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# NEW OSTROWSKI TYPE INEQUALITY FOR TRIPLE INTEGRALS

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**Abstract**. The main aim of this paper is to establish a new Ostrowski type inequality for triple integrals by using a fairly elementary analysis. The discrete version of the result is also given.

## 1. Introduction

The well known Ostrowski's inequality [7] can be stated as follows (see slso [6, p.469]).

Let $f : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on $(a, b)$ whose derivative $f' : (a, b) \to R$ is bounded on $(a, b)$ i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\,\|f'\|_\infty\,, \tag{1.1}$$

for all $x \in [a, b]$.

In 2000, Pachpatte [8] obtained the following Ostrowski type inequality for triple integrals.

Let $\Delta = [a, k] \times [b, m] \times [c, n]$ for $a, b, c, k, m, n$ in $R_+$ and $f(r, s, t)$ be differentiable on $\Delta$. Denote the partial derivatives by $D_1 f(r, s, t) = \frac{\partial}{\partial r} f(r, s, t)$, $D_2 f(r, s, t) = \frac{\partial}{\partial s} f(r, s, t)$, $D_3 f(r, s, t) = \frac{\partial}{\partial t} f(r, s, t)$ and $D_3 D_2 D_1 f(r, s, t) = \frac{\partial^3}{\partial t \partial s \partial r} f(r, s, t)$. Let $F(\Delta)$ be the class of continuous functions $f : \Delta \to R$ for which $D_1 f, D_2 f, D_3 f, D_3 D_2 D_1 f$ exist and are continuous on $\Delta$. For $f \in F(\Delta)$ we have

$$\Big| \int_a^k \int_b^m \int_c^n f(r, s, t)\,dtdsdr - \frac{1}{8}(k-a)(m-b)(n-c)\,[f(a, b, c) + f(k, m, n)]$$
$$+\frac{1}{4}(m-b)(n-c)\int_a^k [f(r, b, c) + f(r, m, n) + f(r, m, c) + f(r, b, n)]\,dr$$
$$+\frac{1}{4}(k-a)(n-c)\int_b^m [f(a, s, c) + f(k, s, n) + f(a, s, n) + f(k, s, c)]\,ds$$

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$$+\frac{1}{4}(k-a)(m-b)\int_c^n [f(a,b,t)+f(k,m,t)+f(k,b,t)+f(a,m,t)]\,dt$$
$$-\frac{1}{2}(k-a)\int_b^m\int_c^n [f(a,s,t)+f(k,s,t)]\,dtds$$
$$-\frac{1}{2}(m-b)\int_a^k\int_c^n [f(r,b,t)+f(r,m,t)]\,dtdr$$
$$-\frac{1}{2}(n-c)\int_a^k\int_b^m [f(r,s,c)+f(r,s,n)]\,dsdr\Bigg|$$
$$\leq \frac{1}{8}(k-a)(m-b)(n-c)\int_a^k\int_b^m\int_c^n |D_3D_2D_1 f(r,s,t)|dtdsdr. \quad (1.2)$$

In [8], the inequality (1.2) and its discrete version are established by using elementary analysis. In [13] Sofo has given a refinement of the inequality (1.2) by using Peano kernels. For Ostrowski type inequalities in several independent variables, we refer the interested readers to [1-5, 9-12]. The main purpose of the present paper is to establish a new Ostrowski type inequality involving functions of three independent variables and their partial derivatives. The discrete version of the main result is also given.

## 2. Statement of Results

In what follows $R$ denotes the set of real numbers. We use the notation $H = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ $(a_1 < b_1, a_2 < b_2, a_3 < b_3)$ for $a_1, a_2, a_3, b_1, b_2, b_3$ in $R$. If $h = h(r,s,t)$ is a differentiable function defined on $H$, then its partial derivatives are denoted by $D_1h = \frac{\partial}{\partial r}h$, $D_2h = \frac{\partial}{\partial s}h$, $D_3h = \frac{\partial}{\partial t}h$, $D_1D_2h = \frac{\partial^2}{\partial r\partial s}h$, $D_2D_3h = \frac{\partial^2}{\partial s\partial t}h$, $D_3D_1h = \frac{\partial^2}{\partial t\partial r}h$, and $D_3D_2D_1h = \frac{\partial^3}{\partial t\partial s\partial r}h$. We denote by $F(H)$ the class of continuous functions $h : H \to R$ for which $D_1h$, $D_2h$, $D_3h$, $D_1D_2h$, $D_2D_3h$ , $D_3D_1h$, $D_3D_2D_1h$ exist and are continuous on $H$. Let $N$ denote the set of natural numbers, $A = \{1, 2, \ldots, a+1\}$, $B = \{1, 2, \ldots, b+1\}$ , $C = \{1, 2, \ldots, c+1\}$ for $a, b, c$ in $N$ and $E = A \times B \times C$. For a function $h = h(x,y,z) : N^3 \to R$ we define the difference operators $\Delta_1 h = h(x+1,y,z) - h(x,y,z)$, $\Delta_2 h = h(x,y+1,z) - h(x,y,z)$, $\Delta_3 h = h(x,y,z+1) - h(x,y,z)$, $\Delta_1\Delta_2 h = \Delta_1(\Delta_2 h)$, $\Delta_2\Delta_3 h = \Delta_2(\Delta_3 h)$, $\Delta_3\Delta_1 h = \Delta_3(\Delta_1 h)$, $\Delta_3\Delta_2\Delta_1 h = \Delta_3(\Delta_2\Delta_1 h)$. We denote by $G(E)$ the class of functions $h = h(x,y,z) : E \to R$ for which $\Delta_1 h, \Delta_2 h, \Delta_3 h, \Delta_1\Delta_2 h, \Delta_2\Delta_3 h, \Delta_3\Delta_1 h, \Delta_3\Delta_2\Delta_1 h$ exist on $E$. We assume that $h(x,y,z) = 0$ for $(x,y,z) \notin E$ and also use the usual convention that, empty sum is taken to be 0.

Our main result is given in the following theorem.

**Theorem 1.** *Let $f \in F(H)$. Then*

$$\Bigg|f(x,y,z) - \Big[\frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(r,y,z)\,dr + \frac{1}{b_2-a_2}\int_{a_2}^{b_2} f(x,s,z)\,ds$$

$$
\begin{aligned}
&+\frac{1}{b_3-a_3}\int_{a_3}^{b_3} f(x,y,t)\,dt\Big] \\
&+\Big[\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(r,s,z)\,dsdr \\
&+\frac{1}{(b_1-a_1)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_3}^{b_3} f(r,y,t)\,dtdr \\
&+\frac{1}{(b_2-a_2)(b_3-a_3)}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,s,t)\,dtds\Big] \\
&-\frac{1}{(b_1-a_1)(b_2-a_2)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr\Bigg| \\
\le\ & \frac{1}{(b_1-a_1)(b_2-a_2)(b_3-a_3)} \\
&\times\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|\int_r^x\int_s^y\int_t^z D_3D_2D_1 f(u,v,w)\,dwdvdu\right| dtdsdr, \qquad (2.1)
\end{aligned}
$$

*for all* $(x,y,z)\in H$.

The following Corollary holds.

**Corollary.** *Let* $f$ *be as in Theorem* 1. *Then*

$$
\begin{aligned}
&\Bigg| f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2},\frac{a_3+b_3}{2}\right)-\Big[\frac{1}{b_1-a_1}\int_{a_1}^{b_1} f\left(r,\frac{a_2+b_2}{2},\frac{a_3+b_3}{2}\right)dr \\
&+\frac{1}{b_2-a_2}\int_{a_2}^{b_2} f\left(\frac{a_1+b_1}{2},s,\frac{a_3+b_3}{2}\right)ds+\frac{1}{b_3-a_3}\int_{a_3}^{b_3} f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2},t\right)dt\Big] \\
&+\Big[\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f\left(r,s,\frac{a_3+b_3}{2}\right)dsdr \\
&+\frac{1}{(b_1-a_1)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_3}^{b_3} f\left(r,\frac{a_2+b_2}{2},t\right)dtdr \\
&+\frac{1}{(b_2-a_2)(b_3-a_3)}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f\left(\frac{a_1+b_1}{2},s,t\right)dtds\Big] \\
&-\frac{1}{(b_1-a_1)(b_2-a_2)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr\Bigg| \\
\le\ & \frac{(b_1-a_1)(b_2-a_2)(b_3-a_3)}{64}\left\|D_3D_2D_1 f\right\|_\infty, \qquad (2.2)
\end{aligned}
$$

*where*

$$\left\|D_3D_2D_1 f\right\|_\infty=\sup_{(u,v,w)\in H}\left|D_3D_2D_1 f(u,v,w)\right|<\infty.$$

*By taking* $x=\frac{a_1+b_1}{2}, y=\frac{a_2+b_2}{2}, z=\frac{a_3+b_3}{2}$ *in* (2.1) *and simple computation, we get the desired inequality in* (2.2).

The discrete version of Theorem 1 is embodied in the following theorem.

**Theorem 2.** *Let* $f \in G(E)$. *Then*

$$\Bigg| f(k,m,n) - \left[ \frac{1}{a}\sum_{r=1}^{a} f(r,m,n) + \frac{1}{b}\sum_{s=1}^{b} f(k,s,n) + \frac{1}{c}\sum_{t=1}^{c} f(k,m,t) \right] + \left[ \frac{1}{ab}\sum_{r=1}^{a}\sum_{s=1}^{b} f(r,s,n) + \frac{1}{ac}\sum_{r=1}^{a}\sum_{t=1}^{c} f(r,m,t) + \frac{1}{bc}\sum_{s=1}^{b}\sum_{t=1}^{c} f(k,s,t) \right] - \frac{1}{abc}\sum_{r=1}^{a}\sum_{s=1}^{b}\sum_{t=1}^{c} f(r,s,t) \Bigg| \le \frac{1}{abc}\sum_{r=1}^{a}\sum_{s=1}^{b}\sum_{t=1}^{c} \left| \sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\sum_{w=t}^{n-1} \Delta_3\Delta_2\Delta_1 f(u,v,w) \right|, \tag{2.3}$$

*for all* $(k,m,n) \in E$.

## 3. Proof of Theorem 1

The proof is based on the following identity

$$I = f(x,y,z) - [f(r,y,z) + f(x,s,z) + f(x,y,t)] + [f(r,s,z) + f(r,y,t) + f(x,s,t)] - f(r,s,t), \tag{3.1}$$

for $(x,y,z), (r,s,t) \in H$, where

$$I = \int_r^x \int_s^y \int_t^z D_3D_2D_1 f(u,v,w)\, dw dv du. \tag{3.2}$$

From (3.2) it is easy to observe that

$$I = \int_r^x \int_s^y D_2D_1 f(u,v,z)\, dv du - \int_r^x \int_s^y D_2D_1 f(u,v,t)\, dv du = I_1 - I_2. \tag{3.3}$$

By simple computation we have

$$I_1 = \int_r^x \int_s^y D_2D_1 f(u,v,z)\, dv du = \int_r^x D_1 f(u,y,z)\, du - \int_r^x D_1 f(u,s,z)\, du = f(x,y,z) - f(r,y,z) - f(x,s,z) + f(r,s,z). \tag{3.4}$$

Similarly, we have

$$I_2 = f(x,y,t) - f(r,y,t) - f(x,s,t) + f(r,s,t). \tag{3.5}$$

Using (3.4) and (3.5) in (3.3) we get (3.1).

Integrating both sides of (3.1) with respect to $(r,s,t)$ over $H$ and rewriting we get

$$\begin{aligned}
& f(x,y,z)-\left[\frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(r,y,z)\,dr+\frac{1}{b_2-a_2}\int_{a_2}^{b_2} f(x,s,z)\,ds\right. \\
&\quad \left.+\frac{1}{b_3-a_3}\int_{a_3}^{b_3} f(x,y,t)\,dt\right] \\
&\quad +\left[\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(r,s,z)\,dsdr\right. \\
&\quad +\frac{1}{(b_1-a_1)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_3}^{b_3} f(r,y,t)\,dtdr \\
&\quad +\frac{1}{(b_2-a_2)(b_3-a_3)}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,s,t)\,dtds \\
&\quad \left.-\frac{1}{(b_1-a_1)(b_2-a_2)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr\right] \\
&=\frac{1}{(b_1-a_1)(b_2-a_2)(b_3-a_3)} \\
&\quad \times\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left\{\int_r^x\int_s^y\int_t^z D_3D_2D_1 f(u,v,w)\,dwdvdu\right\}dtdsdr, \qquad (3.6)
\end{aligned}$$

for $(x,y,z)\in H$. From (3.6) and using the properties of modulus and integrals, we get the desired inequality in (2.1). The proof is complete.

## 4. Proof of Theorem 2

We first prove the following identity

$$\begin{aligned}
S &= f(k,m,n)-[f(r,m,n)+f(k,s,n)+f(k,m,t)] \\
&\quad +[f(r,s,n)+f(r,m,t)+f(k,s,t)]-f(r,s,t), \qquad (4.1)
\end{aligned}$$

for $(k,m,n),(r,s,t)\in E$, where

$$S=\sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\sum_{w=t}^{n-1}\Delta_3\Delta_2\Delta_1 f(u,v,w). \qquad (4.2)$$

From (4.2), by simple calculation we have

$$\begin{aligned}
S &= \sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\left[\sum_{w=t}^{n-1}\{\Delta_2\Delta_1 f(u,v,w+1)-\Delta_2\Delta_1 f(u,v,w)\}\right] \\
&= \sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\Delta_2\Delta_1 f(u,v,n)-\sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\Delta_2\Delta_1 f(u,v,t)
\end{aligned}$$

$$= S_1 - S_2. \tag{4.3}$$

By simple calculation we have

$$\begin{aligned}
S_1 &= \sum_{u=r}^{k-1} \sum_{v=s}^{m-1} \Delta_2 \Delta_1 f(u,v,n) \\
&= \sum_{u=r}^{k-1} \left[ \sum_{v=s}^{m-1} \{\Delta_1 f(u,v+1,n) - \Delta_1 f(u,v,n)\} \right] \\
&= \sum_{u=r}^{k-1} \Delta_1 f(u,m,n) - \sum_{u=r}^{k-1} \Delta_1 f(u,s,n) \\
&= \sum_{u=r}^{k-1} \{f(u+1,m,n) - f(u,m,n)\} - \sum_{u=r}^{k-1} \{f(u+1,s,n) - f(u,s,n)\} \\
&= f(k,m,n) - f(r,m,n) - f(k,s,n) + f(r,s,n). 
\end{aligned} \tag{4.4}$$

Similarly, we have

$$S_2 = f(k,m,t) - f(r,m,t) - f(k,s,t) + f(r,s,t). \tag{4.5}$$

Using (4.4) and (4.5) in (4.3) we get (4.1).

Summing both sides of (4.1) first with respect to $t$ from 1 to $c$, then with respect to $s$ from 1 to $b$ and finally with respect to $r$ from 1 to $a$ and rewriting we have

$$\begin{aligned}
f(k,m,n) &- \left[ \frac{1}{a} \sum_{r=1}^{a} f(r,m,n) + \frac{1}{b} \sum_{s=1}^{b} f(k,s,n) + \frac{1}{c} \sum_{t=1}^{c} f(k,m,t) \right] \\
&+ \left[ \frac{1}{ab} \sum_{r=1}^{a} \sum_{s=1}^{b} f(r,s,n) + \frac{1}{ac} \sum_{r=1}^{a} \sum_{t=1}^{c} f(r,m,t) + \frac{1}{bc} \sum_{s=1}^{b} \sum_{t=1}^{c} f(k,s,t) \right] \\
&- \frac{1}{abc} \sum_{r=1}^{a} \sum_{s=1}^{b} \sum_{t=1}^{c} f(r,s,t) \\
&= \frac{1}{abc} \sum_{r=1}^{a} \sum_{s=1}^{b} \sum_{t=1}^{c} \left\{ \sum_{u=r}^{k-1} \sum_{v=s}^{m-1} \sum_{w=t}^{n-1} \Delta_3 \Delta_2 \Delta_1 f(u,v,w) \right\},
\end{aligned} \tag{4.6}$$

for all $(k,m,n) \in E$. From (4.6) and using the properties of modulus and sums we get the required inequality in (2.3). The proof is complete.

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