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SOME RESULTS CONCERNING FRAMES IN BANACH SPACES

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Abstract. A necessary and sufficient condition for the associated sequence of functionals to a complete minimal sequence to be a Banach frame has been given. We give the definition of a weak-exact Banach frame, and observe that an exact Banach frame is weak-exact. An example of a weak-exact Banach frame which is not exact has been given. A necessary and sufficient condition for a Banach frame to be a weak-exact Banach frame has been obtained. Finally, a necessary condition for the perturbation of a retro Banach frame by a finite number of linearly independent vectors to be a retro Banach frame has been given.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [4], while addressing some deep problems in non-harmonic Fourier series. Gröchenig [6] generalized frames for Banach spaces and called them *atomic decompositions*. He also introduced a more general concept of Banach spaces called *Banach frames*. Banach frames were further studied in [1, 2, 3, 5, 7].

In the present paper, we give a necessary and sufficient condition for the associated sequence of functionals to a complete minimal sequence to be a Banach frame. Also it has been shown that a continuous linear mapping from a Banach space E onto another Banach space F determines a Banach frame for F (Section 3). In Section 4, w-exact Banach frames has been defined and a necessary and sufficient condition for a Banach frame to be a w-exact Banach frame has been given. Finally, in Section 5, we considered perturbation of a retro Banach frame by a finite number of linearly independent vectors and obtained a necessary condition for the perturbed sequence to be a retro Banach frame.

2. Preliminaries

Throughout this paper E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, u the canonical isomorphism of E into $[f_n]^*$, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, $\widetilde{[f_n]}$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$-topology, E_d and $(E^*)_d$, respectively, the associated Banach spaces of

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the scalars-valued sequences indexed by $\mathbb{N}$, and $\gamma_E(V)$ the greatest number r such that the unit ball $\{f \in V : \|f\| \leq 1\}$ of V is $\sigma(E^*, E)$-dense in the ball $\{f \in E^* : \|f\| \leq r\}$ of E^*.

A sequence $\{x_n\}$ in E is said to be complete if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be total over E if $\{x \in E : f_n(x) = 0,\ n \in \mathbb{N}\} = \{0\}$. A pair (x_n, f_n) $(\{x_n\} \subset E, \{f_n\} \subset E^*)$ is called a biorthogonal system if $f_i(x_j) = \delta_{ij}$ (Kronecker's delta) for all $i, j \in \mathbb{N}$ and E-complete biorthogonal system if it is a biorthogonal system such that $[x_n] = E$. A sequence $\{x_n\} \subset E$ is minimal if there exists a sequence $\{f_n\} \subset E^*$ such that (x_n, f_n) is a biorthogonal system. If (x_n, f_n) is a E-complete biorthogonal system, then $\{x_n\}$ is a complete minimal sequence and $\{f_n\}$ is called the associated sequence of functional (a.s.f.) to the sequence $\{x_n\}$.

Definition 2.1.([6]) Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a Banach frame for E with respect to E_d if

(i) $\{f_n(x)\} \in E_d$ for each $x \in E$,

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E\,, \quad x \in E \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x\,, \quad x \in E\,.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.1) is called the *frame inequality.*

The Banach frame $(\{f_n\}, S)$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset E^*$ no longer a Banach frame for E, then $(\{f_n\}, S)$ is called an *exact Banach frame.*

The following results which are refered in this paper are listed in the form of lemmas.

Lemma 2.1.([9]) *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Lemma 2.2.([8]) *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Then $(\{f_n\}, S)$ is exact if and only if $f_n \notin \widetilde{[f_i]}_{i\neq n}$, for all n.*

Proof. Suppose first that the Banach frame $(\{f_n\}, S)$ is exact. Then for some $n \in \mathbb{N}$, there exists no reconstruction operator S_0 such that $(\{f_i\}_{i\neq n}, S_0)$ is a Banach frame for E. Therefore, $\widetilde{[f_i]}_{i\neq n} \neq E^*$. Hence $f_n \notin \widetilde{[f_i]}_{i\neq n}$. Conversely, let $f_n \notin \widetilde{[f_i]}_{i\neq n}$ and let $(\{f_n\}, S)$ be not exact. Then there exists a reconstruction operator S_1 defined by $S_1 : (\{f_i(x)\}_{i\neq n}) = x$, $x \in E$ such that $(\{f_i\}_{i\neq n}, S_1)$ is a Banach frame for E. Therefore, by frame inequality, $\widetilde{[f_i]}_{i\neq n} = E^*$. This gives $f_n \in \widetilde{[f_i]}_{i\neq n}$ which is a contradiction.

Finally, in this section, we give the definition of a retro Banach frame introduced in [7].

Definition 2.2. Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar valued sequences associated with E^* indexed by $\mathbb{N}$. Let$\{x_n\} \subset E$ and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a *retro Banach frame* (RBF) for E^* with respect to $(E^*)_d$ if

(i) $\{f(x_n)\} \in (E^*)_d$ for each $f \in E^*$,

(ii) there exist positive constants A and B with $0 < A \le B < \infty$ such that

$$A\|f\|_{E^*} \le \|\{f(x_n)\}\|_{(E^*)_d} \le B\|f\|_{E^*}\,, \quad f \in E^* \tag{2.2}$$

(iii) T is a bounded linear operator such that $T(\{f(x_n)\}) = f$, $f \in E^*$.

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the retro Banach frame $(\{x_n\}, T)$. The operator $T : (E^*)_d \to E^*$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.2) is called the *retro frame inequality.*

3. Banach frames

We begin this section with a necessary and sufficient condition for the associated sequence of functionals to a complete minimal sequence in E to be a Banach frame for E.

Theorem 3.1. *Let (x_n, f_n) $(\{x_n\} \subset E, \{f_n\} \subset E^*)$ be an E-complete biorthogonal system. Then there exists an associated Banach space E_d and a bounded linear operator $S : E_d \to E$ such that $(\{f_n\}, S)$ is a normalized tight and exact Banach frame for E with respect to E_d if and only if*

$$\lim_{n\to\infty} \alpha_i^{(n)} = 0\,,\ i \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} x_i = 0\,,\ i \in \mathbb{N}\,.$$

Proof. Suppose $\lim_{n\to\infty} \alpha_i^{(n)} = 0$ for each $i \in \mathbb{N}$ and $x = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} x_i$. Then for each $j \in \mathbb{N}$

$$f_j(x) = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_j(x_i) = \lim_{n\to\infty} \alpha_j^{(n)} = 0\,.$$

Therefore, by frame inequality for the Banach frame $(\{f_n\}, S)$, $x = 0$.

Conversely, let $x \in E$ be such that $f_j(x) = 0$ for all $j \in \mathbb{N}$. Since $[x_n] = E$,

$$x = \lim_{n\to\infty} \sum_{i=1}^{p_n} \beta_i^{(n)} x_i \ \text{ and so for all } j \in \mathbb{N}\,, \qquad \lim_{n\to\infty} \beta_j^{(n)} = f_j(x) = 0\,.$$

Then, by hypothesis, $x = 0$. Therefore, by Lemma 2.1, there exists an associated Banach space $E_d = \{\{f_n(x) : x \in E\}\}$ and a bounded linear operator $S : E_d \to E$ given by $S(\{f_n(x)\}) = x$, $x \in E$ such that $(\{f_n\}, S)$ is a normalized tight Banach frame for E with respect to E_d. Further since $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$, $f_n \notin [\widetilde{f_i}]_{i \neq n}$. Indeed, if $f_n \in [\widetilde{f_i}]_{i \neq n}$, then each f_n has the form

$$f_n = \sigma([x_n]^*, [x_n]) - \lim_{p \to \infty} \sum_{\substack{i=1 \\ i \neq n}}^{m_p} \alpha_i^{(p)} f_i \,.$$

Then $f_n(x_n) = 0$, a contradiction. Hence, by Lemma 2.2, $(\{f_n\}, S)$ is an exact Banach frame for E with respect to E_d.

Let v be a continuous linear mapping from E onto another Banach space F. If E has a Banach frame, then the following theorem shows that we may have a Banach frame for F.

Theorem 3.2. *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Let $\{g_n\} \subset F^*$, where F is any other Banach space. If there exists a continuous linear mapping v from E onto F such that $v^*(g_n) = f_n$, $n \in \mathbb{N}$. Then there exists an associated Banach space F_d and a bounded linear operator $U : F_d \to F$ such that $(\{g_n\}, U)$ is a normalized tight Banach frame for F with respect to F_d. Moreover, if $(\{f_n\}, S)$ is exact, then $(\{g_n\}, U)$ is also exact.*

Proof. For each $y \in F$ there exists an $x \in E$ such that $v(x) = y$. Let $g_n(y) = 0$ for all $n \in \mathbb{N}$. Then

$$f_n(x) = v^*(g_n)(x) = g_n(v(x)) = g_n(y) = 0, \quad \text{for all} \quad n \in \mathbb{N} \,.$$

Therefore, by frame inequality for the Banach frame $(\{f_n\}, S)$, $x = 0$ and so $y = 0$. Then by Lemma 2.1 there exists an associated Banach space $F_d = \{\{g_n(y)\} : y \in F\}$ and a bounded linear operator $U : F_d \to F$ given by $U(\{g_n(y)\}) = y$, $y \in F$ such that $(\{g_n\}, U)$ is a normalized tight Banach frame for F with respect to F_d. Further since the Banach frame $(\{f_n\}, S)$ is exact, by Lemma 2.2. $f_n \notin [\widetilde{f_i}]_{i \neq n}$, for all $n \in \mathbb{N}$. Therefore there exists a sequence $\{x_n\} \subset E$ such that $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Put $y_n = v(x_n)$, $n \in \mathbb{N}$. Then $\{y_n\} \subset F$ is such that

$$g_i(y_j) = g_i(v(x_j)) = v^*(g_i)(x_j) = f_i(x_j) = \delta_{ij}, \quad i, j \in \mathbb{N} \,.$$

Thus $g_n \notin [\widetilde{g_i}]_{i \neq n}$. Hence, by Lemma 2.2, $(\{g_n\}, U)$ is an exact Banach frame for F with respect to F_d.

Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be an exact Banach frame for E with respect to E_d. Then, in view of Lemma 2.2, there exists a unique sequence $\{x_n\} \subset E$, called the *admissible sequence* to the Banach frame $(\{f_n\}, S)$, such that for all $i, j \in \mathbb{N}$, $f_i(x_j) = \delta_{ij}$.

If $\sup_{1\le n<\infty}\big\|\sum_{i=1}^{n}\alpha_i f_i\big\|<\infty$, then the following result shows that α_i's are determined by $f(x_i)$ for each $i\in\mathbb{N}$, provided $\gamma_{[x_n]}[f_n]>0$.

Theorem 3.3. *Let $(\{f_n\},S)$ $(\{f_n\}\subset E^*, S: E_d\to E)$ be an exact Banach frame for E with respect to E_d and with the admissible sequence $\{x_n\}\subset E$ such that $\gamma_{[x_n]}[f_n]>0$ and $\sup_{1\le n<\infty}\big\|\sum_{i=1}^{n}\alpha_i f_i\big\|<\infty$. Then there exists an $f\in E^*$ such that $f(x_n)=\alpha_n$, $n\in\mathbb{N}$.*

Proof. Let W be the natural canonical embedding of $[f_n]$ into $[u(x_n)]^*$. Since $[u(x_n)]$ is separable, the $\sigma([u(x_n)]^*,[u(x_n)])$-topology is metrizable on bounded sets of $[u(x_n)]^*$ ([10], Theorem 3.1.1). Therefore there exists a subsequence $\{\sum_{i=1}^{n_k}\alpha_i W(f_i)\}_k$ of $\{\sum_{i=1}^{n}\alpha_i W(f_i)\}_n$ which is $\sigma([u(x_n)]^*,[u(x_n)])$-convergent to some $\psi\in[u(x_n)]^*$. Since $\gamma_{[x_n]}[f_n]>0$, W is an isomorphism of $[f_n]$ onto $[u(x_n)]^*$. Therefore there exists an $f\in[f_n]$ such that $\psi=W(f)$.

Hence

$$\begin{aligned} f(x_n)&=(W(f))(u(x_n))\\ &=\psi(u(x_n))\\ &=\sigma([u(x_n)]^*,[u(x_n)])-\lim_{k\to\infty}\sum_{i=1}^{n_k}\alpha_i W(f_i)(u(x_n))\\ &=\alpha_n\,,\qquad n\in\mathbb{N}\,. \end{aligned}$$

4. Weak-exact Banach frames

Definition 4.1. A Banach frame $(\{f_n\},S)$ $(\{f_n\}\subset E^*, S: E_d\to E)$ for E with respect to E_d is called *weak exact* (in short, w-exact) if there exists a sequence $\{\phi_n\}\subset E^{**}$, called an admissible sequence to $(\{f_n\},S)$, such that $\phi_i(f_j)=\delta_{ij}$, for all $i,j\in\mathbb{N}$.

An admissible sequence to a w-exact Banach frame $(\{f_n\},S)$ need not be unique as in case of exact Banach frames.

Example 4.1. Let $E=l^1$ and let $\{f_n\}$ be the sequence of unit vectors in E^*. Then by Lemma 2.1 there exists an associated Banach space $E_d=\{\{f_n(x)\}: x\in E\}$ and a reconstruction operator $S: E_d\to E$ given by $S(\{f_n\}(x))=x$, $x\in E$ such that $(\{f_n\},S)$ is a Banach frame for E with respect to E_d. Define $\{g_n\}\subset E^{**}$ by

$$g_n(f)=\xi_n \qquad (f=\{\xi_n\}\in E^*, n\in\mathbb{N})\,.$$

Then $\{g_n\}$ is an admissible sequence to the w-exact Banach frame $(\{f_n\},S)$. Let $e\in E^*$ be such that $e\notin[f_n]=c_0$. Let $g_0\in E^{**}$ be such that $g_0(e)\neq 0$ and $g_0([f_n])=\{0\}$. Define $\{h_n\}\subset E^{**}$ by $h_1=g_1-g_0$, $h_n=g_n$, $n=2,3,\ldots$. Then $h_i(f_j)=\delta_{ij}$ for all $i,j\in\mathbb{N}$. So $\{h_n\}$ is another admissible sequence to $(\{f_n\},S)$.

We now give a necessary and sufficient condition for the uniqueness of the admissible sequence to a w-exact Banach frame.

Theorem 4.1. *An admissible sequence to a w-exact Banach frame $(\{f_n\}, S)$ is unique if and only if $(\{f_n\}, S)$ is a retro Banach frame for E^{**} with respect to E_d.*

Proof. Let $\{\phi_n\} \subset E^{**}$ be the unique admissible sequence to $(\{f_n\}, S)$. Suppose $(\{f_n\}, S)$ is not a retro Banach frame for E^{**}. Then by Theorem 3.1 in [7], $[f_n] \neq E^*$. Define $\{\psi_n\} \subset E^{**}$ by $\psi_1 = \phi_1 - \phi_0$ and $\psi_n = \phi_n$, $n = 2, 3, \ldots$. Then $\psi_i(f_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. This is a contradiction.

Conversely, let $\{g_n\}$ and $\{h_n\}$ in E^{**} be two admissible sequences to the w-exact Banach frame $(\{f_n\}, S)$. Then $(g_i - h_i)(f_j) = \delta_{i,j}$, for all $i, j \in \mathbb{N}$. Since $(\{f_n\}, S)$ is a retro Banach frame for E^{**}, $[f_n] = E^*$. Hence $g_i = h_i$ for all $i \in \mathbb{N}$.

In view of Lemma 2.2 one may observe that an exact Banach frame for E is a w-exact Banach frame for E. The converse however need not be true as shown by the following example.

Example 4.2. Let $E = c_0$. Define $\{f_n\} \subset E^*$ by

$$\begin{cases} f_1 = (1, 0, 0, \ldots) \\ f_n = ((-1)^{n+1}, 0, 0, \ldots, \underset{n^{\text{th}}\ \text{place}}{\underset{\downarrow}{1}}, 0, \ldots), \quad (n = 2, 3, \ldots) \end{cases}$$

and $\{\phi_n\} \subset E^{**}$ by

$$\begin{cases} \phi_1 = (1, 1, -1, 1, -1, 1, \ldots) \\ \phi_n = (0, 0, \ldots, \underset{n^{\text{th}}\ \text{place}}{\underset{\downarrow}{1}}, 0, \ldots), \quad (n = 2, 3, \ldots) \end{cases}$$

Then by Lemma 2.1 there exist an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ and a reconstruction operator $S : E_d \to E$ given by $S(\{f_n(x)\}) = x$, $x \in E$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d. Also since $\phi_i(f_j) = \delta_{ij}$ for all $i, j \in \mathbb{N}$, $(\{f_n\}, S)$ is a w-exact Banach frame. Further since $[f_n] = E^*$, by Theorem 3.1 in [7], $(\{f_n\}, S)$ is a retro Banach frame for E^{**}. Therefore, by Theorem 4.1, $\{\phi_n\}$ is the unique admissible sequence to the w-exact Banach frame $(\{f_n\}, S)$. But $\phi_1 \notin \pi(E)$, where π is the canonical isomorphism of E into E^{**}. So there exists no sequence $\{x_n\} \subset E$ such that $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Hence, by Lemma 2.2, $(\{f_n\}, S)$ is not exact.

We now give a necessary and sufficient condition for a Banach frame to be w-exact.

Theorem 4.2. *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Then $(\{f_n\}, S)$ is w-exact if and only if*

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i = 0 \qquad \Rightarrow \qquad \lim_{n\to\infty} \alpha_i^{(n)} = 0, \quad \text{for each } i\,.$$

Proof. Suppose $\{\phi_n\} \subset E^{**}$ be an admissible sequence to the w-exact Banach frame $(\{f_n\}, S)$. Let $\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i = 0$. Then for each $j \in \mathbb{N}$,

$$\lim_{n\to\infty} \alpha_j^{(n)} = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} \phi_j(f_i) = 0\,.$$

Conversely, let $Q = \{\sum_{i=1}^{n} \alpha_i f_i : f_i \in E^*, \alpha_i \in \mathbb{K}; i \in \mathbb{N}\}$. Let $\{g_n\} \subset E^{**}$ be such that for each $j \in \mathbb{N}$, $g_j\,(\sum_{i=1}^{n} \alpha_i f_i) = \alpha_j$. If $f \in \bar{Q}$, the closure of Q, then $f = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i$. So, by hypothesis, $g_j(f) = \lim\limits_{n\to\infty} \alpha_j^{(n)}$ for each $j \in \mathbb{N}$. Thus, for each j, g_j is a continuous linear functional on $\bar{Q}$. Also $g_j(f_i) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Hence $(\{f_n\}, S)$ is w-exact.

5. Perturbation of Banach frames

Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d and let f_0 be a nonzero functional in E^*. If $(\{f_n\}, S)$ is an exact Banach frame for E, then for a non-zero functional $f_0 \in E^*$, the following example shows that there may exists a reconstruction operator S_0 such that $(\{f_n + f_0\}, S_0)$ is a Banach frame for E which is not exact.

Example 5.1. Let $E = c_0$. Define $\{h_n\} \in E^*$ by

$$h_n(x) = \xi_n\,, \quad x = \{\xi_j\} \in E\,.$$

Then by Lemma 2.1 there exists an associated Banach space $E_d = \{\{h_n(x)\} : x \in E\}$ and a bounded linear operator $S : E_d \to E$ such that $(\{h_n\}, S)$ is a normalized tight exact Banach frame for E with respect to E_d. Now observe that the sequence $\{h_n + h_1\} \subset E^*$ is total over E. Therefore, again by Lemma 2.1 there exists an associated Banach space $E_{d_1} = \{\{(h_n + h_1)(x)\} : x \in E\}$ and a reconstruction operator $S_1 : E_{d_1} \to E$ such that $(\{h_n + h_1\}, S_1)$ is a Banach frame for E with respect to E_{d_1}. Further by Lemma 2.2 $(\{h_h + h_1\}, S_1)$ is not exact since $[\widetilde{h_n + h_1}]_{n\neq 1} = E^*$.

It is natural to ask under what condition the Banach frame $(\{h_n + h_1\}, S_1)$ is exact. The following theorem answers this query.

Theorem 5.1. *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be an exact Banach frame for E with admissible sequence $\{x_n\} \subset E$ such that $[x_n] = E$. Let f_0 be a non-zero functional in E^*. If there exists an associated Banach space E_{d_0} and a reconstruction operator $S_0 : E_{d_0} \to E$ such that $(\{f_n + f_0\}, S_0)$ is a Banach frame for E, then the Banach frame $(\{f_n + f_0\}, S_0)$ is non-exact.*

Proof. Since $\{x_n\} \subset E$ is an admissible sequence to the Banach frame $(\{f_n\}, S)$, $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Suppose $(\{f_n + f_0\}, S_0)$ is exact. Then, by Lemma 2.2, there exists a sequence $\{y_n\} \subset E$ such that $(f_i + f_0)(y_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Since

$f_0 \neq 0$, there exists a $p \in \mathbb{N}$ such that $f_0(x_p) \neq 0$. Let $m \geq p$ be a fixed but arbitrary integer and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be any scalars. Then

$$\Big|\sum_{i=1}^{m} \alpha_i (f_i + f_0)(x_p)\Big| = \left|\alpha_p + \sum_{i=1}^{m} \alpha_i f_0(x_p)\right|$$
$$\geq \left|\sum_{i=1}^{m} \alpha_i\right| |f_0(x_p)| - \left|\sum_{i=1}^{m} \alpha_i (f_i + f_0)(y_p)\right|.$$

This gives

$$\left|\sum_{i=1}^{m} \alpha_i\right| \leq (\|x_p\| + \|y_p\|)(f_0(x_p))^{-1} \left\|\sum_{i=1}^{m} \alpha_i (f_i + f_0)\right\| .$$

Therefore, by Helly's theorem ([11], p.109), there exists an element $x \in E$ such that $(f_i + f_0)(x) = 1$, for all $i \in \mathbb{N}$. Put $y = x/(1 - f_0(x))$. Then $y \in E$ is such that $f_i(y) = 1$, for all $i \in \mathbb{N}$. Let $f \in E^*$ be such that $f(y) \neq 0$. Put $f_0 = ((-1)/f(y))f$. Then $0 \neq f_0 \in E^*$ is such that $(f_i + f_0)(y) = 0$, for all $i \in \mathbb{N}$. This is a contradiction.

In the following theorem, we consider perturbation of a *retro Banach frame* (RBF) by a finite number of linearly independent elements and obtain a necessary condition for the perturbed sequence to be a RBF.

Theorem 5.2. *Let* $(\{x_n\}, U)$ $(\{x_n\} \subset E, U : (E^*)_d \to E^*)$ *be a RBF for* E^* *with respect to* $(E^*)_d$*. Let* $\{z_k\}_{k=1}^{m}$ *be a linearly independent set of vectors in* E *and let for each* k $(1 \leq k \leq m)$ *there exists an* $f_k \in E^*$ *such that* $f_k(x_n) = c_k^{(n)}$*, for all* $n \in \mathbb{N}$*. If* $\left(\left\{x_n + \sum_{k=1}^{m} c_k^{(n)} z_k\right\}, V\right)$ *is a RBF for* E^* *with respect to* $(E^*)_d$*, where* $V : (E^*)_d \to E^*$ *is a reconstruction operator, then* -1 *is not an eigen value of the matrix*

$$\begin{pmatrix} f_1(z_1) & f_2(z_1) & \ldots & f_m(z_1) \\ f_1(z_2) & f_2(z_2) & \ldots & f_m(z_2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(z_m) & f_2(z_m) & \ldots & f_m(z_m) \end{pmatrix}.$$

Proof. It is enough to prove the result for the case $m = 2$. Suppose -1 is an eigenvalue of the matrix

$$\begin{pmatrix} f_1(z_1) & f_2(z_1) \\ f_1(z_2) & f_2(z_2) \end{pmatrix}.$$

Then

$$\begin{vmatrix} f_1(z_1) + 1 & f_2(z_1) \\ f_1(z_2) & f_2(z_2) + 1 \end{vmatrix} = 0.$$

So there exists scalars α, β not both zero, such that

$$\alpha f_1(z_1) + \beta f_2(z_1) = -\alpha$$
$$\alpha f_1(z_2) + \beta f_2(z_2) = -\beta\,.$$

Put $g = -\alpha f_1 - \beta f_2$. Then, g is a non-zero functional in E^* such that

$$g(x_n) = -\alpha c_1^{(n)} - \beta c_2^{(n)}, \quad \text{for all} \quad n \in \mathbb{N},$$

where $c_i^{(n)} = f_i(x_n)$, $i = 1, 2$.

Now

$$g(z_k) = -\alpha f_1(z_k) - \beta f_2(z_k), \quad k = 1, 2.$$

Therefore $g(z_1) = \alpha$ and $g(z_2) = \beta$. Thus

$$g(x_n + c_1^{(n)} z_1 + c_2^{(n)} z_2) = 0, \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $\left(\left\{x_n + \sum_{k=1}^{2} c_k^{(n)} z_k\right\}, V\right)$ is a RBF for E^*, it follows from the retro frame inequality that $g = 0$. This is a contradiction.

As an application to the above theorem, we give the following example.

Example 5.2. Let $E = c_0$ and let $\{x_n\} \subset E$ be the sequence of unit vectors. Then, by Lemma 2.1, there exists an associated Banach space $(E^*)_d = \{\{f(x_n)\} : f \in E^*\}$ and a reconstruction operator $U : (E^*)_d \to E^*$ such that $(\{x_n\}, U)$ is a RBF for E^* with respect to $(E^*)_d$.

Let $z_1, z_2 \in E$ be given by

$$z_1 = (1, 0, 0, \ldots), \qquad z_2 = (0, 1, 0, \ldots)$$

and let $f_1, f_2 \in E^*$ be given by

$$f_1 = (-1, 0, 0, \ldots), \qquad f_2 = (0, -1, 0, \ldots).$$

Then -1 is an eigen value of the matrix

$$\begin{pmatrix} f_1(z_1) & f_2(z_1) \\ f_1(z_2) & f_2(z_2) \end{pmatrix}.$$

Therefore, by Theorem 5.2, there exists no reconstruction operator $V : (E^*)_d \to E^*$ such that $\left(\left\{x_n + \sum_{k=1}^{2} c_k^{(n)} z_k\right\}, V\right)$ is a RBF for E^* with respect to $(E^*)_d$, where $c_k^{(n)} = f_k(x_n)$, $k = 1, 2$.

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