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SOFT $K(G)$-ALGEBRAS

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Abstract. The concept of soft sets, introduced by Molodtsov [4] is a mathematical tool for dealing with uncertainties, that is free from the difficulties that have troubled the traditional theoretical approaches. In this paper, we apply the notion of the soft sets of Molodtsov to the theory of $K(G)$-algebras. The notion of soft $K(G)$-algebras, idealistic soft $K(G)$-algebras, soft subalgebras and soft ideals are introduced, and their basic properties are investigated.

1. Introduction

Molodtsov [4] introduced the concept of soft set, which can be seen a new mathematical tool for dealing with uncertainty. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields. For example, the study of smoothness of functions, game theory, operations research, probability, the theory of measurement and so on. At present, work on soft set theory is progressing rapidly. Maji et al. [15] described the application of soft set theory to a decision making problem. In theoretical aspects, Maji et al. [16] defined several operations on soft sets. Chen et al. [3] presented a new definition of soft set parameterization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. Some results on an application of fuzzy-soft-sets in a decision making problem have been given by Roy et al. [1]. Also, some new operations in soft set theory have been given by Irfan Ali et al. [13].

The notion of a $K(G)$-algebra $(G,\cdot,\odot,e)$ was first introduced by Dar and Akram [9] in 2003 and published in 2005. A $K(G)$-algebra built on a group $(G,\cdot)$ by adjoining induced binary operation $\odot$ defined by $x\odot y = x\cdot y^{-1}$ for all $x, y \in G$, where e is the identity of the group. It is attached to an abstract $K(G)$-algebra $(G,\odot,e)$, which is non-commutative and non-associative with right identity element e. Recently, Dar and Akram have proved in [10] that the class of $K(G)$-algebras is a generalized class of B-algebras [7] when G is a non-abelian group, and they also have proved that the $K(G)$-algebra is a generalized class of the class of $BCH/BCI/BCK$-algebras [11], [21] and [17] when G is an abelian group. The algebraic structure of soft sets

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has been studied by some authors. Aktaş et al. [6] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets. They also discussed the notion of soft groups. Jun [18] introduced and investigated the notion of soft BCK/BCI-algebras. Jun and Park [19] discussed the applications of soft sets in the ideal theory of BCK/BCI-algebras. Feng et. al. [5] introduced the notions of soft ideals and idealistic soft semirings. Park et al. [2] also studied several operations on the theory of soft sets. Jun et al. [20] applied the notion of soft sets by Molodtsov to the theory of BCC-algerbas, and introduced the notion of soft BCC-algebras. Kazancı et al. [14] introduced the notions of soft BCH-algebras and soft BCH-homomorphisms.

In this paper, we apply the notion of the soft sets of Molodtsov to the theory of $K(G)$-algebras. We introduce the notion of soft $K(G)$-algebras, idealistic soft $K(G)$-algebras, soft subalgebras and soft ideals, and then derive their basic properties. Moreover, the notion of soft $K(G)$-homomorphism is introduced and its basic properties are investigated.

2. Basic results on K(G)-algebras

In this section we give review of $K(G)$-algebras.

Definition 2.1 ([9])**.** Let $(G, \cdot, e)$ be a group with the identity e such that $x^2 \neq e$ for some $x(\neq e) \in G$. A *right algebra built on G* (briefly, $K(G)$-algebra) is a structure $\mathscr{G} = (G, \cdot, \odot, e)$, where "$\odot$" is a binary operation on G which is derived from the operation "$\cdot$", that satisfies the following:

$(k1)$ $(\forall a, x, y \in G)$ $((a \odot x) \odot (a \odot y) = (a \odot (y^{-1} \odot x^{-1})) \odot a)$,

$(k2)$ $(\forall a, x \in G)$ $(a \odot (a \odot x) = (a \odot x^{-1}) \odot a)$,

$(k3)$ $(\forall a \in G)$ $(a \odot a = e,\ a \odot e = a,\ e \odot a = a^{-1})$.

If the group G is abelian then the axioms $(k1)$ and $(k2)$ of $K(G)$-algebra change to be $(k1^{'})$ and $(k2^{'})$ respectively where

$(k1^{'})$ $(\forall a, x, y \in G)$ $((a \odot x) \odot (a \odot y) = y \odot x)$,

$(k2^{'})$ $(\forall a, x \in G)$ $(a \odot (a \odot x) = x)$

Definition 2.2 ([9])**.** A nonempty subset H of a $K(G)$-algebra $\mathscr{G}$ is called a subalgebra of $\mathscr{G}$ if it satisfies:

(i) $(\forall a, b \in H)$ $(a \odot b \in H)$,

(ii) $e \in H$.

Definition 2.3 ([12])**.** A nonempty subset I of a $K(G)$-algebra $\mathscr{G}$ is called an ideal of $\mathscr{G}$ if it satisfies:

(i) $e \in I$,

(ii) $(\forall x, y \in G)\ (x \odot y \in I,\ y \odot (y \odot x) \in I \Longrightarrow x \in I)$.

Definition 2.4 ([8]). A mapping $\Psi : \mathscr{X} \to \mathscr{Y}$ of $K(G)$-algebras is called a homomorphism if $\Psi(x \odot y) = \Psi(x) \odot \Psi(y)$ for all $x, y \in \mathscr{X}$.

Proposition 2.5 ([8]). *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two $K(G)$-subalgebras (ideals) of a $K(G)$-algebra $(G, \odot, e)$ then $\mathscr{H}_1 \cap \mathscr{H}_2$ is a $K(G)$-subalgebra (ideal) of $(G, \odot, e)$.*

Proposition 2.6 ([8]). *Let $\mathscr{K}_1 = (G_1, \odot, e_1)$ and $\mathscr{K}_2 = (G_2, \odot, e_2)$ be two $K(G)$-algebras and $\Psi \in Hom(\mathscr{K}_1, \mathscr{K}_2)$. Then*

(i) *If H is a subalgebra of $\mathscr{K}_1$ then $\Psi(H)$ is a subalgebra of $\mathscr{K}_2$.*

(ii) *If H is an ideal of $\mathscr{K}_1$ then $\Psi(H)$ is an ideal of $\mathscr{K}_2$.*

3. Basic results on soft sets

Molodtsow [4] defined the notion of a soft set in the following way: Let U be an initial universe set and E be a set of parameters. The power set of U is denoted by $\mathscr{P}(U)$ and A is a subset of E.

Definition 3.1 ([4]). A pair $(\mathfrak{F}, A)$ is called a soft set over U, where $\mathfrak{F}$ is a mapping given by $\mathfrak{F} : A \to \mathscr{P}(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $\mathfrak{F}(x)$ may be considered as the set of x-approximate elements of the soft set $(\mathfrak{F}, A)$. Clearly, a soft set is not a classical set.

Definition 3.2 ([4]). Let $(\mathfrak{F}, A)$, $(\mathfrak{H}, B)$ be two soft sets over a common universe U.

(i) $(\mathfrak{F}, A)$ is said to be a soft subset of $(\mathfrak{H}, B)$, denoted by $(\mathfrak{F}, A) \widetilde{\subseteq} (\mathfrak{H}, B)$, if $A \subseteq B$ and $\mathfrak{F}(x) \subseteq \mathfrak{H}(x)$ for all $x \in A$.

(ii) $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ are said to be soft equal, denoted by $(\mathfrak{F}, A) = (\mathfrak{H}, B)$, if $(\mathfrak{F}, A) \widetilde{\subseteq} (\mathfrak{H}, B)$ and $(\mathfrak{H}, B) \widetilde{\subseteq} (\mathfrak{F}, A)$.

Definition 3.3 ([16]). Let $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ be two soft sets over a common universe U. The intersection of $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ is defined to be the soft set $(\mathscr{M}, C)$ satisfying the following conditions:

(i) $C = A \cap B$,

(ii) $(\forall x \in C)\ (\mathscr{M}(x) = \mathfrak{F}(x)$ or $\mathfrak{H}(x))$.

In this case, we write $(\mathfrak{F}, A) \mathbin{\widetilde{\cap}} (\mathfrak{H}, B) = (\mathscr{M}, C)$.

Definition 3.4 ([16])**.** Let $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ be two soft sets over a common universe U. The union of $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ is defined to be the soft set $(\mathcal{M}, C)$ satisfying the following conditions:

(i) $C = A \cup B$,

(ii) for all $x \in C$,

$$\mathcal{M}(x) = \begin{cases} \mathfrak{F}(x), & \text{if } x \in A - B \\ \mathfrak{H}(x), & \text{if } x \in B - A \\ \mathfrak{F}(x) \cup \mathfrak{H}(x), & \text{if } x \in A \cap B \end{cases}.$$

In this case, we write $(\mathfrak{F}, A) \,\widetilde{\cup}\, (\mathfrak{H}, B) = (\mathcal{M}, C)$.

Definition 3.5 ([16])**.** If $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ are two soft sets over a common universe U, then "$(\mathfrak{F}, A)$ AND $(\mathfrak{H}, B)$" denoted by $(\mathfrak{F}, A) \,\widetilde{\wedge}\, (\mathfrak{H}, B)$ is defined by $(\mathfrak{F}, A) \,\widetilde{\wedge}\, (\mathfrak{H}, B) = (\mathcal{M}, A \times B)$, where $\mathcal{M}(\gamma, \delta) = \mathfrak{F}(\gamma) \cap \mathfrak{H}(\delta)$ for all $(\gamma, \delta) \in A \times B$.

Definition 3.6 ([16])**.** If $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ are two soft sets over a common universe U, then "$(\mathfrak{F}, A)$ OR $(\mathfrak{H}, B)$" denoted by $(\mathfrak{F}, A) \,\widetilde{\vee}\, (\mathfrak{H}, B)$ is defined by $(\mathfrak{F}, A) \,\widetilde{\vee}\, (\mathfrak{H}, B) = (\mathcal{M}, A \times B)$, where $\mathcal{M}(\gamma, \delta) = \mathfrak{F}(\gamma) \cup \mathfrak{H}(\delta)$for all $(\gamma, \delta) \in A \times B$.

4. Soft K(G)-algebras

In what follows let X and A be a $K(G)$-algebra and a nonempty set, respectively, and R will refer to an arbitrary binary relation between an element of A and an element of X. that is, R is a subset of $A \times X$ unless otherwise specified. A set-valued function $\mathfrak{F} : A \to \mathscr{P}(X)$ can be defined as $\mathfrak{F}(x) = \{y \in X \mid xRy\}$ for all $x \in A$. The pair $(\mathfrak{F}, A)$ is then a soft set over X.

Definition 4.1. Let $(\mathfrak{F}, A)$ be a soft set over X. Then $(\mathfrak{F}, A)$ is called a soft $K(G)$-algebra over X if $\mathfrak{F}(x)$ is a subalgebra of X for all $x \in A$.

Definition 4.2. Let $(\mathfrak{F}, A)$ be a soft set over X. Then $(\mathfrak{F}, A)$ is called an idealistic soft $K(G)$-algebra over X if $\mathfrak{F}(x)$ is an ideal of X for all $x \in A$.

Example 4.3. Consider the $K(G)$-algebra $X = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3\}$ is the cyclic group of order 4 and $\odot$ is given by the following Cayley table:

$\odot$	e	a	a^2	a^3
e	e	a^3	a^2	a
a	a	e	a^3	a^2
a^2	a^2	a	e	a^3
a^3	a^3	a^2	a	e

(1) Let $(\mathfrak{F}, A)$ be a soft set over X, where $A = X$ and $\mathfrak{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{F}(x) = \left\{y \in X \mid xRy \Longleftrightarrow x \odot (x \odot y) \in \left\{e, a^2\right\}\right\}$$

for all $x \in A$. Then $\mathfrak{F}(x) = \left\{e, a^2\right\}$ for all $x \in A$. Therefore $(\mathfrak{F}, A)$ is a soft $K(G)$-algebra over X.

(2) Let $(\mathscr{M}, A)$ be a soft set over X, where $A = X$ and $\mathscr{M} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathscr{M}(x) = \left\{y \in X \mid xRy \Longleftrightarrow y \odot (x \odot y) \in \{e, a\}\right\}$$

for all $x \in A$. Then $\mathscr{M}(a) = \left\{a, a^3\right\}$. Since $e \notin \mathscr{M}(a)$, $\mathscr{M}(a)$ is not a subalgebra of X. Therefore $(\mathscr{M}, A)$ is not a soft $K(G)$-algebra over X.

Example 4.4 ([9])**.** Consider the $K(S_3)$-algebra $X = (S_3, \cdot, \odot, e)$ on the symmetric group $S_3 = \left\{e, a, b, x, y, z\right\}$, where $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, $e = (1)$, and $\odot$ is given by the following cayley table:

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

(1) Let $(\mathfrak{F}, A)$ be a soft set over X, where $A = \{a\}$ and $\mathfrak{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{F}(w) = \{s \in X \mid wRs \Longleftrightarrow w \odot (e \odot s) = s \odot (e \odot w)\}$$

for all $w \in A$. Then $\mathfrak{F}(w) = \{e, a, b\}$ for all $w \in A$. By routine calculations, it is easy to check that $\mathfrak{F}(w)$ is an ideal for all $w \in A$. Therefore $(\mathfrak{F}, A)$ is an idealistic soft $K(G)$-algebra over X.

(2) Let $(\mathfrak{L}, H)$ be a soft set over X, where $H = \{b\}$ and $\mathfrak{L} : H \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{L}(w) = \{s \in X \mid wRs \Longleftrightarrow ((s \odot (e \odot s)) \odot (e \odot s)) \odot (e \odot w) \in \{x, b\}\}$$

for all $w \in H$. Then $\mathfrak{L}(b) = \left\{e, a, b, y\right\}$ is not an ideal of X since $x \odot z = b \in \mathfrak{L}(b)$ and $z \odot (z \odot x) = y \in \mathfrak{L}(b)$, but $x \notin \mathfrak{L}(b)$. Hence $(\mathfrak{L}, H)$ is not an idealistic soft $K(G)$-algebra over X.

Theorem 4.5. *Let $(\mathfrak{F}, A)$ be a soft $K(G)$-algebra over X. If B is a subset of A, then $(\mathfrak{F}_{|_B}, B)$ is a soft $K(G)$-algebra over X.*

Proof. Straightforward. □

Example 4.6. Consider the soft set $(\mathscr{M}, A)$ given in Example 4.3.(2). We know that $(\mathscr{M}, A)$ is not a soft $K(G)$-algebra over X. But if we take $B = \{e, a^3\} \subset A$, then $(\mathscr{M}_{|_B}, B)$ is a soft $K(G)$-algebra over X.

Theorem 4.7. *Let $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ be two soft $K(G)$-algebras (resp. idealistic soft $K(G)$-algebras) over X. If $A \cap B = \emptyset$, then $(\mathfrak{F}, A) \tilde{\cap} (\mathfrak{H}, B)$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over X.*

Proof. Using Definition 3.3, we can write $(\mathfrak{F}, A) \tilde{\cap} (\mathfrak{H}, B) = (\mathscr{M}, C)$, where $C = A \cap B$ and $\mathscr{M}(x) = \mathfrak{F}(x)$ or $\mathfrak{H}(x)$ for all $x \in C$. Clearly, $(\mathscr{M}, C)$ is a soft set over X. Since $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ are two soft $K(G)$-algebras (resp. idealistic soft $K(G)$-algebras) over X, $\mathscr{M}(x) = \mathfrak{F}(x)$ is a subalgebra (resp. an ideal) over X or $\mathscr{M}(x) = \mathfrak{H}(x)$ is a subalgebra (resp. an ideal) over X for all $x \in C$. Therefore, $(\mathscr{M}, C) = (\mathfrak{F}, A) \tilde{\cap} (\mathfrak{H}, B)$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over X. □

Theorem 4.8. *Let $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ be two soft $K(G)$-algebras (resp. idealistic soft $K(G)$-algebras) over X. If $A \cap B = \emptyset$, then $(\mathfrak{F}, A) \tilde{\cup} (\mathfrak{H}, B)$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over X.*

Proof. Using Definition 3.4, we can write $(\mathfrak{F}, A) \tilde{\cup} (\mathfrak{H}, B) = (\mathscr{M}, C)$, where $C = A \cup B$ and for all $x \in C$,

$$\mathscr{M}(x) = \begin{cases} \mathfrak{F}(x), & \text{if } x \in A - B \\ \mathfrak{H}(x), & \text{if } x \in B - A \\ \mathfrak{F}(x) \cup \mathfrak{H}(x), & \text{if } x \in A \cap B \end{cases}.$$

Since $A \cap B = \emptyset$, $x \in A - B$ or $x \in B - A$ for all $x \in C$. If $x \in A - B$, then $\mathscr{M}(x) = \mathfrak{F}(x)$ is a subalgebra (resp. an ideal) of X. If $x \in B - A$, $\mathscr{M}(x) = \mathfrak{H}(x)$ is a subalgebra (resp. an ideal) of X. Therefore, $(\mathscr{M}, C) = (\mathfrak{F}, A) \tilde{\cup} (\mathfrak{H}, B)$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over X. □

The following example shows that Theorem 4.8 is not true if omit the condition that A and B are disjoint sets.

Example 4.9. (1) Consider the $K(G)$-algebra $X = (G, \cdot, \odot, e)$, where $G = \mathbb{Z}_6$ is the cyclic group of order 6 generated by a. For any element x of X, we define the order of x, denoted by $\circ(x)$, as $\circ(x) = \min\{n \in \mathbb{N} \mid x^n = e\}$. Let $(\mathfrak{F}, A)$ be a soft set over X, where $A = \{a^2, a^3, a^4\}$ and $\mathfrak{F} : A \to \mathscr{P}(\mathbb{Z}_6)$ is a set-valued function defined by

$$\mathfrak{F}(x) = \{e\} \cup \{y \in \mathbb{Z}_6 \mid xRy \Longleftrightarrow \circ(x) = \circ(y)\}$$

for all $x \in A$. Then $\mathfrak{F}(a^2) = \{e, a^2, a^4\}$, $\mathfrak{F}(a^3) = \{e, a^3\}$, $\mathfrak{F}(a^4) = \{e, a^2, a^4\}$ are subalgebras of X. Hence $(\mathfrak{F}, A)$ is a soft $K(G)$-algebra over X. Let $(\mathfrak{H}, B)$ be a soft set over X, where $B = \{a^3\}$ and $\mathfrak{H} : B \to \mathscr{P}(\mathbb{Z}_6)$ is a set-valued function defined by

$$\mathfrak{H}(x) = \{e\} \cup \left\{y \in \mathbb{Z}_6 \mid xRy \Longleftrightarrow x \odot (e \odot y) \in \left\{a, a^5\right\}\right\}$$

for all $x \in B$. Then $\mathfrak{H}(a^3) = \left\{e, a^2, a^4\right\}$ is a subalgebras of X. Hence $(\mathfrak{H}, B)$ is a soft $K(G)$-algebra over X. Nevertheless, $(\mathcal{M}, C) = (\mathfrak{F}, A) \,\widetilde{\cup}\, (\mathfrak{H}, B)$ is not a soft $K(G)$-algebra over X since $\mathcal{M}(a^3) = \left\{e, a^2, a^3, a^4\right\}$ is not a subalgebra of X.

(2) Consider the soft set $(\mathfrak{F}, A)$ given in Example 4.4.(1). We know that $(\mathfrak{F}, A)$ is an idealistic soft $K(G)$-algebra over X. Let $(\mathfrak{H}, C)$ be a soft set over X, where $C = \{a, x\}$ and $\mathfrak{H} : C \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{H}(w) = \{s \in X \mid wRs \Longleftrightarrow ((w \odot (e \odot w)) \odot (e \odot w)) \odot (e \odot s) \in \{e, x\}\}$$

for all $w \in C$. Then $\mathfrak{H}(w) = \{e, x\}$ is an ideal of X. Hence $(\mathfrak{H}, C)$ is an idealistic soft $K(G)$-algebra over X. Nevertheless, $(\mathcal{N}, D) = (\mathfrak{F}, A) \,\widetilde{\cup}\, (\mathfrak{H}, C)$ is not an idealistic soft $K(G)$-algebra over X since $\mathcal{N}(a) = \{e, x, a, b\}$ is not an ideal of X.

Theorem 4.10. *If $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ are soft $K(G)$-algebras (resp. idealistic soft $K(G)$-algebras) over X, then $(\mathfrak{F}, A) \,\widetilde{\wedge}\, (\mathfrak{H}, B)$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over X.*

Proof. By Definition 3.5, we can write $(\mathfrak{F}, A) \,\widetilde{\wedge}\, (\mathfrak{H}, B) = (\mathcal{M}, A \times B)$, where $\mathcal{M}(\gamma, \delta) = \mathfrak{F}(\gamma) \cap \mathfrak{H}(\delta)$ for all $(\gamma, \delta) \in A \times B$. Since $\mathfrak{F}(\gamma)$ and $\mathfrak{H}(\delta)$ are subalgebras (resp. ideals) of X for all $\gamma \in A$ and $\delta \in B$, $\mathfrak{F}(\gamma) \cap \mathfrak{H}(\delta)$ is also a subalgebra (resp. an ideal) of X. So $\mathcal{M}(\gamma, \delta)$ is a subalgebra (resp. an ideal) of X for all $(\gamma, \delta) \in A \times B$. Therefore $(\mathfrak{F}, A) \,\widetilde{\wedge}\, (\mathfrak{H}, B)$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over X. □

Definition 4.11. Let $(\mathfrak{F}, A)$ be a soft $K(G)$-algebra over X.

(i) $(\mathfrak{F}, A)$ is called trivial soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over X if $\mathfrak{F}(\gamma) = \{e\}$ for all $\gamma \in A$.

(ii) $(\mathfrak{F}, A)$ is called whole soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over X if $\mathfrak{F}(\gamma) = X$ for all $\gamma \in A$.

Example 4.12. (1) Consider the $K(G)$-algebra $X = (G, \cdot, \odot, e)$, where G is any group. Let $(\mathfrak{F}, A)$ be a soft set over X, where A is any proper normal subgroup of G and $\mathfrak{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{F}(x) = \left\{y \in X \mid xRy \Longleftrightarrow y \odot (y \odot x) \in A\right\}$$

for all $x \in A$. Then $\mathfrak{F}(x) = X$ for all $x \in A$. Therefore $(\mathfrak{F}, A)$ is a whole soft $K(G)$-algebra over X.

(2) Consider the $K(G)$-algebra $X = (G, \cdot, \odot, e)$, where G is any group. Let $(\mathfrak{F}, A)$ be a soft set over X, where $A = X$ and $\mathfrak{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{F}(x) = \left\{y \in X \mid xRy \Longleftrightarrow x \odot (e \odot y) = x\right\}$$

for all $x \in A$. Then $\mathfrak{F}(x) = \{e\}$ for all $x \in A$. Therefore $(\mathfrak{F}, A)$ is a trivial soft $K(G)$-algebra over X.

Let $\mathscr{X}, \mathscr{Y}$ be two $K(G)$-algebras and $\Psi : \mathscr{X} \to \mathscr{Y}$ a mapping of $K(G)$-algebras. For a soft set $(\mathfrak{F}, A)$ over $\mathscr{X}$, $(\Psi(\mathfrak{F}), A))$ is a soft over $\mathscr{Y}$ where $\Psi(\mathfrak{F}) : A \to \mathscr{P}(\mathscr{Y})$ is defined by $\Psi(\mathfrak{F})(x) = \Psi(\mathfrak{F}(x))$ for all $x \in A$.

Theorem 4.13. *Let $\Psi : \mathscr{X} \to \mathscr{Y}$ be an onto homomorphism of $K(G)$-algebras. If $(\mathfrak{F}, A)$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) of $\mathscr{X}$, then $(\Psi(\mathfrak{F}), A))$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) of $\mathscr{Y}$.*

Proof. Since $\mathfrak{F}(x)$ is a subalgebra (resp. an ideal) of $\mathscr{X}$ for all $x \in A$, its onto homomorphic image $\Psi(\mathfrak{F}(x))$ is a subalgebra (resp. an ideal) of $\mathscr{Y}$ for all $x \in A$. Therefore $(\Psi(\mathfrak{F}), A))$ is a soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) of $\mathscr{Y}$. □

Theorem 4.14. *Let $\Psi : \mathscr{X} \to \mathscr{Y}$ be a homomorphism of $K(G)$-algebras. Let $(\mathfrak{F}, A)$ be a soft $K(G)$-algebra (resp. an idealistic soft $K(G)$-algebras) over X.*

- (i) *If $\mathfrak{F}(x) = Ker(\Psi)$ for all $x \in A$, then $(\Psi(\mathfrak{F}), A)$ is the trivial soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over $\mathscr{Y}$.*
- (ii) *If Ψ is onto and $(\mathfrak{F}, A)$ is whole, then $(\Psi(\mathfrak{F}), A)$ is the whole soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over $\mathscr{Y}$.*

Proof.

- (i) Assume that $\mathfrak{F}(x) = Ker(\Psi)$ for all $x \in A$. Then $\Psi(\mathfrak{F})(x) = \Psi(\mathfrak{F}(x)) = \{e_{\mathscr{Y}}\}$ for all $x \in A$. It follows from Theorem 4.13 and Definition 4.11 that $(\Psi(\mathfrak{F}), A)$ is the trivial soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over $\mathscr{Y}$.
- (ii) Suppose that Ψ is onto and $(\mathfrak{F}, A)$ is whole. Then $\mathfrak{F}(x) = \mathscr{X}$ for all $x \in A$, and so $\Psi(\mathfrak{F})(x) = \Psi(\mathfrak{F}(x)) = \Psi(\mathscr{X}) = \mathscr{Y}$ for all $x \in A$. It follows from Theorem 4.13 and Definition 4.11 that $(\Psi(\mathfrak{F}), A)$ is the whole soft $K(G)$-algebra (resp. idealistic soft $K(G)$-algebra) over $\mathscr{Y}$. □

Definition 4.15. Let $(\mathfrak{F}, A)$ and $(\mathfrak{H}, B)$ be two soft $K(G)$-algebras over X. Then $(\mathfrak{F}, A)$ is called a soft subalgebra (resp. soft ideal) of $(\mathfrak{H}, B)$, denoted by $(\mathfrak{F}, A)\tilde{<}(\mathfrak{H}, B)$ (resp. $(\mathfrak{F}, A)\tilde{\triangleleft}(\mathfrak{H}, B)$, if it satisfies

- (i) $A \subset B$,
- (ii) $\mathfrak{F}(x)$ is a subalgebra (resp. an ideal) of $\mathfrak{H}(x)$ for all $x \in A$.

Example 4.16. Consider the $K(Q_8)$-algebra $X = (Q_8, \cdot, \odot, e)$ on the Quaternion group $Q_8 = \left\{-1, 1, i, -i, j, -j, k, -k\right\}$ and $\odot$ is given by the following Cayley table:

$\odot$	-1	1	i	$-i$	j	$-j$	k	$-k$
-1	1	-1	i	$-i$	j	$-j$	k	$-k$
1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	$-i$	i	1	-1	$-k$	k	j	$-j$
$-i$	i	$-i$	-1	1	k	$-k$	$-j$	j
j	$-j$	j	k	$-k$	1	-1	$-i$	i
$-j$	j	$-j$	$-k$	k	-1	1	i	$-i$
k	$-k$	k	$-j$	j	i	$-i$	1	-1
$-k$	k	$-k$	j	$-j$	$-i$	i	-1	1

(1) Let $(\mathfrak{F}, A)$ be a soft set over X, where $A = \{i, -i\}$ and $\mathfrak{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{F}(x) = \left\{y \in X \mid xRy \Longleftrightarrow y \odot (y \odot x) \in \{i, -i\}\right\}$$

for all $x \in A$. Then $\mathfrak{F}(x) = X$ for all $x \in A$. Therefore $(\mathfrak{F}, A)$ is a soft $K(G)$-algebra over X. Now let $(\mathfrak{H}, B)$ be a soft set over X, where $B = \{i\}$ and $\mathfrak{H} : B \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{H}(x) = \left\{y \in X \mid xRy \Longleftrightarrow (x \odot (e \odot x)) \odot (e \odot y) \in \{-1, 1\}\right\}$$

for all $x \in B$. Then $\mathfrak{H}(i) = \{-1, 1\}$. Therefore $(\mathfrak{H}, B)$ is a soft $K(G)$-algebra over X. It is easy to verify that $\mathfrak{H}(x)$ is a subalgebra of $\mathfrak{F}(x)$ for all $x \in B$. Hence $(\mathfrak{H}, B) \widetilde{<} (\mathfrak{F}, A)$.

(2) Let $(\mathfrak{J}, E)$ be a soft set over X, where $E = \left\{i, j\right\}$ and $\mathfrak{J} : E \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{J}(x) = \left\{y \in X \mid xRy \Longleftrightarrow y \odot (e \odot x) \in \left\{i, -i, j, -j\right\}\right\}$$

for all $x \in E$. Then $\mathfrak{J}(x) = \{-1, 1, k, -k\}$ for all $x \in E$. Therefore $(\mathfrak{J}, E)$ is a soft $K(G)$-algebra over X. Now let $(\mathfrak{K}, F)$ be a soft set over X, where $F = \{i\}$ and $\mathfrak{K} : F \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathfrak{K}(x) = \left\{y \in X \mid xRy \Longleftrightarrow y \odot (e \odot x) \in \{i, -i\}\right\}$$

for all $x \in F$. Then $\mathfrak{K}(i) = \{-1, 1\}$. Therefore $(\mathfrak{K}, F)$ is a soft $K(G)$-algebra over X. It is easy to verify that $\mathfrak{K}(x)$ is an ideal of $\mathfrak{J}(x)$ for all $x \in F$. Hence $(\mathfrak{K}, F) \widetilde{\lhd} (\mathfrak{J}, E)$.

Theorem 4.17. *Let $(\mathfrak{F}, A)$ be a soft $K(G)$-algebra over X and let $(\mathfrak{J}, E)$ and $(\mathfrak{K}, F)$ be soft subalgebras (resp. soft ideals) of $(\mathfrak{F}, A)$. Then*

(i) $(\mathfrak{J},E)\widetilde{\cap}(\mathfrak{K},F)\widetilde{<}(\mathfrak{F},A)$ (*resp.* $(\mathfrak{J},E)\widetilde{\cap}(\mathfrak{K},F)\widetilde{\lhd}(\mathfrak{F},A)$).

(ii) *If* $E\cap F=\emptyset$, *then* $(\mathfrak{J},E)\widetilde{\cup}(\mathfrak{K},F)\widetilde{<}(\mathfrak{F},A)$ (*resp.* $(\mathfrak{J},E)\widetilde{\cup}(\mathfrak{K},F)\widetilde{\lhd}(\mathfrak{F},A)$).

(iii) $(\mathfrak{J},E)\widetilde{\wedge}(\mathfrak{K},F)\widetilde{<}(\mathfrak{F},A)\widetilde{\wedge}(\mathfrak{F},A)$ (*resp.* $(\mathfrak{J},E)\widetilde{\wedge}(\mathfrak{K},F)\widetilde{<}(\mathfrak{F},A)\widetilde{\wedge}(\mathfrak{F},A)$).

Proof.

(i) Using Definition 3.3, we can write $(\mathfrak{J},E)\widetilde{\cap}(\mathfrak{K},F)=(\mathfrak{L},C)$, where $C=E\cap F$ and $\mathfrak{L}(z)=\mathfrak{J}(z)$ or $\mathfrak{K}(z)$ for all $z\in C$. Let $z\in C$. Then $z\in E$ and $z\in F$. If $z\in E$, then $\mathfrak{L}(z)=\mathfrak{J}(z)$ is a subalgebra (resp. an ideal) of $\mathfrak{F}(z)$. If $z\in F$, then $\mathfrak{L}(z)=\mathfrak{K}(z)$ is a subalgebra (resp. an ideal) of $\mathfrak{F}(z)$. Hence $(\mathfrak{J},E)\widetilde{\cap}(\mathfrak{K},F)=(\mathfrak{L},C)\widetilde{<}(\mathfrak{F},A)$ (resp. $(\mathfrak{J},E)\widetilde{\cap}(\mathfrak{K},F)=(\mathfrak{L},C)\widetilde{\lhd}(\mathfrak{F},A)$).

(ii) Assume $E\cap F=\emptyset$. Using Definition 3.4, we can write $(\mathfrak{J},E)\widetilde{\cup}(\mathfrak{K},F)=(\mathfrak{L},C)$, where $C=E\cup F$ and

$$\mathfrak{L}(z)=\begin{cases}\mathfrak{J}(z), & \text{if } z\in E-F\\ \mathfrak{K}(z), & \text{if } z\in F-E\\ \mathfrak{J}(z)\cup\mathfrak{K}(z), & \text{if } z\in E\cap F\end{cases}.$$

for all $z\in C$. Let $z\in C$. Then $z\in E-F$ or $z\in F-E$. If $z\in E-F$, then $\mathfrak{L}(z)=\mathfrak{J}(z)$ is a subalgebra (resp. an ideal) of $\mathfrak{F}(z)$ since $(\mathfrak{J},E)\widetilde{<}(\mathfrak{F},A)$ (resp. $(\mathfrak{J},E)\widetilde{\lhd}(\mathfrak{F},A)$). If $z\in F-E$, then $\mathfrak{L}(z)=\mathfrak{K}(z)$ is a subalgebra (resp. an ideal) of $\mathfrak{F}(z)$ since $(\mathfrak{K},F)\widetilde{<}(\mathfrak{F},A)$ (resp. $(\mathfrak{K},F)\widetilde{\lhd}(\mathfrak{F},A)$). Since $E\cap F=\emptyset$, $\mathfrak{L}(z)$ is a subalgebra (resp. an ideal) of $\mathfrak{F}(z)$ for all $z\in C$. Therefore $(\mathfrak{J},E)\widetilde{\cup}(\mathfrak{K},F)=(\mathfrak{L},C)\widetilde{<}(\mathfrak{F},A)$ (resp. $(\mathfrak{J},E)\widetilde{\cup}(\mathfrak{K},F)=(\mathfrak{L},C)\widetilde{\lhd}(\mathfrak{F},A)$).

(iii) By Definition 3.5, we can write $(\mathfrak{J},E)\widetilde{\wedge}(\mathfrak{K},F)=(\mathscr{M},E\times F)$, where $\mathscr{M}(\gamma,\delta)=\mathfrak{J}(\gamma)\cap\mathfrak{K}(\delta)$ for all $(\gamma,\delta)\in E\times F$. Clearly, $E\times F\subset A\times A$ and $\mathscr{M}(\gamma,\delta)=\mathfrak{J}(\gamma)\cap\mathfrak{K}(\delta)$ is a subalgebra (resp. an ideal) of $\mathfrak{F}(\gamma)\cap\mathfrak{F}(\delta)$ for all $(\gamma,\delta)\in E\times F$. Therefore $(\mathfrak{J},E)\widetilde{\wedge}(\mathfrak{K},F)=(\mathscr{M},E\times F)\widetilde{<}(\mathfrak{F},A)\widetilde{\wedge}(\mathfrak{F},A)$ (resp. $(\mathfrak{J},E)\widetilde{\wedge}(\mathfrak{K},F)=(\mathscr{M},E\times F)\widetilde{\lhd}(\mathfrak{F},A)\widetilde{\wedge}(\mathfrak{F},A)$). □

Theorem 4.18. *Let* $\Psi:\mathscr{X}\to\mathscr{Y}$ *be a homomorphism of* $K(G)$*-algebras and let* $(\mathfrak{F},A)$ *and* $(\mathfrak{H},B)$ *be soft* $K(G)$*-algebras over* X. *Then*

(i) $(\mathfrak{F},A)\widetilde{<}(\mathfrak{H},B)\Rightarrow(\Psi(\mathfrak{F}),A)\widetilde{<}(\Psi(\mathfrak{H}),B)$.

(ii) $(\mathfrak{F},A)\widetilde{\lhd}(\mathfrak{H},B)\Rightarrow(\Psi(\mathfrak{F}),A)\widetilde{\lhd}(\Psi(\mathfrak{H}),B)$.

Proof.

(i) Assume $(\mathfrak{F},A)\widetilde{<}(\mathfrak{H},B)$. Let $x\in A$. Then $A\subset B$ and $\mathfrak{F}(x)$ is a subalgebra of $\mathfrak{H}(x)$. Since Ψ is a homomorphism, $\Psi(\mathfrak{F})(x)=\Psi(\mathfrak{F}(x))$ is a subalgebra of $\Psi(\mathfrak{H})(x)=\Psi(\mathfrak{H}(x))$. Therefore $(\Psi(\mathfrak{F}),A)\widetilde{<}(\Psi(\mathfrak{H}),B)$.

(ii) Assume $(\mathfrak{F},A)\widetilde{\lhd}(\mathfrak{H},B)$. Let $x\in A$. Then $A\subset B$ and $\mathfrak{F}(x)$ is an ideal of $\mathfrak{H}(x)$. Since Ψ is a homomorphism, $\Psi(\mathfrak{F})(x)=\Psi(\mathfrak{F}(x))$ is an ideal of $\Psi(\mathfrak{H})(x)=\Psi(\mathfrak{H}(x))$. Therefore $(\Psi(\mathfrak{F}),A)\widetilde{\lhd}(\Psi(\mathfrak{H}),B)$. □

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