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# INTEGRAL PROPERTIES OF CERTAIN CLASSES OF MULTIVALENT ANALYTIC FUNCTIONS WITH COMPLEX ORDER

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**Abstract**. In the present paper, the generalized Komatu integral operator and the generalized Jung-Kim-Srivastava integral operator are applied to study the integral properties of two subclasses of analytic and $p-$valent functions of complex order. Some special cases (known or new) of the main theorems are indicated.

## 1. Introduction

Let $A_p(n)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad (a_k \geq 0; n, p \in \mathbb{N}) \tag{1.1}$$

that are analytic and $p-$valent in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$.

Differentiating both sides of (1.1) $m-$times with respect to $z$, we get

$$f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m} \tag{1.2}$$

where $n, p \in \mathbb{N}, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \quad p > m.$

Now, let us recall the subclasses $R^p_{n,m}(\lambda, b)$ and $L^p_{n,m}(\lambda, b)$ introduced and studied recently by Srivastava and Orhan [7] and Güney and Breaz [1], which consists of functions $f(z)$ belonging to $A_p(n)$ and satisfying the inequality, respectively:

$$\left| \frac{1}{b} \left( \frac{z f^{(1+m)}(z) + \lambda z^2 f^{(2+m)}(z)}{\lambda z f^{(1+m)}(z) + (1-\lambda) f^{(m)}(z)} - (p-m) \right) \right| < 1 \tag{1.3}$$

and

$$\left| \frac{1}{b} \left( f^{(1+m)}(z) + \lambda z f^{(2+m)}(z) - (p-m) \right) \right| < p - m. \tag{1.4}$$

where $z \in U$; $p \in \mathbb{N}$, $m \in \mathbb{N}_0$; $p > m$; $0 \leq \lambda \leq 1$ and $b \in \mathbb{C} \backslash \{0\}$.

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Srivastava and Orhan [7] (see, e.g. Güney and Breaz [1]) proved the following lemmas, in which coefficient inequalities are given as necessary and sufficient conditions for functions $f(z) \in A_p(n)$ to belong to the classes $R^p_{n,m}(\lambda,b)$ and $L^p_{n,m}(\lambda,b)$.

**Lemma 1.1.** *Let $f(z) \in A_p(n)$ be given by (1.1). Then $f(z) \in R^p_{n,m}(\lambda,b)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{\left(k+|b|-p\right)k!\,[\lambda(k-m-1)+1]}{(k-m)!} a_k \leq \frac{|b|\,p!\left[\lambda(p-m-1)+1\right]}{(p-m)!}. \tag{1.5}$$

**Lemma 1.2.** *Let $f(z) \in A_p(n)$ be given by (1.1). Then $f(z) \in L^p_{n,m}(\lambda,b)$ if and only if*

$$\sum_{k=n+p}^{\infty} \binom{k}{m}(k-m)\,[\lambda(k-m-1)+1]\,a_k \leq (p-m)\left[\frac{|b|-1}{m!}\binom{p}{m}\left[\lambda(p-m-1)+1\right]\right]. \tag{1.6}$$

The generalized Komatu integral operator [4], $P^{\delta}_{c,p} : A_p(n) \to A_p(n)$ and the generalized Jung-Kim-Srivastava integral operator [2], $Q^{\delta}_{c,p} : A_p(n) \to A_p(n)$ (see,e.g. [3] and [6]) are defined for $\delta > 0$ and $c > -p$ as

$$P^{\delta}_{c,p} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta) z^c} \int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1} f(t)\,dt \tag{1.7}$$

and

$$Q^{\delta}_{c,p} f(z) = \begin{pmatrix} \delta+c+p-1 \\ c+p-1 \end{pmatrix} \frac{\delta}{z^c} \int_0^z t^{c-1}\left(1-\frac{t}{z}\right)^{\delta-1} f(t)\,dt \tag{1.8}$$

Note that for functions $f(z) \in A_p(n)$ of the form (1.1), we have

$$P^{\delta}_{c,p} f(z) = z^p - \sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^{\delta} a_k z^k \tag{1.9}$$

and

$$Q^{\delta}_{c,p} f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{(c+p)_{k-p}}{(c+p+\delta)_{k-p}} a_k z^k \tag{1.10}$$

where

$$(\sigma)_k = \frac{\Gamma(\sigma+k)}{\Gamma(\sigma)} = \begin{cases} 1, & (k=0, \sigma \in \mathbb{C}\setminus\{0\}) \\ \sigma(\sigma+1)\cdots(\sigma+n-1), & (k=n\in\mathbb{N}, \sigma\in\mathbb{C}) \end{cases}$$

is the Pochhammer symbol of the extended factorial function.

Now, let us define the integral operator $H^{\delta,\mu}_{c,p} : A_p(n) \to A_p(n)$ by

$$\begin{aligned} H(z) &= H^{\delta,\mu}_{c,p} f(z) = \mu P^{\delta}_{c,p} f(z) + (1-\mu) Q^{\delta}_{c,p} f(z) \\ &= \int_0^z \frac{t^{c-1}}{\Gamma(\delta) z^c}\left[\mu(c+p)^{\delta}\left(\log\frac{z}{t}\right)^{\delta-1} + (1-\mu)\frac{\Gamma(c+p+\delta)}{\Gamma(c+p)}\left(1-\frac{t}{z}\right)^{\delta-1}\right] f(t)\,dt \end{aligned} \tag{1.11}$$

for $\delta > 0,\ \ c > -p,\ \ p \in \mathbb{N}$ and $0 \le \mu \le 1$.
Hence, in view of (1.9) and (1.10), for functions $f(z) \in A_p(n)$ of the form (1.1), we have

$$H(z) = z^p - \sum_{k=n+p}^{\infty} \left[\mu\left(\frac{c+p}{c+k}\right)^{\delta} + (1-\mu)\frac{(c+p)_{k-p}}{(c+p+\delta)_{k-p}}\right] a_k z^k \tag{1.12}$$

Motivated by the recent investigations by Sălăgean [5] and Güney and Breaz [1] and others, the main object of the present investigation is to establish integral properties of analytic $p-$valently functions with negative coefficients and with complex order belonging to the classes $R^p_{n,m}(\lambda, b)$ and $L^p_{n,m}(\lambda, b)$.

## 2. Integral properties of the class $R^p_{n,m}(\lambda, b)$

We begin by proving the following Lemma which is needed to proceed in proving the main theorems.

**Lemma 2.1.** *Let*

$$B_k = B(k,c,p,\mu,\delta) = \left[\mu\left(\frac{c+p}{c+k}\right)^{\delta} + (1-\mu)\frac{(c+p)_{k-p}}{(c+p+\delta)_{k-p}}\right] \tag{2.1}$$

*for* $k \ge n+p,\ \ \delta > 0,\ \ c > -p,\ \ p \in \mathbb{N}$ *and* $0 \le \mu \le 1$. *Then* $B_k$ *is a decreasing function of* $k$, *and*

$$B_{k+1} \le B_k \le B_{n+p} \le 1. \tag{2.2}$$

**Proof.** Since $k = n+p \ge p$ for $n \in \mathbb{N}$, then

$$\frac{c+p}{c+k+1} \le \frac{c+p}{c+k} \le \frac{c+p}{c+n+p} \le 1,$$

and

$$\frac{(c+p)_{k+1-p}}{(c+p+\delta)_{k+1-p}} = \frac{(c+p)_{k-p}}{(c+p+\delta)_{k-p}}\frac{c+k}{c+k+\delta} \le \frac{(c+p)_{k-p}}{(c+p+\delta)_{k-p}} \le \frac{(c+p)_n}{(c+p+\delta)_n} \le 1$$

which yields (2.2). □

Now we prove

**Theorem 2.2.** *If* $f(z) \in R^p_{n,m}(\lambda, b)$, *then* $H(z) \in R^p_{n,m}(\lambda, \gamma)$, *where*

$$|\gamma| = \frac{n|b|B_{n+p}}{n+|b|(1-B_{n+p})}, \tag{2.3}$$

*and* $|\gamma| < |b|$, *provided that* $p, n \in \mathbb{N}, m \in \mathbb{N}_0,\ \ \delta > 0,\ \ b \in \mathbb{C}\setminus\{0\},\ \ c > -p,\ \ 0 \le \mu \le 1,\ \ 0 \le \lambda \le 1$ *and* $B_{n+p}$ *is given by* (2.1). *The result is sharp.*

**Proof.** From Lemma (1.1) and (1.12), $H(z) \in R^p_{n,m}(\lambda,\gamma)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{\left(k+\left|\gamma\right|-p\right) k!(p-m)!\left[\lambda(k-m-1)+1\right]}{(k-m)!\left|\gamma\right| p!\left[\lambda(p-m-1)+1\right]} B_k a_k \le 1. \tag{2.4}$$

Now for $k \ge n+p$, the inequality

$$\frac{\left(k+\left|\gamma\right|-p\right) k!(p-m)!\left[\lambda(k-m-1)+1\right]}{(k-m)!\left|\gamma\right| p!\left[\lambda(p-m-1)+1\right]} B_k$$

$$\le \frac{\left(k+|b|-p\right) k!(p-m)!\left[\lambda(k-m-1)+1\right]}{(k-m)!|b|\, p!\left[\lambda(p-m-1)+1\right]} \tag{2.5}$$

implies (2.4), since $f \in R^p_{n,m}(\lambda,b)$ and satisfies (1.5). Hence

$$\frac{\left(k+\left|\gamma\right|-p\right) B_k}{\left|\gamma\right|} \le \frac{\left(k+|b|-p\right)}{|b|}$$

which yields

$$\left|\gamma\right| \ge \frac{\left(k-p\right)|b|\, B_k}{\left(k-p\right)+|b|\left(1-B_k\right)} \tag{2.6}$$

Now, we show that $\left|\gamma\right|$ is a decreasing function of $k, \;\; k \ge n+p$. Indeed, let

$$h(k) = \frac{\left(k-p\right)|b|\, B_k}{\left(k-p\right)+|b|\left(1-B_k\right)} \tag{2.7}$$

So, in view of Lemma (2.1), we have

$$\begin{aligned}
h(k+1)-h(k) &= \frac{\left(k+1-p\right)|b|\, B_{k+1}}{\left(k+1-p\right)+|b|\left(1-B_{k+1}\right)} - \frac{\left(k-p\right)|b|\, B_k}{\left(k-p\right)+|b|\left(1-B_k\right)} \\
&= \frac{(k+1-p)(k-p)|b|(B_{k+1}-B_k)+(k+1-p)|b|^2 B_{k+1}(1-B_k)-(k-p)|b|^2 B_k(1-B_{k+1})}{\left[(k+1-p)+|b|(1-B_{k+1})\right]\left[(k-p)+|b|(1-B_k)\right]} \\
&\le \frac{\left(k+1-p\right)\left(k-p\right)|b|\left(B_{k+1}-B_k\right)+\left(k+1-p\right)|b|^2\left(B_{k+1}-B_k\right)}{\left[\left(k+1-p\right)+|b|\left(1-B_{k+1}\right)\right]\left[\left(k-p\right)+|b|\left(1-B_k\right)\right]} \\
&\le 0.
\end{aligned}$$

So $h(k)$ is a decreasing function of $k$ and we have

$$\left|\gamma(p,k,c,b,\mu,\delta)\right| \le \left|\gamma\right| = \left|\gamma(p,p+n,c,b,\mu,\delta)\right|; \quad k \ge n+p. \tag{2.8}$$

Thus (2.3) is proved, and in view of Lemma (2.1), $B_{n+p} \le 1$ directly implies $\left|\gamma\right| < |b|$.

Now , the result is sharp because

$$H^{\delta,\mu}_{c,p}(f_b) = f_\gamma \tag{2.9}$$

where

$$f_b(z) = z^p - \frac{|b|\, p!(n+p-m)[\lambda(p-m-1)+1]}{(p-m)!(n+p)!(n+|b|)[\lambda(p-m-1)+1]} z^{n+p} \tag{2.10}$$

and

$$f_\gamma(z) = z^p - \frac{|\gamma|\, p!(n+p-m)[\lambda(p-m-1)+1]}{(p-m)!(n+p)!(n+|\gamma|)[\lambda(p-m-1)+1]} z^{n+p} \tag{2.11}$$

are extremal functions of $R^p_{n,m}(\lambda, b)$ and $R^p_{n,m}(\lambda,\gamma)$, respectively.

Indeed, we have

$$H^{\delta,\mu}_{c,p}(f_b)(z) = z^p - \frac{|b|\, p!(n+p-m)[\lambda(p-m-1)+1]}{(p-m)!(n+p)!(n+|b|)[\lambda(p-m-1)+1]} B_{n+p} z^{n+p} \tag{2.12}$$

Thus in comparison with (2.11), we deduce

$$\frac{|\gamma|}{n+|\gamma|} = \frac{|b|}{n+|b|} B_{n+p} \tag{2.13}$$

which implies (2.8). □

Setting $\mu = 1$ and $\mu = 0$, respectively in Theorem (2.2), we get the following corollaries:

**Corollary 2.3.** *If $f(z) \in R^p_{n,m}(\lambda, b)$, then $P^\delta_{c,p} f(z) \in R^p_{n,m}(\lambda,\gamma_1)$, where*

$$|\gamma_1| = \frac{n|b|(c+p)^\delta}{n(c+n+p)^\delta + |b|\left[(c+n+p)^\delta - (c+p)^\delta\right]} \tag{2.14}$$

*and $|\gamma_1| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0,\ \ \delta > 0,\ \ b \in \mathbb{C}\setminus\{0\},\ \ c > -p$ and $0 \le \lambda \le 1$. The result is sharp.*

**Corollary 2.4.** *If $f(z) \in R^p_{n,m}(\lambda, b)$, then $Q^\delta_{c,p} f(z) \in R^p_{n,m}(\lambda,\gamma_2)$, where*

$$|\gamma_2| = \frac{n|b|(c+p)_n}{n(c+p+\delta)_n + |b|\left[(c+p+\delta)_n - (c+p)_n\right]} \tag{2.15}$$

*and $|\gamma_2| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0,\ \ \delta > 0,\ \ b \in \mathbb{C}\setminus\{0\},\ \ c > -p$ and $0 \le \lambda \le 1$. The result is sharp.*

Setting $\delta = 1$ in either (2.14) or (2.15), we get the result recently obtained by Güney and Breaz [1, Theorem 1]. Also, if we put $\delta = 1,\ \ p = 1,\ \ m = 0,\ \ b = 1-\alpha$ and $\gamma = 1-\beta$, we obtain the result established by Sălăgean [5]. Several special cases can be obtained by specifying the parameters in Theorem (2.2).

## 3. Integral properties of the class $L^p_{n,m}(\lambda, b)$

**Theorem 3.1.** *If $f(z) \in L^p_{n,m}(\lambda, b)$, then $H(z) \in L^p_{n,m}(\lambda, \beta)$, where*

$$\left|\beta\right| = (|b|-1)\, B_{n+p} + \frac{p!}{(p-m)!}\left[\lambda(p-m-1)+1\right]\left(B_{n+p}-1\right)+1 \tag{3.1}$$

*and $|\beta| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0$ $\delta > 0,\ b \in \mathbb{C} \setminus \{0\},\ c > -p,\ 0 \le \mu \le 1,\ 0 \le \lambda \le 1$ and $B_{n+p}$ is given by (2.1). The result is sharp.*

**Proof.** Following similar steps to that in the proof of Theorem (2.2), we can get the result □

Setting $\mu = 1$ and $\mu = 0$, respectively in Theorem 3.1, we get the following corollaries:

**Corollary 3.2.** *If $f(z) \in L^p_{n,m}(\lambda, b)$, then $P^{\delta}_{c,p} f(z) \in L^p_{n,m}(\lambda, \beta_1)$, where*

$$\left|\beta_1\right| = (|b|-1)\left(\frac{c+p}{c+n+p}\right)^{\delta} + \frac{p!}{(p-m)!}\left[\lambda(p-m-1)+1\right]\left(\left(\frac{c+p}{c+n+p}\right)^{\delta}-1\right)+1 \tag{3.2}$$

*and $|\beta_1| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0,\ \ \delta > 0,\ \ b \in \mathbb{C} \setminus \{0\},\ \ c > -p$ and $0 \le \lambda \le 1$. The result is sharp.*

**Corollary 3.3.** *If $f(z) \in L^p_{n,m}(\lambda, b)$, then $Q^{\delta}_{c,p} f(z) \in L^p_{n,m}(\lambda, \beta_2)$, where*

$$\left|\beta_2\right| = (|b|-1)\,\frac{(c+p)_n}{(c+p+\delta)_n} + \frac{p!}{(p-m)!}\left[\lambda(p-m-1)+1\right]\left(\frac{(c+p)_n}{(c+p+\delta)_n}-1\right)+1 \tag{3.3}$$

*and $|\beta_2| < |b|$, provided that $p, n \in \mathbb{N}, m \in \mathbb{N}_0,\ \ \delta > 0,\ \ b \in \mathbb{C} \setminus \{0\},\ \ c > -p$ and $0 \le \lambda \le 1$. The result is sharp.*

Setting $\delta = 1$ in either (3.2) or (3.3), we get the result recently obtained by Güney and Breaz [1, Theorem 2]. More particular results can be obtained by specifying the parameters in Theorem 3.1.

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