TAMKANG JOURNAL OF MATHEMATICS
Volume 41, Number 4, 379-392, Winter 2010

UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING TWO SETS WITH FINITE WEIGHT II

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Abstract. With the help of the notion of weighted sharing of sets we deal with the well known question of Gross and prove some uniqueness theorems on meromorphic functions sharing two sets. Our results will improve and supplement some recent results of the present author.

1. Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h)=o(T(r,h)) \qquad (r\longrightarrow\infty, r\notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z: f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z: f(z)-a=0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S)=\overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

Inspired by the Nevanlinna's three and four value theorem, in 1970s F. Gross and C.C. Yang started to study the similar but more general questions of two functions that share sets of distinct elements instead of values. For instance, they proved that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets

Received March 19, 2009.
2000 *Mathematics Subject Classification*. 30D35.
Key words and phrases. Meromorphic functions, uniqueness, weighted sharing, shared set.

such that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3$, then $f \equiv g$. In 1976 F. Gross proposed the following question in [9]:

Question A *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

In [9] Gross wrote *If the answer of* Question A *is affirmative it would be interesting to know how large both sets would have to be ?*

Yi [23] and independently Fang and Xu [8] gave the same positive answer in this direction.

Now it is natural to ask the following question [21].

Question B *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

Gradually the research on *Question B* gained pace and today it has become one of the most prominent branches of the uniqueness theory. For the last two decades a considerable amount of research work have executed by different authors to consider the shared value problems relative to a meromorphic function sharing two sets and at the same time give affirmative answers to *Question B* under weaker hypothesis {see [1]-[8], [11], [17]-[23], [28]-[31]}.

In 1994 Yi [22] gave an affirmative answer to *Question B* and prove that there exist two finite sets S_1 (with 2 elements) and S_2 (with 9 elements) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical.

Question B will naturally motivate onself to consider the uniqueness of two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$. This type of result was first obtained by Li and Yang in [19] where they proved that there exists one finite set S with 15 elements such that any two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ must be identical. In 1995, Yi [23] and independently Li and Yang [19] reduced the cardinalities of the set S from 15 to 11 to consider the uniqueness of meromorphic functions. In 1997 Fang and Guo in [7] exhibited a set S of nine elements with this property.

In 2002 Yi [27] proved the following result in which he not only reduced the cardinalities of the set S but also relaxed the sharing of the poles from CM to IM.

Theorem A.([27] see also [30]) *Let n be a positive integer such that $n \geq 8$, and let a, b be two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. Then the polynomial*

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 \tag{1.1}$$

has only simple zeros. Let $S = \{w \mid P(w) = 0\}$. If f and g are two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $\overline{E}_f(\{\infty\}) = \overline{E}_g(\{\infty\})$ then $f \equiv g$.

Dealing with the question of Gross in [7] Fang and Lahiri obtained a unique range set S with smaller cardinalities than that obtained previously imposing some restrictions on the poles of f and g.

Theorem B. ([7]) *Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n(\geq 7)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

Let $S = \{z : z^7 - z^6 - 1 = 0\}$ and

$$f = \frac{e^z + e^{2z} + \ldots + e^{6z}}{1 + e^z + \ldots + e^{6z}}, \quad g = \frac{1 + e^z + \ldots + e^{5z}}{1 + e^z + \ldots + e^{6z}}$$

Obviously $f = e^z g$, $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ but $f \not\equiv g$. So for the validity of *Theorem D*, f and g must not have any simple pole.

In 2001 an idea of gradation of sharing known as weighted sharing has been introduced in {[13], [14]} which measure how close a shared value is to being shared CM or to being shared IM. In the following definition we explain the notion.

Definition 1.1.([13, 14]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2.([13]) Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

With the notion of weighted sharing of sets improving *Theorem B*, Lahiri [17] proved the following theorem.

Theorem C.([17]) *Let S be defined as in Theorem B and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > 1$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ then $f \equiv g$.*

Recently the present author [4] has not only generalized Theorem C by investigating the problem of further relaxation of the nature of sharing the set $\{\infty\}$ in *Theorem C* but also give an exact lower bound of $\Theta(\infty; f) + \Theta(\infty; g)$ at the expense of allowing $n \geq 8$ in the same theorem in which the multiplicities of the poles cease to matter.

In [4] the present author has proved the following results.

Theorem D.([4]) *Let S be defined as in Theorem B and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)+\Theta(\infty;g)>1+\frac{29}{6nk+6n-5}$, $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ where $0\leq k<\infty$ then $f\equiv g$.*

Theorem E.([4]) *Let S be defined as in Theorem B and $n(\geq 8)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n-1}$, $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},0)=E_g(\{\infty\},0)$ then $f\equiv g$.*

Now from the above discussion the following query is natural.

(i) *Is it possible in any way to combine Theorem D and* E *to a single theorem so that Theorem D will be improved to some extent ?*

In this paper we will provide an affirmative answer to the above question. The following theorems are the main results of the paper.

Theorem 1.1. *Let S be defined as in Theorem B and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)+\Theta(\infty;g)>\max\{8-n+\frac{4}{nk+n-1},\ \frac{4}{n-1}\}$, with $\min\{\Theta(\infty;f),\Theta(\infty;f)\}>4-\frac{n}{2}+\frac{2}{nk+n-1}$, $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ where $0\leq k<\infty$ then $f\equiv g$.*

Corollary 1.1. *Let S be defined as in Theorem B and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ and $\Theta(\infty;f)+\Theta(\infty;g)>\max\{8-n,\ \frac{4}{n-1}\}$ with $\min\{\Theta(\infty;f),\Theta(\infty;f)\}>4-\frac{n}{2}$, then $f\equiv g$.*

Theorem 1.2. *Let S be defined as in Theorem B and $n(\geq 8)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)+\Theta(\infty;g)>\max\{9-n+\frac{6}{nk+n-1},\ \frac{4}{n-1}\}$, with $\min\{\Theta(\infty;f),\Theta(\infty;f)\}>\frac{9}{2}-\frac{n}{2}+\frac{3}{nk+n-1}$, $E_f(S,1)=E_g(S,1)$ and $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ where $0\leq k<\infty$ then $f\equiv g$.*

Theorem 1.3. *Let S be defined as in Theorem B and $n(\geq 12)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)+\Theta(\infty;g)>\max\{14-\frac{n}{2}+\frac{4}{nk+n-1},\ \frac{4}{n-1}\}$, with $\min\{\Theta(\infty;f),\Theta(\infty;f)\}>7-\frac{n}{2}+\frac{2}{nk+n-1}$, $E_f(S,0)=E_g(S,0)$ and $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ where $0\leq k<\infty$ then $f\equiv g$.*

Following example shows that the condition $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n-1}$ is sharp in *Theorem* 1.1 when $n\geq 8$.

Example 1.1. {Example 2, [15]} Let $f=-a\frac{1-h^{n-1}}{1-h^n}$ and $g=-ah\frac{1-h^{n-1}}{1-h^n}$, where $h=\frac{\alpha^2(e^z-1)}{e^z-\alpha}$, $\alpha=exp(\frac{2\pi i}{n})$ and $n(\geq 3)$ is an integer.

Then $T(r,f)=(n-1)T(r,h)+O(1)$ and $T(r,g)=(n-1)T(r,h)+O(1)$. Further we see that $h\neq\alpha,\alpha^2$ and a root of $h=1$ is not a pole of f and g. Hence $\Theta(\infty;f)=\Theta(\infty;g)=\frac{2}{n-1}$. Clearly f and g share $(\infty;\infty)$. Also $E_f(S,\infty)=E_g(S,\infty)$ because $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ but $f\not\equiv g$.

It is assumed that the readers are familiar with the standard definitions and notations of the value distribution theory as those are available in [10]. We are still going to explain some notations as these are used in the paper.

Definition 1.3. ([12]) For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f| = 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f| \leq m)(N(r, a; f| \geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r, a; f| \leq m)$ ($\overline{N}(r, a; f| \geq m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f| < m),\ N(r, a; f| > m),\ \overline{N}(r, a; f| < m)$ *and* $\overline{N}(r, a; f| > m)$ are defined analogously.

Definition 1.4. Let f and g be two non-constant meromorphic functions such that f and g share a value a IM where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the reduced counting function of those a-points of f and g where $p > q$ ($q > p$), by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$. Clearly when f and g share (a, m) with $m \geq 1$ then $N_E^{1)}(r, a; f) = N(r, a; f \mid= 1)$.

Definition 1.5.([10]) We denote by $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2)$

Definition 1.6.([9, 10]) Let f, g share $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 1.7.([16]) Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f| \ g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.8.([16]) Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f| \ g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H and V the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1.([26]) *If F, G be two non-constant meromorphic functions such that they share (1,0) and $H \not\equiv 0$ then*

$$N_E^{1)}(r,1;F) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let f and g be two non-constant meromorphic functions sharing $(1,m)$, where $0 \leq m < \infty$. Then*

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) - N_E^{1)}(r,1;f) + \left(m - \frac{1}{2}\right)\overline{N}_*(r,1;f,g) \leq \frac{1}{2}\left[N(r,1;f) + N(r,1;g)\right]$$

Proof. Let z_0 be a 1- point of f of multiplicity p and a 1-point of g of multiplicity q. Since f, g share $(1,m)$, we note that the 1-points of f and g up to multiplicity m are same. When $p = q = 1$, z_0 is counted once, both in left and right hand side of the above inequality but when $2 \leq p = q \leq m$, z_0 is counted 2 times in the left hand side of the above inequality whereas it is counted p times in the right hand side of the same. If $p = m+1$ then the possible values of q are as follows. (i) $q = m+1$, (ii) $q \geq m+2$. When $p = m+2$ then q can take the following possible values (i) $q = m+1$, (ii) $q = m+2$, (iii) $q \geq m+3$. Similar explanations hold if we interchange p and q. Clearly when $p = q \geq m+1$, z_0 is counted 2 times in the left hand side and $p \geq m+1$ times in the right hand side of the above inequality. When $p > q \geq m+1$, in view of *Definition* 1.6 we know z_0 is counted $m + \frac{3}{2}$ times in the left hand side and $\frac{p+q}{2} \geq m + \frac{3}{2}$ times in the right hand side of the above inequality. When $q > p$ we can explain similarly. Hence the lemma follows. □

Lemma 2.3.([16], Lemma 4) *Let F, G share $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} N(r,H) \leq\ & \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,\infty;F,G) \\ & + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Lemma 2.4.([20]) *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Let f and g be two non-constant meromorphic function and for an integer $n \geq 3$

$$F = \frac{f^{n-1}(f+a)}{-b}, \tag{2.1}$$

$$G = \frac{g^{n-1}(g+a)}{-b}. \tag{2.2}$$

Lemma 2.5. *Let F, G be given by (2.1) and (2.2) where $n \geq 7$ is an integer and $H \not\equiv 0$. If F, G share $(1, m)$ and f, g share (∞, k), where $0 \leq m < \infty$. Then*

$$\begin{aligned}\frac{n}{2}\{T(r,f)+T(r,g)\} \leq\ & 2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+N_2(r,-a;f)\\ &+N_2(r,-a;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ &+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\ &+S(r,f)+S(r,g).\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}T(r,F)+T(r,G) \leq\ & \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F)\\ &+\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\ &-N_0(r,0;F^{'})-N_0(r,0;G^{'})+S(r,F)+S(r,G).\end{aligned} \tag{2.3}$$

Using *Lemmas* 2.1, 2.2, 2.3 and 2.4 we see that

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) \leq\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N_E^{1)}(r,1;F)\\ &-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\ \leq\ & \frac{n}{2}\{T(r,f)+T(r,g)\}+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\ &+\overline{N}(r,-a;f\mid\geq 2)+\overline{N}(r,-a;g\mid\geq 2)\\ &+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\ &+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g).\end{aligned} \tag{2.4}$$

Using (2.4)) in (2.3)) the lemma follows in view of *Definition* 1.5. □

Lemma 2.6.([4], Lemma 2.8) *Let F, G be given by (2.1) and (2.2), where $n \geq 7$ is an integer. If $H \equiv 0$ then $f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2$ or $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$.*

Lemma 2.7.([17], Lemma 5) *If f, g share $(\infty, 0)$ then for $n(\geq 2)$*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Lemma 2.8.([15], Lemma 9) *Let f, g be two non-constant meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$, where $n(\geq 4)$ is an integer. Then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, a is a nonzero finite constant.*

Lemma 2.9.([24]) *If F, G share $(\infty,0)$ and $V\equiv 0$ then $F\equiv G$.*

Lemma 2.10.([4], Lemma 2.12) *Let F, G be given by (2.1) and (2.2), where $n\geq 7$ is an integer and $V\not\equiv 0$. If F, G share $(1,2)$, f, g share (∞,k), where $0\leq k<\infty$, then the poles of F and G are zeros of V and*

$$\begin{aligned}(nk+n-1)\,\overline{N}(r,\infty;f\mid\geq k+1) &= (nk+n-1)\overline{N}(r,\infty;g\mid\geq k+1)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,0;f+a)+\overline{N}(r,0;g)+\overline{N}(r,0;g+a)\\ &\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+S(r,f)+S(r,g).\end{aligned}$$

Lemma 2.11. *Let F, G be given by (2.1) and (2.2), F, G share $(1,m)$, $0\leq m<\infty$ and $\omega_1,\omega_2\ldots,\omega_n$ are the distinct roots of the equation $z^n+az^{n-1}+b=0$ and $n\geq 3$. Then*

$$\overline{N}_L(r,1;F)\leq\frac{1}{m+1}\left[\overline{N}(r,0;f)+\overline{N}(r,-a;f)-N_{\otimes}(r,0;f^{'})\right]+S(r,f),$$

where $N_{\otimes}(r,0;f^{'})=N(r,0;f^{'}\mid f\neq 0,-a,\omega_1,\omega_2\ldots\omega_n)$

Proof. We first note that $-a$ does not coincide with any of ω_i, $i=1,2,\ldots,n$. Using *Lemma* 2.4, by the first fundamental theorem we see that

$$\begin{aligned}\overline{N}_L(r,1;F) &\leq \overline{N}(r,1;F\mid\geq m+2)\\ &\leq \frac{1}{m+1}\ \big(N(r,1:F)-\overline{N}(r,1;F)\big)\\ &\leq \frac{1}{m+1}\ \left[\sum_{j=1}^{n}\big(N(r,\omega_j;f)-\overline{N}(r,\omega_j;f)\big)\right]\\ &\leq \frac{1}{m+1}\ \Big(N(r,0;f^{'}\mid f\neq 0,-a)-N_{\otimes}(r,0;f^{'})\Big)\\ &\leq \frac{1}{m+1}\ \left(N\left(r,0;\frac{f^{'}}{f(f+a)}\right)-N_{\otimes}(r,0;f^{'})\right)\\ &\leq \frac{1}{m+1}\ \left(N\left(r,\infty;\frac{f^{'}}{f(f+a)}\right)-N_{\otimes}(r,0;f^{'})\right)+S(r,f)\\ &\leq \frac{1}{m+1}\left[\overline{N}(r,0;f)+\overline{N}(r,-a;f)-N_{\otimes}(r,0;f^{'})\right]+S(r,f).\end{aligned}$$

This proves the lemma. □

Lemma 2.12.([2]) *Let F, G be given by (2.1) and (2.2), F, G share $(1,m)$, $0\leq m<\infty$ and $\omega_1,\omega_2\ldots,\omega_n$ are defined as in* Lemma 2.11. *Then*

$$\overline{N}_L(r,1;F)\leq\frac{1}{m+1}\left[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N^1_{\otimes}(r,0;f^{'})\right]+S(r,f),$$

where $N^1_{\otimes}(r,0;f^{'})=N(r,0;f^{'}\mid f\neq 0,\omega_1,\omega_2\ldots\omega_n)$.

Lemma 2.13. *Let F, G be given by (2.1) and (2.2), F, G share $(1,m)$, $0 \leq m < \infty$ and $\omega_1, \omega_2 \ldots, \omega_n$ are defined as in* Lemma 2.11*. Then*

$$\overline{N}_L(r,1;F) \leq \frac{1}{m+1}\left[\overline{N}(r,-a;f) + \overline{N}(r,\infty;f) - N^2_{\otimes}(r,0;f^{'})\right] + S(r,f),$$

where $N^2_{\otimes}(r,0;f^{'}) = N(r,0;f^{'} \mid f \neq -a, \omega_1, \omega_2 \ldots \omega_n)$.

Proof. We omit the proof since the same can be carried out along the line of proof of *Lemma* 2.12. □

Lemma 2.14. *Let F, G be given by (2.1) and (2.2), where $n \geq 7$ is an integer. Suppose S be given as in* Theorem 1.1*. If $E_f(S,0) = E_g(S,0)$. Then $S(r,f) = S(r,g)$.*

Proof. Since $E_f(S,0) = E_g(S,0)$, it follows that F and G share $(1,0)$. Suppose ω_1, $\omega_2 \ldots, \omega_n$ are the distinct roots of the equation $z^n + az^{n-1} + b = 0$. Since F, G share $(1,0)$ from the second fundamental theorem we have

$$\begin{aligned}(n-2)T(r,g) &\leq \sum_{j=1}^{n} \overline{N}\left(r,w_j;g\right) + S(r,g)\\ &= \sum_{j=1}^{n} \overline{N}\left(r,w_j;f\right) + S(r,g)\\ &\leq nT(r,f) + S(r,g).\end{aligned}$$

Similarly we can deduce

$$(n-2)T(r,f) \leq nT(r,g) + S(r,f).$$

The last inequalities imply $T(r,f) = O\left(T(r,g)\right)$ and $T(r,g) = O\left(T(r,f)\right)$ and so we have $S(r,f) = S(r,g)$. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let F and G be given by (2.1) and (2.2). Since $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ it follows that F, G share $(1,2)$ and $(\infty, nk+n-1)$. So $\overline{N}_*(r,\infty;f,g) = \overline{N}_*(r,\infty;F,G) \leq \overline{N}(r,\infty;F \mid \geq nk+n) = \overline{N}(r,\infty;f \mid \geq k+1)$. If possible let us suppose that $H \not\equiv 0$. Then $F \not\equiv G$. So from *Lemma* 2.9 we get $V \not\equiv 0$.

Hence from *Lemma* 2.5 with $m=2$ and *Lemma* 2.10 we obtain for $\varepsilon(>0)$

$$
\begin{aligned}
&\frac{n}{2}\left\{T(r,f)+T(r,g)\right\}\\
&\quad\le 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\right\}+N_2(r,-a;f)+N_2(r,-a;g)+\overline{N}(r,\infty;f)\\
&\qquad+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\quad\le (4-\Theta(\infty;f)+\varepsilon)\,T(r,f)+(4-\Theta(\infty;g)+\varepsilon)\ T(r,g)\\
&\qquad+\frac{1}{nk+n-1}\ [2T(r,f)+2T(r,g)]+S(r,f)+S(r,g).
\end{aligned}
$$

That is

$$
\begin{aligned}
&\left(\frac{n}{2}-4+\Theta(\infty;f)-\frac{2}{nk+n-1}-\varepsilon\right)\ T(r,f)\\
&\qquad+\left(\frac{n}{2}-4+\Theta(\infty;g)-\frac{2}{nk+n-1}-\varepsilon\right)\ T(r,g)\\
&\quad\le S(r,f)+S(r,g)
\end{aligned}
\tag{3.1}
$$

Without the loss of generality, we may suppose that there exists a set I with infinite linear measure such that

$$T(r,g)\le T(r,f),\qquad r\in I.$$

From (3.1) and *Lemma* 2.11 we have

$$\left[\Theta\ (\infty;f)+\Theta\ (\infty;g)-8+n-\frac{4}{nk+n-1}-2\varepsilon\right]\ T(r,g)\ \le S(r,g),\qquad r\in I\backslash E,$$

which leads to a contradiction for arbitrary $\varepsilon>0$. Hence $H\equiv 0$. Now the theorem follows from *Lemmas* 2.6, 2.7 and 2.8. □

Proof.[Proof of Corollary 1.1] Let F and G be given by (2.1). Let F and G be given by (2.1). Since $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$, it follows that f, g share (∞,k) for all large k. Also since $\Theta\ (\infty;f)+\Theta\ (\infty;g)>8-n$, with $\min\{\Theta(\infty;f),\Theta(\infty;f)\}>4-\frac{n}{2}$, for sufficiently large k we can have $\Theta\ (\infty;f)+\Theta\ (\infty;g)>8-n+\frac{4}{nk+n-1}$ with $\min\{\Theta(\infty;f),\Theta(\infty;f)\}>4-\frac{n}{2}+\frac{2}{nk+n-1}$ and hence by *Theorem 1.1* we get the conclusion of *Corollary* 1.1. So *Corollary* 1.1 can be treated as a special case of *Theorem* 1.1. □

Proof.[Proof of Theorem 1.2] Let F and G be given by (2.1) and (2.2). Since $E_f(S,1)=E_g(S,1)$ and $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ it follows that F, G share $(1,1)$ and $(\infty,nk+n-1)$. If possible let us suppose that $H\not\equiv 0$. Now proceeding in the same way as done in the proof of *Theorem* 1.1, using *Lemma* 2.5 with $m=1$, *Lemma* 2.10 and *Lemma* 2.11

with $m = 1$ we obtain for $\varepsilon(> 0)$

$$\begin{aligned}
&\frac{n}{2}\left\{T(r,f)+T(r,g)\right\}\\
&\quad\le 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\right\}+N_2(r,-a;f)+N_2(r,-a;g)+\overline{N}(r,\infty;f)\\
&\qquad+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)+\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\quad\le (4-\Theta(\infty;f)+\varepsilon)\,T(r,f)+(4-\Theta(\infty;g)+\varepsilon)\;T(r,g)\\
&\qquad+\frac{1}{nk+n-1}\;\left[\frac{3}{2}(\overline{N}(r,0;f)+\overline{N}(r,-a;f))+\frac{3}{2}(\overline{N}(r,0;g)+\overline{N}(r,-a;g))\right]\\
&\qquad+\frac{1}{4}\left[\overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,0;g)+\overline{N}(r,-a;g)\right]+S(r,f)+S(r,g).
\end{aligned}$$

That is

$$\begin{aligned}
&\left(\frac{n}{2}-\frac{9}{2}+\Theta(\infty;f)-\frac{3}{nk+n-1}-\varepsilon\right)\;T(r,f)\\
&\qquad+\left(\frac{n}{2}-\frac{9}{2}+\Theta(\infty;g)-\frac{3}{nk+n-1}-\varepsilon\right)\;T(r,g)\\
&\quad\le S(r,f)+S(r,g)
\end{aligned}\tag{3.2}$$

Without the loss of generality, we may suppose that there exists a set I with infinite linear measure such that

$$T(r,g)\le T(r,f),\qquad r\in I.$$

From (3.2) and *Lemma* 2.14 we have

$$\left[\Theta\;(\infty;f)+\Theta\;(\infty;g)-9+n-\frac{6}{nk+n-1}-2\varepsilon\right]\;T(r,g)\;\le S(r,g),\qquad r\in I\backslash E,$$

which leads to a contradiction for arbitrary $\varepsilon > 0$. Hence $H \equiv 0$. Now the theorem follows from *Lemmas* 2.6, 2.7 and 2.8. □

Proof.[Proof of Theorem 1.3] Let F and G be given by (2.1) and (2.2). Since $E_f(S,0) = E_g(S,0)$ and $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ it follows that F, G share $(1,0)$ and $(\infty, nk+n-1)$. If possible let us suppose that $H \not\equiv 0$. Now proceeding in the same way as done in the proof of *Theorem* 1.2, using *Lemma* 2.5 with $m = 0$

$$\begin{aligned}
&\frac{n}{2}\left\{T(r,f)+T(r,g)\right\}\\
&\quad\le 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\right\}+N_2(r,-a;f)+N_2(r,-a;g)+\overline{N}(r,\infty;f)\\
&\qquad+\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f\mid\ge k+1)+\frac{3}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}\tag{3.3}$$

Now using *Lemmas* 2.11, 2.12 and 2.13 with $m=0$ we get from (3.3)

$$\begin{aligned}\frac{n}{2}\left\{T(r,f)+T(r,g)\right\} \leq \frac{7}{2}\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\right\}+N_2(r,-a;f)+N_2(r,-a;g)\\ +\frac{3}{2}\left[\overline{N}(r,-a;f)+\overline{N}(r,-a;g)\right]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ +\overline{N}(r,\infty;f\mid\geq k+1)+S(r,f)+S(r,g). \qquad (3.4)\end{aligned}$$

$$\begin{aligned}\frac{n}{2}\left\{T(r,f)+T(r,g)\right\} \leq \frac{7}{2}\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\right\}+N_2(r,-a;f)+N_2(r,-a;g)\\ +\frac{5}{2}\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]+\overline{N}(r,\infty;f\mid\geq k+1)\\ +S(r,f)+S(r,g). \qquad (3.5)\end{aligned}$$

$$\begin{aligned}\frac{n}{2}\left\{T(r,f)+T(r,g)\right\} \leq 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\right\}+N_2(r,-a;f)+N_2(r,-a;g)\\ +\frac{3}{2}\left[\overline{N}(r,-a;f)+\overline{N}(r,-a;g)\right]+\frac{5}{2}\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]\\ +\overline{N}(r,\infty;f\mid\geq k+1)+S(r,f)+S(r,g). \qquad (3.6)\end{aligned}$$

Adding (3.4), (3.5) and (3.6) we get

$$\begin{aligned}\frac{n}{2}\left\{T(r,f)+T(r,g)\right\} \leq 3\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\right\}+2T(r,f)\\ +2T(r,g)+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]\\ +\overline{N}(r,\infty;f\mid\geq k+1)+S(r,f)+S(r,g). \qquad (3.7)\end{aligned}$$

So using *Lemma* 2.11 we obtain from (3.7) for $\varepsilon>0$ that

$$\begin{aligned}&\left(\frac{n}{2}-7+2\,\Theta(\infty;f)-\frac{4}{nk+n-1}-\varepsilon\right)\,T(r,f)\\ &\quad+\left(\frac{n}{2}-7+2\,\Theta(\infty;g)-\frac{4}{nk+n-1}-\varepsilon\right)\,T(r,g)\\ &\leq S(r,f)+S(r,g) \qquad (3.8)\end{aligned}$$

Without the loss of generality, we may suppose that there exists a set I with infinite linear measure such that

$$T(r,g)\leq T(r,f), \qquad r\in I.$$

From (3.8) and *Lemma* 2.14 we have

$$\left[2\Theta\,(\infty;f)+2\Theta\,(\infty;g)-14+n-\frac{8}{nk+n-1}-2\varepsilon\right]\,T(r,g)\;\leq S(r,g), \qquad r\in I\backslash E,$$

which leads to a contradiction for arbitrary $\varepsilon>0$. Hence $H\equiv 0$. Now the lemma follows from *Lemmas* 2.6, 2.7 and 2.8. □

References

[1] A. Banerjee, *Uniqueness of meromorphic functions that share two sets*, Southeast Asian Bull. Math., **31** (2007), 7–17.

[2] —— , *Some uniqueness results on meromorphic functions sharing three sets*, Ann. Polon. Math., **92**(2007), 261–274.

[3] —— , *On uniqueness of meromorphic functions that share two sets*, Georgian Math. J., **15**(2008), 21–38.

[4] —— , *Uniqueness of meromorphic functions sharing two sets with finite weight*, Portugal. Math. J., (N.S.), **65**(2008), 81–93.

[5] —— and S. Mukherjee, *Uniqueness of meromorphic functions sharing two or three sets*, Hokkaido Math. J., **37**(2008), 507–530.

[6] M.Fang and H.Guo, *On meromorphic functions sharing two values*, Analysis, **17** (1997), 355–366.

[7] —— and I. Lahiri, *Unique range set for certain meromorphic functions*, Indian J. Math., **45**(2003), 141–150.

[8] —— and W. Xu, *A note on a problem of Gross*,(Chinese) Chin. Ann. Math., **18(A)**(1997), 563–568; English transl. : Chinese J. Contempt. Math. **18**(1997), 395–402.

[9] F. Gross, *Factorization of meromorphic functions and some open problems*, Proc. Conf. Univ. Kentucky, Leixngton, Ky(1976); Lecture Notes in Math., **599**(1977), 51–69, Springer(Berlin).

[10] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[11] I.Lahiri, *The range set of meromorphic derivatives*, Northeast J. Math., **14**(1998), 353–360.

[12] —— , *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sci., **28** (2001), 83–91.

[13] —— , *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161** (2001), 193–206.

[14] —— , *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46** (2001), 241–253.

[15] —— and A. Banerjee, *Uniqueness of meromorphic functions with deficient poles*, Kyungpook Math. J., **44** (2004), 575–584.

[16] —— and —— , *Weighted sharing of two sets*, Kyungpook Math. J., **46** (2006), 79–87.

[17] —— , *On a question of Hong Xun Yi*, Arch. Math. (Brno), **38**(2002), 119–128.

[18] P. Li and C. C. Yang, *On the unique range sets for meromorphic functions*, Proc. Amer. Math. Soc., **124** (1996), 177–185.

[19] —— and —— , *Some further results on the unique range sets for meromorphic functions*, Kodai Math. J., **18** (1995) 437–450.

[20] C. C. Yang, *On deficiencies of differential polynomials II*, Math. Z., **125** (1972), 107–112.

[21] W. C. Lin and H. X. Yi, *Some further results on meromorphic functions that share two sets*, Kyungpook Math. J., **43** (2003), 73–85.

[22] H. X. Yi, *Uniqueness of meromorphic functions and a question of Gross*, Science in China (A), **37** (1994), 802–813.

[23] —— , *Unicity theorems for meromorphic or entire functions II*, Bull. Austral. Math. Soc., **52** (1995), 215–224.

[24] —— , *Meromorphic functions that share three sets*, Kodai Math. J., **20** (1997), 22–32.

[25] —— , *On a question of Gross concerning uniqueness of entire functions*, Bull. Austral. Math. Soc., **57** (1998), 343–349.

[26] —— , *Meromorphic functions that share one or two values II*, Kodai Math. J., **22** (1999), 264–272.

[27] —— , *Meromorphic functions that share two sets*, Acta Mat. Sinica, **45**(2002), 75–82. (in Chinese).

[28] Q. Han and H. X. Yi, *Some further results on meromorphic functions that share two sets*, Ann. Polon. Math., **93**(2008), 17–31.

[29] H. X. Yi and W. C. Lin, *Uniqueness of meromorphic functions and a question of Gross*, Kyungpook Math. J, **46** (2006), 437–444.

[30] —— and W.R.Lü, *Meromorphic functions that share two sets II*, Acta Math. Sci. Ser. B Engl. Ed., **24**(2004), 83–90.

[31] W. C. Lin and H. X. Yi, *Some further results on meromorphic functions that share two sets*, Kyungpook Math. J., **43** (2003), 73–85.

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