{\delta-1} f(t) dt
\] (1.8)

Note that for functions \( f(z) \in A_p(n) \) of the form (1.1), we have

\[
P^\delta_{c,p} f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{(c+p)^\delta}{c+k} a_k z^k
\] (1.9)

and

\[
Q^\delta_{c,p} f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{(c+p)_{k-p}}{(c+p+\delta)_{k-p}} a_k z^k
\] (1.10)

where

\[
(\sigma)_k = \frac{\Gamma(\sigma+k)}{\Gamma(\sigma)} = \begin{cases} 1, & (k = 0, \sigma \in \mathbb{C} \setminus \{0\}) \\ \sigma(\sigma+1) \cdots (\sigma+n-1), & (k = n \in \mathbb{N}, \sigma \in \mathbb{C}) \end{cases}
\]

is the Pochhammer symbol of the extended factorial function.

Now, let us define the integral operator \( H^\delta_{c,p} : A_p(n) \to A_p(n) \) by

\[
H(z) = H^\delta_{c,p} f(z) = \mu P^\delta_{c,p} f(z) + (1 - \mu) Q^\delta_{c,p} f(z)
\]

\[
= \int_0^z t^{c-1} \frac{\mu(c+p)^\delta}{\Gamma(\delta)} \left( \log \frac{z}{t} \right)^{\delta-1} + \frac{(1-\mu)\Gamma(c+p+\delta)}{\Gamma(c+p)} \left( 1 - \frac{t}{z} \right)^{\delta-1} f(t) dt
\] (1.11)
for $\delta > 0$, $c > -p$, $p \in \mathbb{N}$ and $0 \leq \mu \leq 1$.

Hence, in view of (1.9) and (1.10), for functions $f(z) \in A_p(n)$ of the form (1.1), we have

$$H(z) = z^p - \sum_{k=n+p}^{\infty} \left[ \mu \left( \frac{c + p}{c + k} \right)^{\delta} + (1 - \mu) \frac{(c + p)_{k-p}}{(c + p + \delta)_{k-p}} \right] a_k z^k$$  \hspace{1cm} (1.12)

Motivated by the recent investigations by Sălăgean [5] and Güney and Breaz [1] and others, the main object of the present investigation is to establish integral properties of analytic $p-$valently functions with negative coefficients and with complex order belonging to the classes $R^p_{n,m}(\lambda, b)$ and $P^p_{n,m}(\lambda, b)$.

2. Integral properties of the class $R^p_{n,m}(\lambda, b)$

We begin by proving the following Lemma which is needed to proceed in proving the main theorems.

**Lemma 2.1.** Let

$$B_k = B(k, c, p, \mu, \delta) = \left[ \mu \left( \frac{c + p}{c + k} \right)^{\delta} + (1 - \mu) \frac{(c + p)_{k-p}}{(c + p + \delta)_{k-p}} \right]$$  \hspace{1cm} (2.1)

for $k \geq n + p$, $\delta > 0$, $c > -p$, $p \in \mathbb{N}$ and $0 \leq \mu \leq 1$. Then $B_k$ is a decreasing function of $k$, and

$$B_{k+1} \leq B_k \leq B_{n+p} \leq 1.$$  \hspace{1cm} (2.2)

**Proof.** Since $k = n + p \geq p$ for $n \in \mathbb{N}$, then

$$\frac{c + p}{c + k + 1} \leq \frac{c + p}{c + k} \leq \frac{c + p}{c + n + p} \leq 1,$$

and

$$\frac{(c + p)_{k+1-p}}{(c + p + \delta)_{k+1-p}} = \frac{(c + p)_{k-p}}{(c + p + \delta)_{k-p}} \frac{c + k}{c + k + \delta} \leq \frac{(c + p)_{k-p}}{(c + p + \delta)_{k-p}} \leq \frac{(c + p)_n}{(c + p + \delta)_n} \leq 1$$

which yields (2.2). \hfill $\Box$

Now we prove

**Theorem 2.2.** If $f(z) \in R^p_{n,m}(\lambda, b)$, then $H(z) \in R^p_{n,m}(\lambda, \gamma)$, where

$$|\gamma| = \frac{n|b|B_{n+p}}{n + b(1 - B_{n+p})},$$  \hspace{1cm} (2.3)

and $|\gamma| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0$, $\delta > 0$, $b \in \mathbb{C} \setminus \{0\}$, $c > -p$, $0 \leq \mu \leq 1$, $0 \leq \lambda \leq 1$ and $B_{n+p}$ is given by (2.1). The result is sharp.
Proof. From Lemma (1.1) and (1.12), \( H(z) \in R_{n,m}^p(\lambda, \gamma) \) if and only if
\[
\sum_{k=n+p}^{\infty} \frac{(k + |\gamma| - p)k!(p - m)![\lambda(k - m - 1) + 1]}{(k - m)!|\gamma| p![\lambda(p - m - 1) + 1]}B_k a_k \leq 1. \tag{2.4}
\]
Now for \( k \geq n + p \), the inequality
\[
\frac{(k + |\gamma| - p)k!(p - m)!|\lambda(k - m - 1) + 1]}{(k - m)!|\gamma| p![\lambda(p - m - 1) + 1]}B_k
\leq \frac{(k + |b| - p)k!(p - m)!|\lambda(k - m - 1) + 1]}{(k - m)!|b| p![\lambda(p - m - 1) + 1]} \tag{2.5}
\]
implies (2.4), since \( f \in R_{n,m}^p(\lambda, b) \) and satisfies (1.5). Hence
\[
\left|\frac{(k + |\gamma| - p)B_k}{|\gamma|} \leq \frac{(k + |b| - p)}{|b|} \right.
\]
which yields
\[
|\gamma| \geq \frac{(k - p)|b|B_k}{(k - p) + |b|(1 - B_k)} \tag{2.6}
\]
Now, we show that \( |\gamma| \) is a decreasing function of \( k, \ k \geq n + p \). Indeed, let
\[
h(k) = \frac{(k - p)|b|B_k}{(k - p) + |b|(1 - B_k)} \tag{2.7}
\]
So, in view of Lemma (2.1), we have
\[
h(k+1) - h(k) = \frac{(k + 1 - p)|b|B_{k+1}}{(k + 1 - p) + |b|(1 - B_{k+1})} - \frac{(k - p)|b|B_k}{(k - p) + |b|(1 - B_k)}
\leq \frac{(k + 1 - p)(k - p)|b|(B_{k+1} - B_k) + (k + 1 - p)|b|^2B_{k+1}(1 - B_k) - (k - p)|b|^2B_k(1 - B_k)}{([k + 1 - p] + |b|(1 - B_{k+1}))([k - p] + |b|(1 - B_k))}
\leq 0.
\]
So \( h(k) \) is a decreasing function of \( k \) and we have
\[
|\gamma(p, k, c, b, \mu, \delta)| \leq |\gamma| = |\gamma(p, p + n, c, b, \mu, \delta)|; \ k \geq n + p. \tag{2.8}
\]
Thus (2.3) is proved, and in view of Lemma (2.1), \( B_{n+p} \leq 1 \) directly implies \( |\gamma| < |b| \).

Now, the result is sharp because
\[
H_{c,p}^{\delta, \mu}(f_b) = f_\gamma \tag{2.9}
\]
where
\[ f_b(z) = z^p - \frac{|b| p!(n + p - m)[\lambda(p - m - 1) + 1]}{(p - m)!(n + p)!(n + |b|)[\lambda(p - m - 1) + 1]} z^{n+p} \] (2.10)
and
\[ f_\gamma(z) = z^p - \frac{|\gamma| p!(n + p - m)[\lambda(p - m - 1) + 1]}{(p - m)!(n + p)!(n + |\gamma|)[\lambda(p - m - 1) + 1]} z^{n+p} \] (2.11)
are extremal functions of \( R_{n,m}^p(\lambda, b) \) and \( R_{n,m}^p(\lambda, \gamma) \), respectively.

Indeed, we have
\[ H_{c,p}^{\delta,\mu}(f_b)(z) = z^p - \frac{|b| p!(n + p - m)[\lambda(p - m - 1) + 1]}{(p - m)!(n + p)!(n + |b|)[\lambda(p - m - 1) + 1]} B_{n+p} z^{n+p} \] (2.12)
Thus in comparison with (2.11), we deduce
\[ \frac{|\gamma|}{n + |\gamma|} = \frac{|b|}{n + |b|} B_{n+p} \] (2.13)
which implies (2.8).

Setting \( \mu = 1 \) and \( \mu = 0 \), respectively in Theorem (2.2), we get the following corollaries:

**Corollary 2.3.** If \( f(z) \in R_{n,m}^p(\lambda, b) \), then \( P_{c,p}^\delta f(z) \in R_{n,m}^p(\lambda, \gamma_1) \), where
\[ |\gamma_1| = \frac{n |b| (c + p)^\delta}{n(c + n + p)^\delta + |b| [(c + n + p)^\delta - (c + p)^\delta]} \] (2.14)
and \( |\gamma_1| < |b| \), provided that \( p, n \in \mathbb{N}, m \in \mathbb{N}_0, \ \delta > 0, \ b \in \mathbb{C} \setminus \{0\}, \ c > -p \) and \( 0 \leq \lambda \leq 1 \). The result is sharp.

**Corollary 2.4.** If \( f(z) \in R_{n,m}^p(\lambda, b) \), then \( Q_{c,p}^\delta f(z) \in R_{n,m}^p(\lambda, \gamma_2) \), where
\[ |\gamma_2| = \frac{n |b| (c + p)_n}{n(c + p + \delta)_n + |b| [(c + p + \delta)_n - (c + p)_n]} \] (2.15)
and \( |\gamma_2| < |b| \), provided that \( p, n \in \mathbb{N}, m \in \mathbb{N}_0, \ \delta > 0, \ b \in \mathbb{C} \setminus \{0\}, \ c > -p \) and \( 0 \leq \lambda \leq 1 \). The result is sharp.

Setting \( \delta = 1 \) in either (2.14) or (2.15), we get the result recently obtained by Güney and Breaz [1, Theorem 1]. Also, if we put \( \delta = 1, \ p = 1, \ m = 0, \ b = 1 - \alpha \) and \( \gamma = 1 - \beta \), we obtain the result established by Sălăgean [5]. Several special cases can be obtained by specifying the parameters in Theorem (2.2).
3. Integral properties of the class $L_{n,m}^{p}(\lambda, b)$

**Theorem 3.1.** If $f(z) \in L_{n,m}^{p}(\lambda, b)$, then $H(z) \in L_{n,m}^{p}(\lambda, \beta)$, where

$$|\beta| = (|b| - 1) B_{n+p} + \frac{p!}{(p-m)!} \left[ \lambda(p-m-1) + 1 \right] (B_{n+p} - 1) + 1 \quad (3.1)$$

and $|\beta| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0$, $\delta > 0$, $b \in \mathbb{C} \setminus \{0\}$, $c > -p$, $0 \leq \mu \leq 1$, $0 \leq \lambda \leq 1$ and $B_{n+p}$ is given by (2.1). The result is sharp.

**Proof.** Following similar steps to that in the proof of Theorem (2.2), we can get the result \(\square\)

Setting $\mu = 1$ and $\mu = 0$, respectively in Theorem 3.1, we get the following corollaries:

**Corollary 3.2.** If $f(z) \in L_{n,m}^{p}(\lambda, b)$, then $P_{c,p}^{\delta} f(z) \in L_{n,m}^{p}(\lambda, \beta_1)$, where

$$|\beta_1| = (|b| - 1) \left( \frac{c+p}{c+n+p} \right)^{\delta} + \frac{p!}{(p-m)!} \left[ \lambda(p-m-1) + 1 \right] \left( \frac{c+p}{c+n+p} \right)^{\delta} - 1 + 1 \quad (3.2)$$

and $|\beta_1| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0$, $\delta > 0$, $b \in \mathbb{C} \setminus \{0\}$, $c > -p$ and $0 \leq \lambda \leq 1$. The result is sharp.

**Corollary 3.3.** If $f(z) \in L_{n,m}^{p}(\lambda, b)$, then $Q_{c,p}^{\delta} f(z) \in L_{n,m}^{p}(\lambda, \beta_2)$, where

$$|\beta_2| = (|b| - 1) \left( \frac{c+p}{c+p+\delta} \right)^{n} + \frac{p!}{(p-m)!} \left[ \lambda(p-m-1) + 1 \right] \left( \frac{c+p}{c+p+\delta} \right)^{n} - 1 + 1 \quad (3.3)$$

and $|\beta_2| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0$, $\delta > 0$, $b \in \mathbb{C} \setminus \{0\}$, $c > -p$ and $0 \leq \lambda \leq 1$. The result is sharp.

Setting $\delta = 1$ in either (3.2) or (3.3), we get the result recently obtained by Güney and Breaz [1, Theorem 2]. More particular results can be obtained by specifying the parameters in Theorem 3.1.

**References**


Department of Mathematics, Al-Azhar University-Gaza, P.O. Box 1277, Gaza, Palestine.

E-mail: trsalim@yahoo.com; t.salim@alazhar.edu.ps