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CERTAIN NEW CLASSES OF MEROMORPHIC FUNCTIONS RELATED WITH A CONVOLUTION OPERATOR

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Abstract. Recently, Hussain [Hussain, Saqib (2010). Some applications of Miller-Mocanu lemma on certain classes of meromorphic functions, "Appl. Math. Comput.", 216(2010), 3016-3021], introduced certain new classes of meromorphic functions using newly defined convolution operators. The purpose of present investigation is to extend the work of Hussain by considering some new classes of meromorphic functions. We derive some inclusion relationships, coefficient bound, integral representation and extreme point theorem.

1. Introduction

Let Σ denotes the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured open unit disc $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Further, let $P_k(\gamma)$ be the class of functions $p(z)$ analytic in $E = D \cup \{0\}$ satisfying $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re}\, p(z) - \gamma}{1-\gamma} \right| d\theta \leq k\pi, \tag{1.2}$$

where $z = re^{i\theta}, k \geq 2,\ 0 \leq \gamma < 1$. This class was introduced by Padmanabhan and Parvatham [7]. For $\gamma = 0$ we obtain the class P_k defined by Pinchuk [9] and for $k = 2$, $P_2(\gamma) = P(\gamma)$ is the class with real part greater than γ.

Also from (1.2) it can be seen that $p \in P_k(\gamma)$ if and only if

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z),$$

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where $p_1, p_2 \in P(\gamma)$ for $z \in E$. By $MC(\gamma)$ and $MS^*(\gamma)$, we mean the subclasses of meromorphic convex and meromorphic starlike functions of order γ respectively. The class Σ is closed under the convolution denoted and defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k\, b_k\, z^k,$$

where

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k\, z^k \;, \quad g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k\, z^k.$$

In [10] Yuan, et. all defined an operator $I_{n,\mu} : \Sigma \to \Sigma$ as follows:

$$I_{n,\mu} f(z) = f_{n,\mu}(z) * f(z), \tag{1.3}$$

where

$$f_{n,\mu}(z) * \frac{1}{z(1-z)^{n+1}} = \frac{1}{z(1-z)^{\mu}}\;, n > -1, \mu > 0, z \in D. \tag{1.4}$$

From (1.3) and (1.4), it implies that

$$I_{n,\mu} f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\mu)_{k+1}}{(n+1)_{k+1}} a_k\, z^k, \tag{1.5}$$

where $(a)_k$ is the Pochhammer symbol defined as

$$(a)_0 = 1, \qquad (a)_k = a(a+1)(a+2)\cdots(a+k-1),\; k \in \mathbb{N}.$$

From (1.5), it can be easily verified that

$$z(I_{n+1,\mu} f(z))' = (n+1) I_{n,\mu} f(z) - (n+2) I_{n+1,\mu} f(z),$$

and

$$z(I_{n,\mu} f(z))' = \mu I_{n,\mu+1} f(z) - (\mu+1) I_{n,\mu} f(z). \tag{1.6}$$

The operator $I_{n,\mu}$ was extensively studied by several authors and obtained numerous results. For details, see [1-5,7] and [9]. Furthermore, for $c > 0$ the generalized Bernardi operator is defined as

$$J_c f(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\, dt. \tag{1.7}$$

From (1.7), we have

$$(c+1) J_c f(z) + z\left(J_c f(z)\right)' = c f(z),\; z \in D. \tag{1.8}$$

Using the operator $I_{n,\mu}$, we define some new classes of meromorphic functions as follows:

Definition 1.1. Let $f(z) \in \Sigma, n > -1, \mu > 0, \lambda \ge 0, 0 \le \gamma < 1, z \in D$, then $f(z) \in MQ_k(n,\mu,\lambda,\gamma)$ if and only if

$$-\left[\frac{z\left(I_{n,\mu} f(z)\right)' + \lambda z^2 \left(I_{n,\mu} f(z)\right)''}{(1-\lambda)\left(I_{n,\mu} f(z)\right) + \lambda z \left(I_{n,\mu} f(z)\right)'}\right] \in P_k(\gamma).$$

Definition 1.2. Let $f(z) \in \Sigma, n > -1, \mu > 0, 0 \le \gamma < 1, z \in D$, then $f \in MR_k(n,\mu,\gamma)$ if and only if

$$-\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}f(z)} \in P_k(\gamma).$$

Definition 1.3. Let $f(z) \in \Sigma, n > -1, \mu > 0, \lambda \ge 0, 0 \le \gamma < 1, 0 \le \delta < 1, m > 0, z \in D$, then $f \in MM_k(n,\mu,\lambda,\gamma,\delta)$ if and only if there exists $g \in MR_2(n,\mu,\delta)$ such that

$$\left[(1-\lambda)\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^m + \lambda\left[\frac{I_{n,\mu+1}f(z)}{I_{n,\mu+1}g(z)}\right]\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^{m-1}\right] \in P_k(\gamma).$$

Remark 1. For special values of parameters n, μ, γ and k, we have many known classes of meromorphic functions, see [3, 8, 9] .

2. Preliminaries Results

Lemma 2.1 ([4])**.** *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u,v)$ be a complex valued function satisfying the conditions:*

(i) $\Psi(u,v)$ *is continuous in* $D \subset C^2$.
(ii) $(1,0) \in D$ *and* $\mathrm{Re}\Psi(1,0) > 0$.
(iii) $\mathrm{Re}\Psi(iu_2, v_1) \leqslant 0$ *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$.

If $h(z)$ is a function analytic in E such that $(h(z), zh'(z) \in D$ and

$\mathrm{Re}\Psi(h(z), zh'(z) > 0$ *for* $z \in E$, *then* $\mathrm{Re}\,h(z) > 0$ *in* E.

Note: Throughout in this article, we will use $n > -1,\ \mu > 0,\ \lambda \ge 0,\ 0 \le \gamma < 1,\ 0 \le \delta < 1,\ m > 0$ unless otherwise stated.

3. Main Results

Theorem 3.1.

$$MQ_k(n,\mu+1,\lambda,\gamma) \subset MQ_k(n,\mu,\lambda,\beta),$$

where

$$\beta = \frac{1}{4}\left[(3+2\mu+2\gamma) - \sqrt{(3+2\mu+2\gamma)^2 - 8\left(2\gamma+2\mu\gamma+1\right)}\right].$$

Proof. Let $f(z) \in MQ_k(n,\mu+1,\lambda,\gamma)$ and let

$$-\left[\frac{z\left(I_{n,\mu}f(z)\right)' + \lambda z^2\left(I_{n,\mu}f(z)\right)''}{(1-\lambda)\left(I_{n,\mu}f(z)\right) + \lambda z\left(I_{n,\mu}f(z)\right)'}\right] = H(z), \tag{3.1}$$

where $H(z)$ is analytic in E and $H(0)=1$. Using (1.6) and (3.1) we obtain

$$-\left[\frac{z\left(I_{n,\mu+1}f(z)\right)'+\lambda z^2\left(I_{n,\mu+1}f(z)\right)''}{(1-\lambda)\left(I_{n,\mu+1}f(z)\right)+\lambda z\left(I_{n,\mu+1}f(z)\right)'}\right]=H(z)+\frac{zH'(z)}{-H(z)+\left(\mu+1\right)}\in P_k\left(\gamma\right). \tag{3.2}$$

Let

$$\Phi_\mu(z)=\frac{1}{\mu+1}\left[\frac{1}{z}+\sum_{k=0}^{\infty}z^k\right]+\frac{\mu}{\mu+1}\left[\frac{1}{z}+\sum_{k=0}^{\infty}kz^k\right],$$

then

$$H(z)*z\Phi_\mu(z)=H(z)+\frac{zH'(z)}{-H(z)+\left(\mu+1\right)}. \tag{3.3}$$

Let

$$H(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z). \tag{3.4}$$

From (3.1), (3.3) and (3.4), we have

$$\begin{aligned}&-\left[\frac{z\left(I_{n,\mu+1}f(z)\right)'+\lambda z^2\left(I_{n,\mu+1}f(z)\right)''}{(1-\lambda)\left(I_{n,\mu+1}f(z)\right)+\lambda z\left(I_{n,\mu+1}f(z)\right)'}\right]\\&=\left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z)+\frac{zh_1'(z)}{-h_1(z)+\left(\mu+1\right)}\right\}-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z)+\frac{zh_2'(z)}{-h_2(z)+\left(\mu+1\right)}\right\}.\end{aligned} \tag{3.5}$$

Since $f(z)\in MQ_k(n,\mu+1,\lambda,\gamma)$, therefore

$$h_i(z)+\frac{zh_i'(z)}{-h_i(z)+\left(\mu+1\right)}\in P\left(\gamma\right)\text{ for } i=1,2\,;\,z\in E.$$

Let $h_i(z)=\beta+\left(1-\beta\right)p_i(z)$ for $i=1,2$. Then

$$\left[\left[\beta-\gamma\right]+\left(1-\beta\right)p_i(z)+\frac{\left(1-\beta\right)zp_i'(z)}{-\left(1-\beta\right)p_i(z)+\left(1+\mu-\beta\right)}\right]\in P\text{ for } i=1,2\,;\,z\in E.$$

We formulate a functional $\Psi(u,\ v)$ by taking $u=u_1+iu_2=p_i(z)$ and $v=v_1+iv_2=zp_i'(z)$, then

$$\Psi(u,\ v)=(\beta-\gamma)+(1-\beta)u+\frac{(1-\beta)v}{-(1-\beta)u+(1+\mu-\beta)}.$$

The first two conditions of Lemma 2.1 are obviously satisfied for $\Psi(u,\ v)$. For the third condition, we proceed as follows:

$$\operatorname{Re}\Psi(iu_2,\ v_1)=(\beta-\gamma)+\frac{(1-\beta)(1+\mu-\beta)v_1}{(1+\mu-\beta)^2+(1-\beta)^2u_2^2}.$$

By putting $v_1\le-\frac{1}{2}(1+u_2^2)$, we have

$$\operatorname{Re}\Psi(iu_2,\ v_1)\le\frac{A+Bu_2^2}{2C},$$

where

$$\begin{aligned} A &= 2(\beta-\gamma)(1+\mu-\beta)^2-(1-\beta)(1+\mu-\beta),\\ B &= 2(\beta-\gamma)(1-\beta)^2-(1-\beta)(1+\mu-\beta),\\ C &= (1+\mu-\beta)^2+(1-\beta)^2u_2^2. \end{aligned}$$

We note that $\operatorname{Re}\Psi(iu_2,\ v_1)\leq 0$ if $A\leq 0$ and $B\leq 0$.From $A\leq 0$, we obtain

$$\beta=\frac{1}{4}\left[(3+2\mu+2\gamma)-\sqrt{(3+2\mu+2\gamma)^2-8\left(2\gamma+2\mu\gamma+1\right)}\right].$$

By virtue of Lemma 2.1, we see that $p_i\in P$, for $i=1,2$ and $z\in E$. Hence $h_i(z)\in P\left(\beta\right)$ which implies $H(z)\in P_k(\beta)$ and consequently $f(z)\in MQ_k(n,\mu,\lambda,\beta)$. □

Next we show that the class $MQ_k(n,\mu,\lambda,\gamma)$ is closed under the Bernardi integral operator.

Theorem 3.2. *Let $f(z)\in\Sigma$. If $f(z)\in MQ_k(n,\mu,\lambda,\gamma)$ then $J_cf(z)\in MQ_k(n,\mu,\lambda,\gamma)$.*

Proof. Let $f(z)\in MQ_k(n,\mu,\lambda,\gamma)$ and set

$$-\left[\frac{zJ_c\left(I_{n,\mu}f(z)\right)'+\lambda z^2J_c\left(I_{n,\mu}f(z)\right)''}{(1-\lambda)J_c\left(I_{n,\mu}f(z)\right)+\lambda zJ_c\left(I_{n,\mu}f(z)\right)'}\right]=H(z), \tag{3.6}$$

where $H(z)$ is analytic in E and $H(0)=1$. Using (1.8) and (3.4) we get

$$-\left[\frac{zJ_c\left(I_{n,\mu+1}f(z)\right)'+\lambda z^2J_c\left(I_{n,\mu+1}f(z)\right)''}{(1-\lambda)J_c\left(I_{n,\mu+1}f(z)\right)+\lambda zJ_c\left(I_{n,\mu+1}f(z)\right)'}\right]=H(z)+\frac{zH'(z)}{-H(z)+(c+1)}\in P_k\left(\gamma\right) \tag{3.7}$$

Now using the same steps as in Theorem 3.1, we can prove that $J_cf(z)\in MQ_k(n,\mu,\lambda,\gamma)$, which completes the proof. □

Theorem 3.3. *The necessary and sufficient condition for $f(z)$ defined by (1.1) to be in the class $MQ_2(n,\mu,\lambda,\gamma)$ is*

$$\sum_{k=0}^{\infty}\left[k^2+\gamma(1+k)\right]\frac{(\mu)_{k+1}}{(n+1)_{k+1}}|a_k|\leq\left(\gamma-1\right)(2\lambda-1). \tag{3.8}$$

Proof. Let $f(z)\in MQ_2(n,\mu,\lambda,\gamma)$, then

$$\operatorname{Re}\left[\frac{z\left(I_{n,\mu}f(z)\right)'+\lambda z^2\left(I_{n,\mu}f(z)\right)''}{(1-\lambda)\left(I_{n,\mu}f(z)\right)+\lambda z\left(I_{n,\mu}f(z)\right)'}\right]<-\gamma,$$

which implies using (1.5)

$$\operatorname{Re}\left[\frac{\frac{1}{z}(2\lambda-1)+\sum_{k=0}^{\infty}k^2\frac{(\mu)_{k+1}}{(n+1)_{k+1}}a_kz^k}{-\frac{1}{z}(2\lambda-1)+\sum_{k=0}^{\infty}(1+k)\frac{(\mu)_{k+1}}{(n+1)_{k+1}}a_kz^k}\right]<-\gamma.$$

Letting $z \to 1^-$ along the real axis, we have

$$\left[\frac{(2\lambda-1)+\sum_{k=0}^{\infty} k^2 \frac{(\mu)_{k+1}}{(n+1)_{k+1}} a_k}{-(2\lambda-1)+\sum_{k=0}^{\infty}(1+k)\frac{(\mu)_{k+1}}{(n+1)_{k+1}} a_k}\right]+\gamma<0.$$

By maximum modulus principle, the above equation leads to the desired inequality

$$\sum_{k=0}^{\infty}\left[k^2+\gamma(1+k)\right]\frac{(\mu)_{k+1}}{(n+1)_{k+1}}|a_k| \leq (\gamma-1)(2\lambda-1).$$

Conversely, suppose (3.8) is true for $z \in D$. Then

$$\operatorname{Re}\left[\frac{z\left(I_{n,\mu} f(z)\right)'+\lambda z^2\left(I_{n,\mu} f(z)\right)''}{(1-\lambda)\left(I_{n,\mu} f(z)\right)+\lambda z\left(I_{n,\mu} f(z)\right)'}\right]+\gamma<0,$$

if

$$\operatorname{Re}\left[\frac{\frac{1}{z}(2\lambda-1)+\sum_{k=0}^{\infty} k^2 \frac{(\mu)_{k+1}}{(n+1)_{k+1}} a_k z^k}{-\frac{1}{z}(2\lambda-1)+\sum_{k=0}^{\infty}(1+k)\frac{(\mu)_{k+1}}{(n+1)_{k+1}} a_k z^k}\right]+\gamma<0,$$

that is if

$$\sum_{k=0}^{\infty}\left[k^2+\gamma(1+k)\right]\frac{(\mu)_{k+1}}{(n+1)_{k+1}}|a_k| \leq (\gamma-1)(2\lambda-1),$$

which completes the proof. □

Corollary 3.1. *Let the function $f(z)$ defined by (1.1) belongs to the class $MQ_2(n,\mu,\lambda,\gamma)$. Then*

$$|a_k| \leq \frac{(\gamma-1)(2\lambda-1)}{\left[k^2+\gamma(1+k)\right]}\frac{(n+1)_{k+1}}{(\mu)_{k+1}}.$$

This inequality is sharp and the extremal function is given by

$$f(z)=\frac{1}{z}+\frac{(\gamma-1)(2\lambda-1)}{\left[k^2+\gamma(1+k)\right]}\frac{(n+1)_{k+1}}{(\mu)_{k+1}} z^k.$$

Theorem 3.4. *Let the functions $f_j(z) \in MQ_2(n,\mu,\lambda,\gamma)$ be defined as*

$$f_0(z)=\frac{1}{z} \text{ and } f_n(z)=\frac{1}{z}+\frac{(\gamma-1)(2\lambda-1)}{\left[k^2+\gamma(1+k)\right]}\frac{(n+1)_{k+1}}{(\mu)_{k+1}} z^k \text{ for } n \in \mathbb{N}.$$

Then $f(z) \in MQ_2(n,\mu,\lambda,\gamma)$ if and only if $f(z)$ can be expressed in the form

$$f(z)=\sum_{k=0}^{\infty}\xi_k f_k(z) \text{ where } \xi_k \geq 0 \text{ and } \sum_{k=0}^{\infty}\xi_k=1. \tag{3.9}$$

Proof. Suppose that $f(z)$ can be expressed in the form (3.9), then

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{(\gamma-1)(2\lambda-1)}{\left[k^2+\gamma(1+k)\right]}\xi_k z^k.$$

Consider

$$\sum_{k=0}^{\infty}\left[\left[k^2+\gamma(1+k)\right]\frac{(\mu)_{k+1}}{(n+1)_{k+1}}\right]\xi_k\frac{(\gamma-1)(2\lambda-1)}{\left[k^2+\gamma(1+k)\right]}\frac{(n+1)_{k+1}}{(\mu)_{k+1}}\le(\gamma-1)(2\lambda-1),$$

so by (3.8), we have $f(z)\in MQ_2(n,\mu,\lambda,\gamma)$. Conversely let $f(z)\in MQ_2(n,\mu,\lambda,\gamma)$, and set

$$\xi_k=\frac{\left[k^2+\gamma(1+k)\right]}{(\gamma-1)(2\lambda-1)}\frac{(\mu)_{k+1}}{(n+1)_{k+1}}|a_k|,\ k=1,2,\ldots$$

with $\xi_0=1-\sum_{k=1}^{\infty}\xi_k$. We obtain

$$\sum_{k=1}^{\infty}\xi_k=\sum_{k=1}^{\infty}\frac{\left[k^2+\gamma(1+k)\right]}{(\gamma-1)(2\lambda-1)}\frac{(\mu)_{k+1}}{(n+1)_{k+1}}|a_k|\le 1,$$

then $f(z)=\sum_{k=0}^{\infty}\xi_k f_k(z)$ where $\xi_k\ge 0$ and $\sum_{k=0}^{\infty}\xi_k=1$. This completes the proof. □

Theorem 3.5. *Let $f(z)\in MM_k(n,\mu,\lambda,\gamma,\delta)$ w.r.t $g(z)\in MR_2(n,\mu,\gamma)$, then $\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^m\in P_k(\beta)$, where*

$$\beta=\frac{q_1+2\gamma}{q_1+2}\ \textit{and}\ q_1(z)=\frac{\lambda}{\mu m}\mathrm{Re}\left(\frac{I_{n,\mu}g(z)}{I_{n,\mu+1}g(z)}\right).$$

Proof. Let $f(z)\in MM_k(n,\mu,\lambda,\gamma,\delta)$ and let

$$\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^m=H(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z).\tag{3.10}$$

It can be easily verified that

$$\begin{aligned}&\left[(1-\lambda)\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^m+\lambda\left[\frac{I_{n,\mu+1}f(z)}{I_{n,\mu+1}g(z)}\right]\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^{m-1}\right]\\&=H(z)+\frac{\lambda zH'(z)Q(z)}{\mu m}\in P_k(\gamma),\end{aligned}\tag{3.11}$$

where $Q(z)=\frac{I_{n,\mu}g(z)}{I_{n,\mu+1}g(z)}$. Let $q(z)=\frac{\lambda Q(z)}{\mu m}$, then by (3.10) and (3.11)

$$\begin{aligned}&\left[(1-\lambda)\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^m+\lambda\left[\frac{I_{n,\mu+1}f(z)}{I_{n,\mu+1}g(z)}\right]\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^{m-1}\right]\\&=\left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z)+zh_1'(z)q(z)\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z)+zh_2'(z)q(z)\right].\end{aligned}\tag{3.12}$$

Since $f(z) \in MM_k(n,\mu,\lambda,\gamma,\delta)$, so by (3.12),

$$h_i(z) + zh_i'(z)\,q(z) \in P(\gamma) \text{ for } i = 1,2\,;\, z \in E$$

Let $h_i(z) = \beta + (1-\beta)\,p_i(z)$ for $i = 1,2$. Then

$$\left[(\beta-\gamma) + (1-\beta)\,p_i(z) + (1-\beta)\,zp_i'(z)\,q(z)\right] \in P \text{ for } i = 1,2\,;\, z \in E.$$

We formulate a functional $\Psi(u,\ v)$ by taking $u = u_1 + iu_2 = p_i(z)$ and $v = v_1 + iv_2 = zp_i'(z)$, then

$$\Psi(u,\ v) = (\beta-\gamma) + (1-\beta)u + (1-\beta)v\left[q_1 + iq_2\right].$$

The first two conditions of Lemma 2.1 are obviously satisfied for $\Psi(u,\ v)$. For the third condition, we proceed as follows:

$$\mathrm{Re}\Psi(iu_2,\ v_1) = (\beta-\gamma) + (1-\beta)v_1 q_1.$$

By putting $v_1 \le -\frac{1}{2}(1+u_2^2)$, we have

$$\mathrm{Re}\Psi(iu_2,\ v_1) \le A + Bu_2^2,$$

where

$$A = (\beta-\gamma) - \frac{1}{2}(1-\beta)q_1,$$
$$B = -\frac{1}{2}(1-\beta)q_1 \le 0$$

We note that $\mathrm{Re}\Psi(iu_2,\ v_1) \le 0$ if $A \le 0$. From $A \le 0$, we obtain

$$\beta = \frac{q_1 + 2\gamma}{q_1 + 2}$$

Using Lemma 2.1, we see that $p_i \in P$, for $i = 1,2$ and $z \in E$. Hence $h_i(z) \in P(\beta)$ which implies $H(z) \in P_k(\beta)$. Our proof is complete. □

Theorem 3.6. *For* $0 \le \lambda_1 < \lambda_2$,

$$MM_k(n,\mu,\lambda_2,\gamma,\delta) \subset MM_k(n,\mu,\lambda_1,\gamma,\delta)$$

Proof. Let $f(z) \in MM_k(n,\mu,\lambda_2,\gamma,\delta)$, then

$$\left[(1-\lambda_2)\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^m + \lambda_2\left[\frac{I_{n,\mu+1}f(z)}{I_{n,\mu+1}g(z)}\right]\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^{m-1}\right] = H_2(z) \in P_k(\gamma),$$

by Theorem **??**,

$$\left[\frac{I_{n,\mu}f(z)}{I_{n,\mu}g(z)}\right]^m = H_1(z) \in P_k(\beta) \subset P_k(\gamma).$$

If $\lambda_1 = 0$, the proof is immediate from Theorem **??**. Therefore we consider $\lambda_1 > 0$,

$$\left[(1-\lambda_1)\left[\frac{I_{n,\mu}f}{I_{n,\mu}g}\right]^m + \lambda_1\left[\frac{I_{n,\mu+1}f}{I_{n,\mu+1}g}\right]\left[\frac{I_{n,\mu}f}{I_{n,\mu}g}\right]^{m-1}\right] = \left(1-\frac{\lambda_1}{\lambda_2}\right)H_1(z) + \frac{\lambda_1}{\lambda_2}H_2(z). \tag{3.13}$$

Since $P_k(\gamma)$ is a convex set [5], thus right side of (3.13) belongs to $P_k(\gamma)$. Thus $f(z) \in MM_k$ $(n,\mu,\lambda_1,\gamma,\delta)$. □

Theorem 3.7. *Let* $f(z) \in MR_2(n,\mu,\gamma)$, *then* $J_c f(z) \in MR_2(n,\mu,\gamma)$.

Proof. Let $f(z) \in MR_2(n,\mu,\gamma)$ and set

$$-\frac{zJ_c(I_{n,\mu}f(z))'}{J_c(I_{n,\mu}f(z))} = H(z). \tag{3.14}$$

(3.14) together with (1.8) implies that

$$-\frac{z(I_{n,\mu}f(z))'}{(I_{n,\mu}f(z))} = H(z) + \frac{zH'(z)}{-H(z)+(c+1)}. \tag{3.15}$$

Now following the same steps as in Theorem 3.1, we obtain the required result.

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