TAMKANG JOURNAL OF MATHEMATICS
Volume 41, Number 4, 317-323, Winter 2010

ON INTEGRAL SUM LABELING OF DENSE GRAPHS

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Abstract. A graph is said to be a *sum graph* if there exists a set S of positive integers as its vertex set with two vertices adjacent whenever their sum is in S. An integral sum graph is defined just as the sum graph, the difference being that the label set S is a subset of Z instead of set of positive integers. The sum number of a given graph G is defined as the smallest number of isolated vertices which when added to G results in a sum graph. The integral sum number of G is analogous.

In this paper, we mainly prove that any connected graph G of order n with at least three vertices of degree $(n-1)$ is not an integral sum graph. We characterise the integral sum graph G of order n having exactly two vertices of degree $(n-1)$ each and hence give an alternative proof for the existence theorem of sum graphs.

1. Introduction

F. Harary [8, 9] introduced sum graphs and integral sum graphs. We denote the set of all positive integers as N^*. By the sum $u+v$, we mean the sum of the labels of the vertices u and v. The *sum graph* $G^+(S)$ of a finite subset $S \subset N^+$ is the graph (V, E) where $V = S$ and an edge $uv \in E$ if and only if the sum $u+v \in S$. A graph G is called a *sum graph* if it is isomorphic to the sum graph $G^+(S)$ of some $S \subset N^+$. The *integral sum graph* $G^+(S)$ is defined just as the sum graph, the difference being that $S \subset Z$ instead of $S \subset N^*$. The *sum number* of a given graph G, denoted by $\sigma(G)$, is defined as the smallest nonnegative integer s such that $G \cup sK_1$ is a sum graph. Analogously, the *integral sum number* of G denoted by $\zeta(G)$, is the smallest nonnegative integer s such that $G \cup sK_1$ is an integral sum graph. For convenience, an integral sum graph is written as $\int\Sigma$-graph. By definition, it is clear that $\zeta(G) \leq \sigma(G)$ for all graphs, and G is an integral sum graph iff $\zeta(G) = 0$.

It is very difficult to determine $\sigma(G)$ and $\zeta(G)$ for a given graph in general. But for some special classes of graphs, such as cycles, trees [4], complete graphs [1] and complete bipartite graphs, etc., the sum numbers have still been derived. F. Harary [9] conjectured that every tree T with $\zeta(T) = 0$ is a caterpillar. Z. Chen [2] disproved this conjecture by showing an integral sum tree that is not a caterpillar. Also, Z. Chen [2] conjectured that

Received December 28, 2009.
2000 *Mathematics Subject Classification*. 05C78.
Key words and phrases. Sum graph, sum number, integral sum graph, integral sum number.

all trees are $\int\Sigma$-graphs. The trees proved to have integral sum labeling include: paths, stars, generalised stars, caterpillars, Banana trees [14] and trees on 10 vertices or less [5].

F. Harary [9] proposed an open problem of characterising the graph G satisfying the graph equation $\zeta(G) = \sigma(G)$. It can be easily noted that any non-trivial connected $\int\Sigma$-graph does not satisfy this equation. Among the graphs known to satisfy this graph equation are: the 4-cycle C_4, the cocktail party graph $H_{2,n}$ for $n \geq 2$, $K_3 \cup K_1$, the complete graph K_n with $n \geq 4$, the complete bipartite graph $K_{n,n}$ when $n \geq 2$ and the ladder graph $P_n \times P_2$, $n \geq 2$ [13, 14].

In a sum labeling of a graph G a vertex w is said to label an edge $uv \in E(G)$ if and only if $w = u + v$. In this case the vertex w is called a working vertex. A sum labeling L is called an exclusive sum labeling with respect to a subgraph H of a graph G if none of the vertices of H are working vertices. In this case H is said to be labeled exclusively. The least number r of isolated vertices such that $G = H \cup \overline{K}$, is a sum graph and H is labeled exclusively is called the exclusive sum number $\varepsilon(H)$ of graph H [12]. Obviously $\varepsilon(H) \geq \sigma(G) \geq \zeta(G)$. More on sum graphs, integral sum graphs and exclusive sum number can be found in [1, 2, 3, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. For a detailed account on variations of sum graphs and integral sum graphs, one can refer to Gallian [5].

In [3], Z. Chen proved several properties of the integral sum labeling of the graph G with $\Delta(G) < |V(G)|-1$. In this paper, we characterise the integral sum graph G of order n having exactly two vertices of degree $(n-1)$ each. We also prove that any connected graph G with at least three vertices of degree $(n-1)$ is not an integral sum graph. Using the above characterisation, we give a simple alternate proof for the existence theorem of sum graphs, which has already been proved by Hao [6] using degree sequences.

Throughout this paper, by a graph G we mean a connected graph with n vertices unless otherwise mentioned. We denote $V_{\max}(G) = \{x \in V(G) \mid \deg(x) = |V(G)| - 1\}$.

2. Main results

We use the following result from [3].

Theorem 1.([3]) *Let G be a non trivial graph with an integral sum labeling f. Then $f(x) \neq 0$ for every vertex x of G if and only if the maximum degree $\Delta(G) < |V(G)| - 1$.*

Lemma 2. *Let f be an integral sum labeling of a graph G with $\Delta(G) = n - 1$. If $x \in V_{\max}(G)$, then for all $v \in V(G)\backslash\{x\}$, $f(v) \cdot f(x) < 0$.*

Proof. When $f(x) = 0$, the result is obvious. Let $f(x) \neq 0$. If $f(v)f(x) > 0$ for some $v \in V(G)\backslash\{x\}$, then both $f(v)$ and $f(x)$ are positive (the case when both are negative is analogous). Since $xv \in E(G)$ for every $v(\neq x)$, $f(v) + f(x)$ must be a label of some vertex and the sum is maximum for a vertex, say y. In this case, y is an isolated vertex that leads to a contradiction. □

Theorem 3. *Let f be an $\int\Sigma$-labeling of a graph G with $\Delta(G) = n - 1$, $n \geq 4$. If $|V_{\max}(G) \geq 2$, then*

(a) *there exists a vertex* $x \in V_{\max}(G)$ *such that* $f(x) = 0$.

(b) *For every vertex* $y \in V_{\max}(G)\backslash\{x\}$, *there exists a vertex* $z \in V(G)\backslash V_{\max}(G)$ *such that* $f(y) + f(z) = 0$ *and for every vertex* $u \in V(G)$, $f(u) = r \cdot f(y)$ *where* $r \in \{0, 1, -1, -2, \ldots, -(n-2)\}$.

Proof. That there exists a vertex $x \in V(G)$ such that $f(x) = 0$, is ensured by Theorem 1. Since f is an inegral sum labeling and $f(x) = 0$, then $x \in V_{\max}(G)$. Now, let $x, y \in V_{\max}(G)$ with $f(x) = 0$. Let $V(G) = \{x, y, v_1, v_2, \ldots, v_{n-2}\}$. Let $f(y) < 0$. Then $f(v_i) > O$ for all $i = 1, 2, \ldots, n-2$, using Lemma 2. Without loss of generality, assume that $f(v_1) > f(v_2) > \cdots > f(v_{n_2})$. Since y is adjacent to every other vertex of G we get

$$f(y) + f(v_i) = f(v_{i+1}), \qquad 1 \leq i \leq n-3 \tag{1}$$

In particular, $f(y) + f(v_{n-2}) = 0$. Put $z = v_{n-2}$. Clearly $f(z) \neq 0$. If $z \in V_{\max}(G)$, then what is true for $f(y)$ holds for $f(z)$ also. This is a contradiction to Lemma 2 as $f(z)$ and each $f(v_i)$ are of the same parity. Hence $z \notin V_{\max}(G)$. The rest of the proof is immediate using (1). □

In fact, every vertex label of G is an integral multiple of $f(y)$. This leads to the following main theorem that any graph G with $|V_{\max}(G)| \geq 3$ is not an integral sum graph.

Theorem 4. *If* $G(\neq K_3)$ *is an* $\int\Sigma$*-graph with* $\Delta(G) = |V(G)| - 1$, *then* $|V_{\max}(G)| \leq 2$.

Proof. If $|V_{\max}(G)| > 2$, we take any two vertices $u, v \in V_{\max}(G)$ with non-zero labels. By Theorem 3, every vertex label of G is an integer multiple of $f(u)$ and in particular, $f(v) = k \cdot f(u)$ where k is an integer $\neq 0$ and ± 1. Then $f(u) = (1/k)f(v)$ which is not an integral multiple of $f(v)$, a contradiction. Hence $V_{\max}(G) \leq 2$. □

Theorem 5. *Integral sum graph* G *of order* n *with* $|V_{\max}(G)| = 2$ *is unique up to isomorphism.*

Proof. Let $V(G) = \{x, y, v_1, v_2, \ldots, v_{n-2}\}$, where $V_{\max}(G) = \{x, y\}$. Let f be an integral sum labeling of G with $f(x) = 0$ and $f(y) = -a$ $(a > 0)$. Then $f(v_i) = (n-i-1)a$, $1 \leq i \leq n-2$, using (1). Two vertices v_i and v_j, $i \neq j$ are adjacent in G if and only if $f(v_i) + f(v_j) \leq (n-2)a$. That is, $(n-i-1) + (n-j-1) \leq (n-2)$ which implies that $i + j \geq n$. This adjacency condition is independent of choice of the label a. Hence the proof is complete. □

We denote the integral sum graph G of order n with $|V_{\max}(G)| = 2$ as $G\Delta_n$.

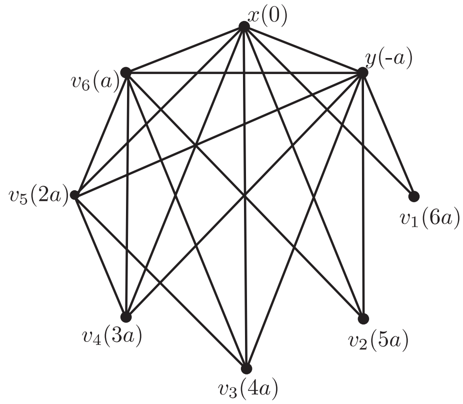

Figure. $G\Delta_8$

Theorem 6.

$$|E(G\Delta_n)| = \begin{cases} \frac{1}{4}(n^2+2n-4) & \textit{when } n \textit{ is even and} \\ \frac{1}{4}(n^2+2n-3) & \textit{when } n \textit{ is odd.} \end{cases}$$

Proof. With the earlier notations for the graph $G\Delta_n$., the degree sequence of the vertices $x, y, v_1, v_2, \ldots, v_{n-2}$, is $n-1, n-1, 2, 3, \ldots, n-3, n-2$ where the vertices $v_{(n-2)/2}$ and $v_{n/2}$ would have equal degrees $n/2$ each when n is even; or the vertices $v_{(n-1)/2}$ and $v_{(n+1)}/2$ would have equal degrees $(n+1)/2$ each when n is odd. Therefore when n is even.

$$\begin{aligned} |E(G\Delta_n)| &= \frac{1}{2}[(n-1)+(n-1)+2+3+\cdots+(n-2)/2+n/2+n/2+(n+2)/2+\cdots+(n-2)] \\ &= \frac{1}{4}(n^2+2n-4). \end{aligned}$$

Similarly, when n is odd, $|E(G\Delta_n)| = \frac{1}{4}(n^2+2n-3)$. □

Theorem 7. *The graph $G\Delta_n$, is Hamiltonion.*

Proof. Since $|V_{\max}(G)| = 2$, $G\Delta_n$, is 2-connected and we can always form a Hamiltonion cycle $xv_1\ y\ v_2\ v_{n-2}\ v_3\ \cdots\ v_{(n+4)/2}\ v_{(n-2)/2}\ v_{(n+2)/2}\ v_{n/2}\ x$, when n is even and a cycle $x\ v_1\ y\ v_2\ v_{n-2}\ v_3\ \cdots\ v_{(n+3)/2}\ V_{(n-1)/2}\ V_{(n+1)/2}\ x$ when n is odd. It can be easily verified that the edges involved in the cycle are distinct. □

Harary [9] defined a graph G_n, called Harary's sum graph, as a sum graph over a set of positive integers $N_n = \{1, 2, \ldots, n\}$. Let $< N_{n-2} >$ denote the induced subgraph of G_n induced by the set $N_{n-2} = \{1, 2, \ldots, n-2\}$. Clearly $|E(< N_{n-2} >)| = |E(G_n)| - 1$.

Theorem 8. *The induced subgraph $< N_{n-2} >$ of G_n is isomorphic to $G\Delta_{n-2}$.*

Proof. With the earlier notations, let $V(G\Delta_{n-2}) = \{x, y, v_1, v_2, \dots, v_{n-4}\}$. Consider a bijective function $f : V(< N_{n-2} >) \to V(G\Delta_{n-2})$ defined by $f(1) = x$, $f(2) = y$ and $f(i) = v_{n-i-1}$, $3 \le i \le n-2$. By construction, two vertices labeled i and j, $1 \le i$, $j \le n-2$, are adjacent in $< N_{n-2} >$ if and only if they are adjacent in G_n. This happens if and only if $i+j \le n$, Now $f(i)$ and $f(j)$ are adjacent in $G\Delta_{n-2}$ if and only if v_{n-i-1} and v_{n-j-1} are adjacent in $G\Delta_{n-2}$. This is possible if and only if $(n-i-1)+(n-j-1) \ge n-2$ (since $G\Delta_{n-2}$ is of order $n-2$). This implies that $i + j \le n$. Hence the theorem is proved. □

Theorem 9. *For the graph G_n, $|E(G_n)| = \frac{1}{2}(\binom{n}{2} - \lfloor n/2 \rfloor)$.*

Proof. $|E(G_n)| = |E(< N_{n-2} >)| + 1$. That is, $|E(G_n)| = |E(G_{n-2})| + 1$. Replacing n with $n-2$ in Theorem 6, we get $|E(G_n)| = \frac{1}{4}n(n-2)$ when n is even and $\frac{1}{4}(n-1)^2$ when n is odd. Hence $|E(G_n)| = \frac{1}{2}(\binom{n}{2} - \lfloor n/2 \rfloor)$. □

Remark 10. We have proved that $|E(G_n)| = \frac{1}{2}(\binom{n}{2} - \lfloor n/2 \rfloor)$. Equivalently we have $|E(G_n)| = \lfloor (n-1)^2/4 \rfloor$. To avoid confusion, we denote the vertex labeled i as v_i in G_n. Then $\deg(v_i) = n - i - 1$ if $i \le \lfloor n/2 \rfloor$ and $n - i$ if $i > \lfloor n/2 \rfloor$.

Now we prove in the following lemma that G_n is the sum graph on n vertices with the maximum number of edges.

Lemma 11. *Let G be a sum graph on n vertices. Then $|E(G)| \le |E(G_n)|$.*

Proof. We use induction on n. Clearly the result is true when $n = 1, 2, 3$. Assume that it is true for all graphs having less than n vertices. Let G be a sum graph on n vertices. Let $f : V(G) \to \{u_1, u_2, \dots, u_n\}$ be a vertex labeling function giving rise to sum graph structure, where $u_1 < u_2 < \dots < u_n$. Clearly the vertex with label u_n, is an isolated vertex in G. Let $e_1, e_2, \dots, e_r$ be the edges arising from the vertex label u_n. That is, $e_1 = u_{1i}u_{1j}$, where $u_{1i} + u_{1j} = u_n$. Obviously the number of such edges is $\le \lfloor (n-1)/2 \rfloor$. Let $G_1 = G \backslash \{u_n\} \backslash \{e_1, e_2, \dots, e_r\}$. Then G_1 is a sum graph on $n-1$ vertices. By induction, $|E(G_1)| \le \lfloor (n-2)^2/4 \rfloor$. Therefore, $|E(G)| \le \lfloor (n-2)^2/4 \rfloor + \lfloor (n-1)/2 \rfloor$. If $n = 2k+1$, then $|E(G)| \le k(k-1) + k = k^2 = (n-1)^2/4 = |E(G_n)|$, by Theorem 9. If $n = 2k$, then $|E(G)| \le (k-1)^2 + (k-1) = k(k-1) = n(n-2)/4 = |E(G_n)|$. Hence the proof is complete by induction. □

The following lemma will be useful to prove the existence theorem of sum graphs.

Lemma 12. *Let $H = G_{m+1}$, be the sum graph on the set $\{1, 2, \dots, m+1\}$. For i, $1 \le i \le m+1$, let H_i denote the subgraph $H \backslash \{u_i\} \backslash \{u_j u_k \mid j + k = i\}$. Then H_i is a sum graph with $|E(H_i)| = \lfloor m^2/4 \rfloor - (m - i) - \lfloor (i-1)/2 \rfloor$ if $i \le \lfloor (m+1)/2 \rfloor$ and $\lfloor m^2/4 \rfloor - (m - i + 1) - \lfloor (i-1)/2 \rfloor$ if $i > \lfloor (m+1)/2 \rfloor$.*

Proof. The number of edges through $u_i = m - i$ if $i \leq \lfloor (m+1)/2 \rfloor$ and $m - i + 1$ if $i > \lfloor (m+1)/2 \rfloor$. The number of edges arising from the label $i = \lfloor (i-1)/2 \rfloor$. Hence

$$|E(H_i)| = \begin{cases} \lfloor m^2/4 \rfloor - (m-i) - \lfloor (i-1)/2 \rfloor & \text{if } i \leq \lfloor (m+1)/2 \rfloor \quad \text{and} \\ \lfloor m^2/4 \rfloor - (m-i+1) - \lfloor (i-1)/2 \rfloor & \text{if } i > \lfloor (m+1)/2 \rfloor. \end{cases} \qquad \square$$

The above lemma underlines the fact that one of these H_i is a sum graph on m vertices with the given number of q edges such that $|E(G_{m-1})| < |E(H_i)| < -|E(G_m)|$.

Now the above lemmas give an alternate proof for the existence theorem of sum graphs proved by Hao [6].

Theorem 13. *A sum graph of order p and size q exists if and only if $0 \leq q \leq \frac{1}{2}(\binom{p}{2}) - \lfloor p/2 \rfloor)$.*

Proof. The necessary part is proved in Lemma 11. The sufficient part is proved by induction on p. Let $0 \leq q \leq \frac{1}{2}(\binom{p}{2} - \lfloor p/2 \rfloor) = \lfloor (p-1)^2/4 \rfloor$. We have to prove that there exists a sum graph on p vertices with q edges. For $p = 1, 2, 3$, the result is clear. Assume that the result is true for all integers $m < p$ and number of edges $\leq \lfloor (m-1)^2/4 \rfloor$.

If $q \leq \lfloor (p-2)^2/4 \rfloor$, then by induction hypothesis there is a sum graph on $(p-1)$ vertices with q edges. This implies that there exists a sum graph on p vertices with q edges by just adding an isolated vertex with sufficiently larger label. So we may assume $q \leq \lfloor (p-1)^2/4 \rfloor$. Now the result follows from the above Lemma 12. $\square$

Example 14. Suppose $p = 10$. We have to prove that there exists a sum graph on 10 vertices with $q \leq \lfloor 9^2/4 \rfloor = 20$ edges.

If $q < 16$, then by induction we can prove that there is a sum graph on 9 vertices with q edges and then we can add one isolated vertex with very large label. So, we may assume $16 < q \leq 20$. Now, let $G_{11} = G^+(N_{11})$ where $N_{11} = \{1, 2, \ldots, 11\}$ and each vertex labeled i is denoted as v_i. Then $|E(G_{11})| = 25$. As in Lemma 12, we construct the sum graphs H_i by dropping the vertex v_i and all the edges adding up to i. Then $|E(H_1)| = 25 - 9 = 16$, $|E(H_2)| = 25 - 8 - \lfloor 1/2 \rfloor = 17$, $|E(H_3)| = 25 - 7 - 1 = 17$, $|E(H_4)| = 18$, $|E(H_5)| = 18$, $|E(H_6)| = 18$, $|E(H_7)| = 18$, $|E(H_8)| = 19$, $|E(H_9)| = 19$, $|E(H_{10})| = 20$, $|E(H_{11})| = 20$. Thus, for every q between 16 and 20, we can obtain a sum graph as one of H_i for some i.

Harary [7, p.21] defined the join of two graphs G_1 and G_2 denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$, and all lines joining $V(G_1)$ with $V(G_2)$. Harary [9] defined an integral sum graph $G_{n,n}$ over the set of integers $\{-n, \ldots, -2, -1, 0, 1, 2, \ldots, n\}$. He specified the structure of the graphs $G_{n,n}$ in terms of G_n. That is, $G_{n,n} = K_1 + (G_n + G_n)$. In this equation, the K_1-term is realized by the integer 0, which obviously is adjacent to all other elements of S. The two G_n-terms are $G^+(\{-1, -2, \ldots, -n\})$ and $G^+(\{1, 2, \ldots, n\})$. Clearly $G_{n,n}$ is of odd order and it is the integral sum graph of order $2n + 1$. It can be easily found that $|E(G_{n,n})| = 3n(n+1)/2 - \lfloor n/2 \rfloor$, for any $n \geq 1$.

Now, let $H = K_1 + (G_{m-1} + G_m)$, where m is any positive integer. Then it can be easily verified that $|E(H)| = m(3m-1)/2$ and H is an integral sum graph of even order. The structure of $G_{n,n}$ and H defined above suggests us to propose the following conjecture:

Conjecture: An integral sum graph of order p and size q exists if and only if $q \leq 3(p^2-1)/8 - \lfloor (p-1)/4 \rfloor$ if p is odd and $q \leq p(3p-2)/8$ if p is even.

References

[1] D. Bergstr and F. Harary et al., *The sum number of a complete graph*, Bull. Malaysian Math. Soc., **12** (1989), 25–28

[2] Z. Chen, *Harary's conjectures on integral sum graphs*, Discrete Math., **160** (1996) 241–244.

[3] Z. Chen, *Integral sum graphs from identification*, Discrete Math., **181**(1998), 77–90.

[4] M. N. Ellingham, *Sum graphs from trees*, Ars Combinatoria, **35** (1993), 335–349.

[5] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, **5** (1998), #DS6.

[6] T. Hao, *On sum graphs*. J. Combina. Math. Combin. Comput., **6** (1989), 207–212.

[7] F. Harary, Graph Theory, Addison-Wersley, Reading, 1969.

[8] F. Harary, *Sum graphs and difference graphs*, Cong. Numer., **72** (1990), 101–108.

[9] F. Harary, *Sum graphs over all integers*, Discrete Math., **124** (1994), 99–105.

[10] N. Hartsfield and W. F. Smyth, *A family of sparse graphs of large sum number*, Discrete Math., **141** (1995), 163–171.

[11] M. Miller, J. Ryan, Slamin and W. F. Smyth, *Labeling wheels for minimum sum number*, JCMCC, **28** (1998), 289–297.

[12] M. Miller, J. Ryan, Slamin, K. Sugeng and M. Tuga. *Exclusive sum graph labeling*, preprint, 2003.

[13] T. Nicholas, Some labeling problems in graph theory. Ph.D. Thesis, Manonmaniam Sundaranar University, Tirunelveli, 2001.

[14] T. Nicholas et al., *Some results on sum graphs*, Journal of Combinatorics, Information & System Sciences, **26** (2001), 135–142.

[15] T. Nicholas et al., *More results on integral sum graphs*, Proceedings of the Conference on Graph Theory and Its Applications Held at Anna University, Chermai, published by Narosa publications, New Delhi, India (2004), 75–83.

[16] A. V. Pyatkin, *New formula for the sum number for the complete bipartite graphs*, Discrete Math., **239** (2001), 155–160.

[17] Y. Wang and and B. Liu, *The sum number and integral sum number of complete bipartite graph*, Discrete Math., **239**(2001), 69–82.

[18] B. Xu, *Note on integral sum graphs*, Discrete Math., **194** (1999), 285–294.

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