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# AN ELEMENTARY APPROACH TO A LATTICE-VALUED BANACH-STONE THEOREM

JINXI CHEN

**Abstract**. Let $X$ and $Y$ be compact Hausdorff spaces, and $E$ be a nonzero real Banach lattice. In this note, we give an elementary proof of a lattice-valued Banach-Stone theorem by Cao, Reilly and Xiong [3] which asserts that if there exists a Riesz isomorphism $\Phi : C(X,E) \to C(Y,\mathbb{R})$ such that $\Phi(f)$ has no zeros if $f$ has none, then $X$ is homeomorphic to $Y$ and $E$ is Riesz isomorphic to $\mathbb{R}$.

## 1. Introduction

Let $X$ and $Y$ be compact Hausdorff spaces, and $E$, $F$ be nonzero real Banach lattices. Let $C(X,E)$ denote the Banach lattice of all continuous $E$-valued functions on $X$ equipped with the pointwise ordering and the sup norm. Let $\mathbb{R}$ be the Banach lattice of real numbers with the usual norm and order. Note that in general, Riesz isomorphism (i.e., lattice isomorphism) of $C(X,E)$ and $C(X,F)$ does not necessarily imply topological homeomorphism of $X$ and $Y$ (see [3]). Recently, Cao, Reilly and Xiong [3] established the following lattice-valued Banach-Stone theorem:

**Theorem A**.([3], Theorem 3.3) *Suppose there is a Riesz isomorphism $\Phi : C(X,E) \to C(Y,\mathbb{R})$ such that $\Phi(f)$ has no zeros if $f$ has none. Then $X$ is homeomorphic to $Y$ and $E$ is Riesz isomorphic to $\mathbb{R}$.*

Towards their proof of Theorem A, they considered the support for a Riesz homomorphism and gave the following:

**Theorem B.**([3], Theorem 2.2) *If $\Phi : C(X,E) \to \mathbb{R}$ is a Riesz homomorphism such that $\Phi(\mathbf{1}_X \otimes e) \neq 0$ if $e \neq 0$, then $\Phi$ has a unique support.*

In this short note we claim that Theorems A and B mentioned above can be deduced from the following two well-known results, respectively:

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**Theorem A′.**([8]; cf. [1], Theorem 7.22) *If $C(X,\mathbb{R})$ and $C(Y,\mathbb{R})$ are Riesz isomorphic, then $X$ and $Y$ homeomorphic.*

**Theorem B′.**([8]; cf. [1], Theorem 7.21) *For any Riesz homomorphism $\Phi : C(X,\mathbb{R}) \to \mathbb{R}$ with $\Phi(\mathbf{1}_X) = 1$ there exists a unique $a \in X$ such that $\Phi(f) = f(a)$ for each $f \in C(X,\mathbb{R})$.*

Our elementary proof also establishes the conjecture posed by Cao, Reilly and Xiong which asserts that Theorem A′ implies Theorem A.

In this note we mostly follow the notion and notations used in [3]. For $\omega \in C(X,\mathbb{R})$ and $e \in E$, let $\omega \otimes e \in C(X,E)$ be defined by $(\omega \otimes e)(x) = \omega(x)e$ for each $x \in X$. We call $a \in X$ a support for a Riesz homomorphism $\Phi : C(X,E) \to F$ if $\Phi(f) = \Phi(\mathbf{1}_X \otimes f(a))$ for every $f \in C(X,E)$, where $\mathbf{1}_X \in C(X,\mathbb{R})$ is defined by $\mathbf{1}_X(x) = 1$ for all $x \in X$. If $\Phi : C(X,E) \to C(Y,\mathbb{R})$ is a Riesz homomorphism, then define $\hat{\Phi}(y)(u) = \Phi(\mathbf{1}_X \otimes u)(y)$ for each $u \in E$ and each $y \in Y$. Clearly, $\hat{\Phi}(y)$ is a linear functional on $E$ for each $y \in Y$. For undefined terms and notions refer to [1] and [3].

## 2. The Elementary Proofs of Theorems A and B

We start with the proof of Theorem B.

**Proof of Theorem B.** First we claim that the Banach lattice $E$ is Riesz isomorphic to $\mathbb{R}$, in notation $E \cong \mathbb{R}$. Suppose on the contrary that $E$ is not Riesz isomorphic to $\mathbb{R}$. Then there would exist two elements $e_1, e_2 \in E^+ \setminus \{0\}$ such that $e_1 \wedge e_2 = 0$ (see [4], P. 19). Since $(\mathbf{1}_X \otimes e_1) \wedge (\mathbf{1}_X \otimes e_2) = 0$ in $C(X,E)$ and $\Phi$ is a Riesz isomorphism, we have $\Phi(\mathbf{1}_X \otimes e_1) \wedge \Phi(\mathbf{1}_X \otimes e_2) = 0$ in $\mathbb{R}$, which implies that $\Phi(\mathbf{1}_X \otimes e_1) = 0$ or $\Phi(\mathbf{1}_X \otimes e_2) = 0$. This is impossible. Let $u$ be an arbitrary nonzero element of $E^+$. Then $E = \{\lambda u : \lambda \in \mathbb{R}\}$. Next, we note that to every $f \in C(X,E)$ there corresponds a unique $\omega_f \in C(X,\mathbb{R})$ such that $f = \omega_f \otimes u$. Clearly, this correspondence is a Riesz isomorphism of $C(X,E)$ onto $C(X,\mathbb{R})$.

Now, let $\Gamma(\omega) = \Phi(\omega \otimes u)$ for $\omega \in C(X,\mathbb{R})$. It is obvious that $\Gamma : C(X,\mathbb{R}) \to \mathbb{R}$ is a Riesz homomorphism and $\Gamma(\mathbf{1}_X) = \Phi(\mathbf{1}_X \otimes u) > 0$. By Theorem B′, there exists a unique $a \in X$ such that $\Gamma(\omega) = \omega(a)\Phi(\mathbf{1}_X \otimes u)$ for each $\omega \in C(X,\mathbb{R})$. Therefore, for every $f = \omega_f \otimes u \in C(X,E)$ we have

$$\begin{aligned}\Phi(f) = \Phi(\omega_f \otimes u) &= \Gamma(\omega_f)\\ &= \omega_f(a)\Phi(\mathbf{1}_X \otimes u)\\ &= \Phi(\mathbf{1}_X \otimes (\omega_f(a)u))\\ &= \Phi(\mathbf{1}_X \otimes f(a)).\end{aligned}$$

So $a$ is also a support for $\Phi$.

It remains to show that $\Phi$ has a unique support. Let $a_1, a_2 \in X$ be such that $\Phi(f) = \Phi(\mathbf{1}_X \otimes f(a_1)) = \Phi(\mathbf{1}_X \otimes f(a_2))$ for all $f \in C(X,E)$. Then we have $f(a_1) = f(a_2)$ for every $f \in C(X,E)$, which implies $a_1 = a_2$.

**The proof of Theorem A.** As proved in Lemma 3.1 of [3], $E \cong \mathbb{R}$ (The proof of surjectivity of $\hat{\Phi}(y)$ is superfluous since the range space is $\mathbb{R}$). Therefore it follows from the proof of Theorem B that $C(X,\mathbb{R}) \cong C(X,E) \cong C(Y,\mathbb{R})$. In view of Theorem A′ we can see that $X$ and $Y$ are homeomorphic.

We can say more about the Riesz isomorphism $\Phi$. As done in the proof of Theorem B, let $u \in E^+ \setminus \{0\}$ be fixed, and let the Riesz isomorphism $\Psi : C(X,\mathbb{R}) \to C(X,E)$ be defined by $\Psi(\omega) = \omega \otimes u$ for each $\omega \in C(X,\mathbb{R})$. Clearly, $\Phi \circ \Psi$ is a Riesz isomorphism of $C(X,\mathbb{R})$ onto $C(Y,\mathbb{R})$. Then there exists a unique positive function $\pi \in C(Y,\mathbb{R})$ and an (onto) homeomorphism $\phi : Y \to X$ such that

$$[(\Phi \circ \Psi)(\omega)](y) = \pi(y)\,\omega(\phi(y))$$

for all $y \in Y$ and all $\omega \in C(X,\mathbb{R})$ ( see, e.g. [1], Theorem 7.22). Here $\pi = (\Phi \circ \Psi)(\mathbf{1}_X)$, and $\pi(y) > 0$ for every $y \in Y$. Now, for every $f = \omega_f \otimes u \in C(X,E)$ and $y \in Y$, we have

$$[\Phi(f)](y) = [(\Phi \circ \Psi)(\omega_f)](y) = \pi(y)\,\omega_f(\phi(y)) = \Pi(y)\,f(\phi(y)),$$

where $\Pi(y)$ is a Riesz isomorphism of $E$ onto $\mathbb{R}$ satisfying $\Pi(y)(\lambda u) = \lambda\,\pi(y)$, $\lambda \in \mathbb{R}$. That is, $\Phi$ can be written as a weighted composition operator.

**Remark 1.** Under the hypothesis of Theorem A, $\Phi$ and $\Phi^{-1}$ are disjointness preserving operators. Then the homeomorphism of $Y$ onto $X$ and the representation of $\Phi$ as a weighted composition operator can be obtained from Gau, Jeang and Wong [5, Theorem 2.3].

**Remark 2.** In the above discussion and [3], the compactness of $X$ and $Y$ plays a key role. If we weaken the compactness of $X$ and $Y$ to realcompactness, then the conclusion of Theorem A is still valid. Indeed, we still have $E \cong \mathbb{R}$ and $C(X,\mathbb{R}) \cong C(X,E) \cong C(Y,\mathbb{R})$ as we did in the above proof of Theorem A. Then the desired conclusion follows from Proposition 3 of [2].

If $\Phi : C(X,E) \to C(Y,\mathbb{R})$ is a linear bijection such that $\Phi f$ has no zeros if, and only if, $f$ has no zeros, then $X$, $Y$ are homeomorphic even without the hypothesis that $\Phi$ is a Riesz isomorphism, which is required for Theorem A. To prove this we need the following proposition. Recall that a continuous scalar function is invertible whenever it has no zeros.

**Proposition.** *Let $X, Y$ be compact Hausdorff connected topological spaces. Let $T : C(X,\mathbb{R}) \to C(Y,\mathbb{R})$ be a linear bijection such that $Tf$ is invertible in $C(Y,\mathbb{R})$ if, and only if, $f$ is invertible in $C(X,\mathbb{R})$. Then there is a homeomorphism $\sigma$ from $Y$ onto $X$ and a strictly positive or negative function $h$ in $C(Y,\mathbb{R})$ such that*

$$Tf = h \cdot f \circ \sigma, \quad \forall f \in C(X,\mathbb{R}).$$

**Proof.** First note that the invertible function $T\mathbf{1}_X$ is either strictly positive or strictly negative on $Y$. Assume $0 < m < T\mathbf{1}_X(y) < M, \forall y \in Y$. We claim that $\Phi f \geq 0$ whenever $f \geq 0$. Indeed, $Tf$ must assume positive values at some points. For else, $Tf - \delta T\mathbf{1}_X < 0$ for all $\delta > 0$. Then $f - \delta$ is invertible. But this is impossible for some $\delta$. Suppose $Tf$ also assumed negative values. Let $\epsilon > 0$ be small enough that $Tf + \epsilon T\mathbf{1}_X$ still assumes both positive and negative values. In particular, $Tf + \epsilon T\mathbf{1}_X$ is not invertible. Thus $f + \epsilon$ is not invertible, a contradiction.

Let $x \in X$ and let $M_x$ be the subspace of $C(X, \mathbb{R})$ consisting of all functions $f$ in $C(X, \mathbb{R})$ with $f(x) = 0$. Let

$$\operatorname{Ker} TM_x = \{y \in Y : Tf(y) = 0, \forall f \in M_x\}.$$

We claim that $\operatorname{Ker} TM_x$ is non-empty. Suppose, on contrary, that $\operatorname{Ker} TM_x$ is empty. Then for each $y$ in $Y$ there is an $f_y$ in $M_x$ such that $Tf_y$ is nonzero at $y$, and thus in a neighborhood of $y$. We can assume further that both $f_y$ and $Tf_y$ are non-negative, by replacing them by their positive parts or negative parts. By compactness of $Y$, we can choose finitely many positive $f_1, \ldots, f_n$ from $M_x$ such that the positive functions $Tf_1, \ldots, Tf_n$ have no common zero in $Y$. Hence $T(f_1 + \cdots + f_n)$ is strictly positive, and thus invertible. This conflicts with the fact that $f_1 + \cdots + f_n$ vanishes at $x$.

Next, we claim that $\operatorname{Ker} TM_x$ is indeed a singleton. Indeed, if $y_1, y_2 \in \operatorname{Ker} TM_x$ then we have $TM_x \subseteq M_{y_i}, \quad i = 1, 2$. Applying the above argument for $T^{-1}$, we shall have $T^{-1}M_{y_i} \subseteq M_{x_i}$ for some $x_i$ in $X$, $i = 1, 2$. However, this gives $TM_x \subseteq M_{y_i} \subseteq TM_{x_i}, \quad i = 1, 2$. It follows from the bijectivity of $T$ that $x = x_1 = x_2$. Thus,

$$TM_x = M_{y_1} = M_{y_2} \quad \text{and} \quad y_1 = y_2.$$

We can now define a bijective map $\sigma : Y \to X$ such that

$$TM_{\sigma(y)} = M_y, \quad \forall y \in Y.$$

Hence there is a (real) scalar function $h$ on $Y$ such that

$$Tf(y) = h(y) f(\sigma(y)), \quad \forall y \in Y.$$

Clearly, $h = T\mathbf{1}_X$ is a strictly positive function in $C(Y, \mathbb{R})$. It is then routine to see that $\sigma$ is a homeomorphism from $Y$ onto $X$. For the proof refer to [6], [7]. $\square$

Now let $E$ be a Banach space. Let $\Phi : C(X, E) \to C(Y, \mathbb{R})$ is a linear bijection such that $\Phi f$ has no zeros if, and only if, $f$ has no zeros. Then, for each $y \in Y$, $\hat{\Phi}(y)$ defined by $\hat{\Phi}(y)(u) = \Phi(\mathbf{1}_X \otimes u)(y)$ is a linear isomorphism from $E$ onto $\mathbb{R}$. Let $u$ be an arbitrary nonzero element of $E$. Let $\Psi : C(X, \mathbb{R}) \to C(X, E)$ be defined by $\Psi(\omega) = \omega \otimes u$ for each $\omega \in C(X, \mathbb{R})$. Clearly, $\Phi \circ \Psi$ is a linear isomorphism of $C(X, \mathbb{R})$ onto $C(Y, \mathbb{R})$ such that $(\Phi \circ \Psi)\omega$ is invertible in $C(Y, \mathbb{R})$ if and only if $\omega$ is invertible in $C(X, \mathbb{R})$. Therefore, we have proved the following corollary.

**Corollary.** *Let $X, Y$ be compact Hausdorff connected topological spaces. If $\Phi : C(X,E) \to C(Y,\mathbb{R})$ is a linear bijection such that $\Phi f$ has no zeros if, and only if, $f$ has no zeros, then $X$, $Y$ are homeomorphic.*


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Department of Mathematics, Southwest Jiaotong University, Chengdu 610031, P.R. China.

E-mail: chenjinxi@mail.nankai.edu.cn