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THE CONTINUED FRACTION EXPANSION OF EULER'S CONSTANT

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Abstract. The paper presents the continued fraction expansion of $1-\gamma$ and $\zeta(n)$.

1. Introduction and statement results

We find the expansion Euler's constant in series.

Theorem 1. *The following relation holds*

$$1-\gamma=\frac{1}{4}+\frac{1}{2\cdot 3}\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3}\end{vmatrix}+\frac{1}{3\cdot 4}\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{4}\end{vmatrix}+\frac{1}{4\cdot 5}\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5}\end{vmatrix}$$
$$+\frac{1}{5\cdot 6}\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6}\end{vmatrix}+\cdots \tag{1}$$

Proof. We transform the integral

$$-\gamma=\int_0^1 \ln|\ln x|dx$$
$$-\gamma=\int_0^1 \ln|\ln(1-x)|dx$$

and have

$$1-\gamma=\int_0^1 \ln(1+\frac{x^1}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\cdots)dx. \tag{2}$$

We need the lemma. □

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Lemma 1. *The following expansion holds*

$$\begin{aligned}
&\ln(1 + A_1x^1 + A_2x^2 + A_3x^3 + \cdots) \\
&= A_1x^1 + \frac{1}{2}\begin{vmatrix} 1 & A_1 \\ A_1 & 2A_2 \end{vmatrix} x^2 + \frac{1}{3}\begin{vmatrix} 1 & A_1 & A_2 \\ 0 & 1 & A_1 \\ A_1 & 2A_2 & 3A_3 \end{vmatrix} x^3 \\
&+ \frac{1}{4}\begin{vmatrix} 1 & A_1 & A_2 & A_3 \\ 0 & 1 & A_1 & A_2 \\ 0 & 0 & 1 & A_1 \\ A_1 & 2A_2 & 3A_3 & 4A_4 \end{vmatrix} x^4 + \cdots
\end{aligned} \tag{3}$$

Proof of Lemma 1. Let

$$\ln(1 + A_1x^1 + A_2x^2 + A_3x^3 + \cdots) = b_0x^1 + \frac{1}{2}b_1x^2 + \frac{1}{3}b_2x^3 + \cdots$$

then

$$A_1 + 2A_2x^1 + 3A_3x^2 + \cdots = (1 + A_1x^1 + A_2x^2 + A_3x^3 + \cdots)(b_0 + b_1x^1 + b_2x^2 + b_3x^3 + \cdots).$$

The coefficient b_n we find by solving a system of equations

$$\begin{cases}
b_0 &= A_1 \\
b_0A_1 + b_1 &= 2A_2 \\
b_0A_2 + b_1A_1 + b_2 &= 3A_3 \\
\vdots &= \vdots \\
b_0A_n + b_1A_{n-1} + b_2A_{n-2} + \cdots + b_n &= (n+1)A_{n+1}.
\end{cases}$$

Hence

$$b_n = \begin{vmatrix}
1 & A_1 & A_2 & A_3 & \cdots & & A_n \\
0 & 1 & A_1 & A_2 & \cdots & & A_{n-1} \\
0 & 0 & 1 & A_1 & \cdots & & A_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \cdots & & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & A_1 \\
A_1 & 2A_2 & 3A_3 & 4A_4 & \cdots & & (n+1)A_{n+1}
\end{vmatrix}.$$

The proves the Lemma 1. □

Application of (3) into (2) gives (1).
We use now the well known formula

$$\begin{aligned}
&a_1 + a_1a_2 + a_1a_2a_3 + a_1a_2a_3a_4 + \cdots \\
&= \cfrac{a_1}{1 - \cfrac{a_2}{1 + a_2 - \cfrac{a_3}{1 + a_3 - \cfrac{a_4}{1 + a_4 - \cdots}}}}.
\end{aligned} \tag{4}$$

The restriction: The above series is convergent.

Hence the formula (1) can be written in the following form

$$1-\gamma=\cfrac{\frac{1}{4}}{1-\cfrac{\frac{1}{2\cdot3}\begin{vmatrix}1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}\end{vmatrix}}{\frac{1}{4}+\frac{1}{2\cdot3}\begin{vmatrix}1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}\end{vmatrix}-\cfrac{\frac{1}{4}\cdot\frac{1}{3\cdot4}\begin{vmatrix}1&\frac{1}{2}&\frac{1}{3}\\ 0&1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}&\frac{3}{4}\end{vmatrix}}{\frac{1}{2\cdot3}\begin{vmatrix}1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}\end{vmatrix}+\frac{1}{3\cdot4}\begin{vmatrix}1&\frac{1}{2}&\frac{1}{3}\\ 0&1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}&\frac{3}{4}\end{vmatrix}-\cfrac{\frac{1}{2\cdot3}\begin{vmatrix}1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}\end{vmatrix}\cdot\frac{1}{4\cdot5}\begin{vmatrix}1&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}\\ 0&1&\frac{1}{2}&\frac{1}{3}\\ 0&0&1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}&\frac{3}{4}&\frac{4}{5}\end{vmatrix}}{\frac{1}{3\cdot4}\begin{vmatrix}1&\frac{1}{2}&\frac{1}{3}\\ 0&1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}&\frac{3}{4}\end{vmatrix}+\frac{1}{4\cdot5}\begin{vmatrix}1&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}\\ 0&1&\frac{1}{2}&\frac{1}{3}\\ 0&0&1&\frac{1}{2}\\ \frac{1}{2}&\frac{2}{3}&\frac{3}{4}&\frac{4}{5}\end{vmatrix}-\cdots}}}} \tag{5}$$

We consider the Riemann zeta function

$$\zeta(n)=\sum_{k=1}^{\infty}\frac{1}{k^n}\qquad n=2,3,4,\dots. \tag{6}$$

Application of (4) leads to the following formula

$$\zeta(n)=\cfrac{1}{1-\cfrac{1^{2n}}{2^n+1^n-\cfrac{2^{2n}}{3^n+2^n-\cfrac{3^{2n}}{4^n+3^n-\cdots}}}}. \tag{7}$$

The formula (7) is the special case of the expansion

$$\zeta(n;a)=\cfrac{1}{a^n-\cfrac{a^{2n}}{(a+1)^n+a^n-\cfrac{(a+1)^{2n}}{(a+2)^n+(a+1)^n-\cfrac{(a+2)^{2n}}{(a+3)^n+(a+2)^n-\cdots}}}}. \tag{8}$$

where

$$\zeta(n;a)=\sum_{k=0}^{\infty}\frac{1}{(a+k)^n}\qquad a>0.$$

Proof of (8). It is enough to show that the expression $a^n-\frac{1}{\zeta(n;a)}$ satisfies the recurrential relation

$$a^n-\frac{1}{\zeta(n;a)}=\frac{a^{2n}}{(a+1)^n+a^n-\left[(a+1)^n-\frac{1}{\zeta(n;a+1)}\right]}. \tag{9}$$

We have

$$\begin{aligned}\frac{a^{2n}}{(a+1)^n + a^n - \left[(a+1)^n - \frac{1}{\zeta(n;a+1)}\right]} &= \frac{a^n\Big(a^n + \dfrac{1}{\zeta(n;a+1)} - \dfrac{1}{\zeta(n;a+1)}\Big)}{a^n + \dfrac{1}{\zeta(n;a+1)}} \\ &= a^n - \frac{a^n}{a^n\zeta(n;a+1)+1} \\ &= a^n - \frac{1}{\zeta(n;a)}.\end{aligned}$$

Putting $a = 1$ in (8) we obtain (7).

References

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