TAMKANG JOURNAL OF MATHEMATICS
Volume 39, Number 4, 341-346, Winter 2008

ON THE INVERSE LAPLACE TRANSFORM OF $\overline{H}$-FUNCTION ASSOCIATED WITH FEYNMAN TYPES INTEGRALS

V. B. L. CHAURASIA AND HARI SINGH PARIHAR

Abstract. The Laplace transform and its inverse are fundamental and powerful tools in solving boundary value problems occurring in the diverse fields of engineering. Here we will establish some useful formulas giving the inverse Laplace transform of various products of algebraic powers and $\overline{H}$-function, involving one and more variables, which are unified and likely to have applications in several different areas.

1. Introduction

The $\overline{H}$-function, due to Inayat-Hussain [1, 2] which is a generalization of the familiar Fox H-function, contains as special cases, a certain class of Feynman integrals, the exact partition function of the Gaussian model in statistical mechanics, the polylogarithm of order p and several other functions, is given by (cf. [8])

$$\overline{H}^{m,n}_{p,q}[z] = \overline{H}^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,p} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{matrix} \right] = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \overline{\phi}(s) z^s \, ds, \tag{1.1}$$

where

$$\overline{\phi}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \{\Gamma(1 - a_j + \alpha_j s)\}^{A_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \beta_j s)\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)}. \tag{1.2}$$

This contains the fractional powers of some of the Gamma functions. Here and throughout the paper a_j $(j = 1, \ldots, p)$ and b_j $(j = 1, \ldots q)$ are complex parameters, $\alpha_j \geq 0$ $(j = 1, \ldots, p)$, $\beta_j \geq 0$ $(j = 1, \ldots, q)$ (not all zero simultaneously) and the exponents A_j $(j = 1, \ldots, n)$ and B_j $(j = m+1, \ldots, q)$ can take non-integer values.

The contour in (1.1) is imaginary axis $Re(s) = 0$. It is suitably indented in order to avoid the singularities of the Gamma functions and to keep those singularities on appropriate sides. Again, for A_j $(j = 1, \ldots, n)$ not an integer, the poles of the Gamma functions of the numerator in (1.2) are converted to the branch points. However, as long as there is no coincidence of

Received January 19, 2007; revised August 23, 2007.
2000 *Mathematics Subject Classification*. xxx.
Key words and phrases. Feynman integals, $\overline{H}$-function, Mittage-Leffler function, Wright Bessel function, Wright hyper geometric function.

poles from any $\Gamma(b_j-\beta_j s)$ $(j=1,\ldots,m)$ and $\Gamma(1-a_j+\alpha_j s)$ $(j=1,\ldots,n)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

The convergence conditions and other details of the above function are given by Buschman and Srivastava [8]. We assume that the convergence and sufficient condition of above function, given by equation (1.1) is satisfied by each of the various $\overline{H}$-function involved throughout the present work.

The behavior of the $\overline{H}$-function for small values of $|z|$ follows easily from a result recently given by Rathie [5, p.306, Eq. (6.9)], we have

$$\overline{H}_{p,q}^{m,n}[z]=O(|z|^{\alpha}),\quad \alpha=\min_{1\le j\le m}[Re(b_j/B_j)],\quad |z|\to 0. \tag{1.3}$$

2. Main Results

In this section, we will obtain the inverse Laplace transform of various products of algebraic powers and $\overline{H}$-function.

$$\begin{aligned}(\mathbf{A})L^{-1}&\left\{(p^2+b^2)^{-(\frac{2\mu+1}{2})}\overline{H}_{P,Q}^{M,N}\left[z(p^2+b^2)^{-\eta}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right];x\right\}\\&=\sqrt{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{x}{2}\right)^{2(\mu+r)}b^{2r}\\&\times\overline{H}_{P,Q+2}^{M,N}\left[\frac{zx^{2\eta}}{2^{2\eta}}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\left(\frac{1}{2}-\mu,\eta;1\right),(-r-\mu,\eta;1)\end{matrix}\right.\right],\end{aligned}\tag{2.1}$$

provided that $\eta\ge 0$, $Re(p)>0$ and $Re(2\mu+1)>0$

$$\begin{aligned}(\mathbf{B})L^{-1}&\Big\{\{p+(p^2+b^2)^{-1/2}\}^{-\nu}\\&\times\overline{H}_{P,Q}^{M,N}\left[z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right];x\Big\}\\&=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{b^{2r}x^{\nu+2r-1}}{2^{\nu+2r}}\right)\overline{H}_{P+1,Q+2}^{M,N+1}\left[z\left(\frac{x}{2}\right)^{\rho}\left|\begin{matrix}(-\nu,\rho;1),(a_j,\alpha_j;A_j)_{1,N},\\(b_j,\beta_j)_{1,M},\end{matrix}\right.\right.\\&\qquad\left.\begin{matrix}(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j;B_j)_{M+1,Q}(1-\nu,\rho;1),(-\nu-r,\rho;1)\end{matrix}\right],\end{aligned}\tag{2.2}$$

provided that $\rho\ge 0$, $Re(p)>0$ and $Re(\nu+\rho s)>0$.

$$\begin{aligned}(\mathbf{C})L^{-1}&\left\{p^{-2\lambda}(p^2+b^2)^{-\nu}\overline{H}_{P,Q}^{M,N}\left[zp^{-2\rho}(p^2+b^2)^{-\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right];x\right\}\\&=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{b}{2}\right)^{2r}x^{2(\lambda+\nu+r)-1}\overline{H}_{P+3,Q+4}^{M,N+3}\left[zx^{\rho+\sigma}\left|\begin{matrix}(1-\nu-r,\sigma;1),(1-\lambda-\nu,\rho+\sigma;1),\\(b_j,\beta_j)_{1,M},\end{matrix}\right.\right.\end{aligned}$$

$$\Big(\frac{1}{2}-\lambda-\nu,\rho+\sigma;1\Big),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}$$
$$(b_j,\beta_j;B_j)_{M+1,Q},(1-\nu,\sigma;1),(1-\lambda-\nu-r,\rho+\sigma;1),$$
$$\Big(\frac{1}{2}-\lambda-\nu-r,\rho+\sigma;1\Big),(1-2\lambda-2\nu,2\rho+2\sigma;1)\Big], \tag{2.3}$$

provided that $Re(p)>0$, $Re(\lambda)>0$, $Re(\nu)\geq 0$.

$$(\mathbf{D})\,L^{-1}\Big\{(p^2+b^2)^{-1/2}\{p+(p^2+b^2)^{1/2}\}^{-\lambda}$$
$$\times\overline{H}^{M,N}_{P,Q}\Big[z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\Big|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\Big];x\Big\}$$
$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\Big(\frac{b^{2r}x^{\lambda+2r}}{2^{\lambda+2r}}\Big)$$
$$\times\overline{H}^{M,N}_{P,Q+1}\Big[\frac{zx^{\rho}}{2^{\rho}}\Big|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\lambda-r,\rho;1)\end{matrix}\Big], \tag{2.4}$$

provided that $Re(p)>0$, $Re(\lambda)>-1$ and $Re(\lambda+\rho s)>-1$.

$$(\mathbf{E})\,L^{-1}\Big\{p^{-\nu}q^{-\mu}\overline{H}^{M,N}_{P,Q}\Big[zp^{-\rho}q^{-\sigma}\Big|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\Big];x,y\Big\}$$
$$=x^{\nu-1}y^{\mu-1}$$
$$\times\overline{H}^{M,N}_{P,Q+2}\Big[zx^{\rho}y^{\sigma}\Big|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\nu,\rho;1),(1-\mu,\sigma;1)\end{matrix}\Big], \tag{2.5}$$

provided that $Re(\nu)>0$, $Re(\mu)>0$, $Re(p)>0$, $Re(q)>0$.

Proof.

By using the definition of $\overline{H}$-function {Eqn. (1.1) and (1.2)} in the L. H. S. of (2.1), we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(s)z^sL^{-1}\Big\{(p^2+b^2)^{-(\frac{(2\mu+1)}{2}+\eta s)};x\Big\}ds.$$

Now, by using the known result [4, p.239, Eq. (18)], we get

$$\sqrt{\pi}\Big(\frac{x}{2b}\Big)^{\mu}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\overline{\phi}(s)\frac{1}{\Gamma\Big(\mu+\eta s+\frac{1}{2}\Big)}z^s\Big(\frac{x}{2b}\Big)^{\eta s}J_{\mu+\eta s}(bx)ds.$$

Now expanding $J_{\mu+\eta s}(bx)$ in summation form and after a little simplification, we have

$$\sqrt{\pi}\Big(\frac{x}{2b}\Big)^{\mu}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\Big(\frac{bx}{2}\Big)^{\mu+2r}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(s)\frac{1}{\Gamma\Big(\mu+\eta s+\frac{1}{2}\Big)\Gamma(\mu+\eta s+r+1)}\Big(\frac{zx^{2\eta}}{2^{x\eta}}\Big)^{s}ds.$$

Finally, on reinterpreting the contour integral thus obtained in terms of $\overline{H}$-function, we get R. H. S. of (2.1) after a little simplification.

Similarly, the proofs of (2.2) through (2.5) can be established by using the definition of $\overline{H}$-function and known results [4, p.240, Eqn. (23)], [4, p.238, Eqn. (10)], [4, p.240, Eqn. (21)] and [4, p.239, Eqn. (18)].

3. Particular Cases

(a) Replacing $\overline{H}^{M,N}_{P,Q}\left[z(p^2+b^2)^{-\eta}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]$

by $\overline{H}^{1,P}_{P,Q-1}\left[-z(p^2+b^2)^{-\eta}\middle|\begin{matrix}(a_j,\alpha_j;1)_{1,P}\\(0,1),(b_j,\beta_j;1)_{1,Q-2}\end{matrix}\right]$

in L. H. S. of Eqn. (2.1), we get

$$L^{-1}\left\{(p^2+b^2)^{-\left(\frac{2\mu+1}{2}\right)}\overline{H}^{1,P}_{P,Q-1}\left[-z(p^2+b^2)^{-\eta}\middle|\begin{matrix}(a_j,\alpha_j;1)_{1,PN}\\(0,1),(b_j,\beta_j;1)_{1,Q-2}\end{matrix}\right];x\right\}$$

$$=\sqrt{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{x}{2}\right)^{2(\mu+r)}b^{2r}\ {}_P\overline{\Psi}_Q\left[\begin{matrix}(a_j,\alpha_j)_{1,P};\\(b_j,\beta_j)_{1,Q-2},\left(\mu+\frac{1}{2},\eta\right),(\mu+r+1,\eta);\end{matrix}\ \frac{zx^{2\eta}}{2^{2\eta}}\right],$$

$$=\sqrt{\pi}\frac{\prod_{j=1}^{P}\{\Gamma(a_j)\}}{\prod_{j=1}^{Q}\{\Gamma(b_j)\}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{x}{2}\right)^{2(\mu+r)}b^{2r}$$

$$\times\ {}_P\overline{F}_Q\left[\begin{matrix}(a_j)_{1,P},(\mu+r+1);\\(b_j)_{1,Q-2},\left(\mu+\frac{1}{2}\right),(\mu+r+1);\end{matrix}\ \frac{zx^{2\eta}}{2^{2\eta}}\right],\tag{3.1}$$

where ${}_P\overline{\Psi}_Q$ [7, p.271, Eqn. (6)] is the generalized Wright hypergeometric function and ${}_P\overline{F}_Q$ is a particular case of ${}_P\overline{\Psi}_Q$ [7, Eqn. (9)].

Similarly, we can find special cases for (2.2), (2.3), (2.4) and (2.5).

(b) Replacing $\overline{H}^{M,N}_{P,Q}\left[z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]$

by $\overline{H}^{1,0}_{0,1}\left[z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\middle|\begin{matrix}-\\(0,1)\end{matrix}\right]$ in L. H. S. of Eqn. (2.4), we get

$$L^{-1}\left\{(p^2+b^2)^{1/2}\{p+(p^2+b^2)^{1/2}\}^{-\lambda}\overline{H}^{1,0}_{0,1}\left[z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\middle|_{(0,1)}\right];x\right\}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{b^{2r}x^{\lambda+2r}}{2^{\lambda+2r}}J^{\rho}_{\lambda+r}\left(\frac{zx^{\rho}}{2^{\rho}}\right),\tag{3.2}$$

where $J^{\rho}_{\lambda+r}$ is a Wright-Bessel function [6, p.19, Eq. (2.6.10)]

(c) Replacing $\overline{H}^{M,N}_{P,Q}\left[z(p^2+b^2)^{-\eta}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]$
by $\overline{H}^{1,3}_{3,1}\left[-z(p^2+b^2)^{-\eta}\middle|\begin{matrix}(1-\lambda,1;1),(1-\lambda+\mu-\frac{1}{2},1;1),(1-\lambda-\mu,1;1)\\(0,1)\end{matrix}\right]$ in Eqn. (2.1), we get

$$L^{-1}\left\{(p^2+b^2)^{-2(\frac{2\mu+1}{2})}\overline{H}^{1,3}_{3,1}\left[-z(p^2+b^2)^{-\eta}\middle|\begin{matrix}(1-\lambda,1;1),(1-\lambda+\mu-\frac{1}{2},1;1),\\(0,1)\end{matrix}\right.\right.$$
$$\left.\left.\begin{matrix}(1-\lambda-\mu,1;1)\end{matrix}\right];x\right\}$$
$$=\frac{4\pi\Gamma(\lambda)\Gamma\left(\lambda-\mu+\frac{1}{2}\right)}{k_{d-1}\Gamma(\mu)}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{x}{2}\right)^{2(\mu+r)}b^{2r}g(\gamma,r+\mu,2\mu-1,0;\frac{zx^{2\eta}}{4^\eta}),\tag{3.3}$$

where $k_d=\dfrac{2^{1-d}\pi^{-d/2}}{\Gamma(d/2)}$ [2, p.4121, Eqn (5)].

The above function is connected with certain class of Feynman integrals [2].

Similarly, we can reduce Eqn. (2.2) in a function, which is connected with certain class of Feynman integrals [2].

(d) Replacing $\overline{H}^{M,N}_{P,Q}\left[z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]$
by $\overline{H}^{1,1}_{1,1}\left[-z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\middle|\begin{matrix}(0,1)\\(0,1)\end{matrix}\right]$ in Eqn. (2.4), we get

$$L^{-1}\left\{(p^2+b^2)^{1/2}\{p+(p^2+b^2)^{1/2}\}^{-\lambda}\overline{H}^{1,1}_{1,1}\left[-z\{p+(p^2+b^2)^{1/2}\}^{-\rho}\middle|\begin{matrix}(0,1)\\(0,1)\end{matrix}\right];x\right\}$$
$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{b^{2r}x^{\lambda+2r}}{2^{\lambda+2r}}E_{\rho,\lambda+r+1}\left(\frac{zx^\rho}{2^\rho}\right),\tag{3.4}$$

where $E_{\lambda,\lambda+r+1}\left(\frac{zx^\rho}{2^\rho}\right)$ is Mittage - Leffler function [3, p.65, Eq. (2.9.28)].

A number of other special cases can also be obtained with the help of our Eqn. (2.1) through (2.5) but we do not record them here due to lack of space.

Acknowledgement

The authors are thankful to Prof. H. M. Srivastava, University of Victoria, Canada for his kind help and valuable suggestions in the preparation of this paper and also to the worthy referee for fruitful suggestions.

References

[1] A. A. Inayat-Hussain, *New properties of hyper geometric series derivable from Feynman integrals: I Transformation and reduction formulae*, J. Phys. A.: Math. Gen. **20**(1987), 4109–4117.

[2] A. A. Inayat-Hussain, *New Properties of hyper geometric series derivable from Feynman integrals: II A generalization of H-function*, J, Phys. A.: Math Gen. **20**(1987), 4119–4128.

[3] A. A. Kilbas and M. Saigo, H-Transforms: Theory and Applications, Chapman & Hall / CRC, New York, 2004.

[4] A. Erdélyi et al., Tables of Integral Transforms, Vol. I. McGraw-Hill, New York, 1954.

[5] A. K. Rathie, *A new generalization of generalized hyper-geometric functions*, Le Matematiche Fasc. II **52**(1997), 297–310.

[6] H. M. Srivastava, K. C. Gupta and S. P. Goyal, The H-function of One and Two Variables with Applications, South Asian Publishers, New Delhi, Madras, (1982).

[7] K. C. Gupta, Rashmi Jain and Aarti Sharma, *A study of unified finite integral transforms with applications*, J. Rajasthan Acad. Phy. Sci. **2**(2003), 269–282.

[8] R. G. Buschman and H. M. Srivastava, *The $\overline{H}$-function associated with a certain class of Feynman integrals*, J. Phys. A: Math. Gen. **23**(1990), 4707–4710.

University of Rajasthan-jaipur, India.

E-mail:

Poornima Institute of Engineering & Technology, Jaipur, India.

E-mail: harisingh.p@rediffmail.com