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EDGE DOMINATING GRAPH OF A GRAPH

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Abstract. The edge dominating graph $E_D(G)$ of a graph $G = (V,E)$ is a graph with $V(E_D(G)) = E(G) \cup S(G)$, where $S(G)$ is the set of all minimal edge dominating sets of G with two vertices $u, v \in V(E_D(G))$ adjacent if $u \in E$ and v is a minimal edge dominating set of G containing u. In this paper, we establish the bounds on order and size, diameter and vertex(edge) connectivity.

1. Introduction

All graphs considered here are finite, nontrivial, undirected, connected, without loops or multiple edges or isolated vertices. For undefined terms or notations in this paper, may be found in Harary [1].

Let $G = (V,E)$ be a graph. A set $S \subseteq E$ is an edge dominating set of G, if every edge in $E - S$ is adjacent to at least one edge in S. The edge domination number $\gamma^{'}(G)$ of G is the minimum cardinality of an edge dominating set.

The minimal dominating graph of G is an intersection graph on the minimal dominating sets of vertices of G. This concept was introduced by Kulli and Janakiram [4].

In [5], the concept of common minimal dominating graph of G was defined as the graph having same vertex set as G with two vertices adjacent if there is a minimal dominating set containing them. The concept of vertex minimal dominating graph $M_V D(G)$ of G was introduced in [6], as the graph having $V(M_V D(G)) = V(G) \cup S(G)$, where $S(G)$ is the set of all minimal dominating sets of G with two vertices u, v adjacent if they are adjacent in G or $v = D$ is a minimal dominating set containing u.

In this paper, we introduce the concept of edge dominating graph $E_D(G)$ of G as the graph with $V(E_D(G)) = E(G) \cup S(G)$, where $S(G)$ is the set of all minimal edge dominating sets of G with two vertices $u, v \in V(E_D(G))$ adjacent, if $u \in E$ and $v = S$ is a minimal edge dominating set containing u.

In Figure 1, a graph G and its edge dominating graph $E_D(G)$ are shown.

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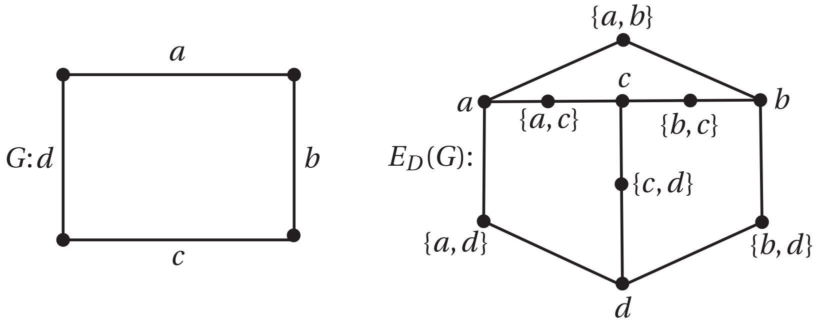

Figure 1:

2. Preliminary results

We need the following theorems for our further results.

Remark 1. For any graph G, $E_D(G)$ is bipartite.

Remark 2. $V(E_D(G)) = E \cup S$, where S is the set of all minimal edge dominating sets of G. No two edges of G and $S(G)$ are adjacent vertices in $E_D(G)$.

Remark 3. Let G be a (p, q)-graph. Then $\deg_{E_D(G)}(e_i)$= number of minimal edge dominating sets containing e_i, where $i = 1, 2, \ldots, q$.

Theorem A [3]**.** *If G is a graph without isolated edges, then for every minimal edge dominating set S, $E - S$ is also an edge dominating set.*

3. Main results

First, we obtain necessary and sufficient conditions on G for which $E_D(G)$ is connected.

Theorem 3.1. *For any graph G with at least one edge, the edge dominating graph $E_D(G)$ of G is connected if and only if $\Delta'(G) < q - 1$, where $\Delta'(G)$ is the maximum edge degree of G.*

Proof. Suppose there is no minimal edge dominating set containing both x and y. Then there exists an edge $z \in E$ not adjacent to both x and y. Let S and S' be two maximal edge independent sets containing the edges xz and yz respectively. Since every maximal edge independent set is a minimal edge dominating set, therefore x and y are connected in $E_D(G)$ by a path (x, S, z, S', y). Thus, $E_D(G)$ is connected. From this it follows that for any two edges $x, y \in E(G)$ either there exists a minimal edge dominating set containing x and y or there exist two nondisjoint minimal edge dominating sets S_1 and S_2 containing x and y respectively. This implies that in $E_D(G)$, x and y are connected by a path of length at most four.

Conversely, suppose $E_D(G)$ is connected. On the contrary $\Delta'(G) = q-1$ and x is an edge of degree $q-1$. Then $S = x$ is a minimum edge dominating set of G. Since every minimum edge dominating set is a minimal edge dominating set and further G has at least two edges with x adjacent to every other edge of G, G has no isolated edges. Thus by Theorem A, $E-S$ contains a minimal edge dominating set S'. Since $S \cap S' = \phi$, in $E_D(G)$ there is no path joining x to any edge of $E-S$. This implies that $E_D(G)$ is disconnected, a contradiction. Hence $\Delta'(G) < q-1$. □

In the next result, we obtain the bounds on the order of $E_D(G)$.

Theorem 3.2. *For any graph G*

$$q + d'(G) \leq p' \leq \frac{q(q+1)}{2},$$

where $d'(G)$ is the edge domatic number of G and p' denotes the number of vertices in $E_D(G)$. Further, the lower bound is attained if and only if $G = K_{1,p-1}$ and upper bound is attained if and only if G is $q-2$ regular graph.

Proof. The lower bound follows from the fact that every graph has at least $d'(G)$ number of minimal edge dominating sets of G and the upper bound follows from the fact that every edge is in $(q-1)$ minimal edge dominating sets of G.

Suppose the lower bound is attained. Then every edge is in exactly one minimal edge dominating set of G and hence, every minimal edge dominating set of G is independent. This implies the necessity.

Suppose the upper bound is attained. Then each edge is in exactly $(q-1)$ minimal edge dominating sets and hence G is $(q-2)$ regular. This implies the sufficiency. □

In the following result, we obtain the bounds on the size of $E_D(G)$.

Theorem 3.3. *For any graph G*

$$q \leq q' \leq q(q-1),$$

where q' is the number of edges in $E_D(G)$. Further, the lower bound is attained if and only if $G = K_{1,p-1}$ and the upper bound is attained if and only if G is a $(q-2)$ regular graph.

Proof. Proof follows from Theorem 3.2. □

Theorem 3.4. *For any graph G*

$$\alpha_0(E_D(G)) = \min\{|E|, |S|\},$$

where S is the set of all minimal edge dominating sets of G.

Proof. Let $V(E_D(G) = E \cup S$. By Remark 2, E and S are independent sets and every element of E is adjacent with some elements of S. Also we observe that $|E| < |S|$, then elements of E covers all the edges of $E_D(G)$. Otherwise, elements of S covers all edges of $E_D(G)$. Hence $\alpha_0(E_D(G)) = \min\{|E|, |S|\}$. □

Theorem 3.5. *For any graph G*

$$\beta_0(E_D(G)) = \max\{|E|, |S|\},$$

where S is the set of all minimal edge dominating sets of G.

Proof. For any graph G, clearly E and S are independent sets of $E_D(G)$. If $|E| > |S|$, then $\beta_0(E_D(G)) = |V|$. Otherwise $\beta_0(E_D(G)) = |S|$.

$$\beta_0(E_D(G)) = \max\{|E|, |S|\}.$$ □

Theorem 3.6. *For any graph G with* $\Delta'(G) < q - 1$.

$$diam(E_D(G)) \leq 5,$$

where $diam(G)$ *is the diameter of G.*

Proof. By Theorem 3.1, $E_D(G)$ is connected. Let $u, v \in V(E_D(G))$. We consider the following cases:

Case 1. Suppose $u, v \in E(G)$, then by Theorem 3.1, $d(u,v)_{E_D(G)} \leq 4$, where $d(u,v)_{E_D(G)}$ is the distance between u and v in $E_D(G)$.

Case 2. Suppose $u \in E(G)$ and $v \in S(G)$. Then $v = S$ is a minimal edge dominating set of G. If $u \in S$, then u and v are adjacent in $E_D(G)$. Otherwise there exist a vertex $w \in S$. This implies

$$d(u,v)_{E_D(G)} \leq d(u,w)_{E_D(G)} + d(w,S)_{E_D(G)} \leq 4 + 1 = 5$$

Case 3. Suppose $u, v \in S(G)$. Then $u = S_1$ and $u = S_2$ are two minimal edge dominating sets of G. If S_1 and S_2 are nonadjacent, then $d(u,v)_{E_D(G)} = d(S_1,S_2)_{E_D(G)} = 2$. Otherwise there exist a minimal edge dominating set S_3 containing the vertices of S_1 and S_2. Thus

$$d(u,v)_{E_D(G)} = d(S_1,S_2)_{E_D(G)} \leq d(S_1,S_3)_{E_D(G)} + d(S_3,S_2)_{E_D(G)} \leq 2 + 2 = 4.$$

Thus from all the above cases we get $diam(E_D(G) \leq 5$. □

Theorem 3.7. *For any graph G*

$$\gamma(\overline{E_D(G)}) = 2.$$

Proof. Let $x \in E$ and S be a minimal edge dominating set containing x. Then x and S together form a minimal edge dominating set of $\overline{E_D(G)}$. Since $\Delta(\overline{E_D(G)}) \le p-2$, the result holds. □

Theorem 3.8. *For any graph G, $E_D(G)$ is bicolorable.*

Proof. For any graph G, $E_D(G)$ is bipartite by Remark 1. Since chromatic number of any bipartite graph is two, therefore $E_D(G)$ is bicolorable. □

Theorem 3.9. *For any graph G*

$$\kappa(E_D(G)) = \min\{\min_{1\le i\le q}(\deg(e_i)), \min_{1\le j\le n}|S_j|\},$$

where S'_j are the minimal edge dominating sets of G.

Proof. Let G be a (p,q) graph. Clearly edge set and minimal edge dominating sets of G are independent. We consider the following cases:

Case 1. Let $x \in e_i$ for some i, having minimum edge degree among all $e'_i s$ in $E_D(G)$. If the degree of x is less than any other edge in $E_D(G)$, then by deleting those edges of $E_D(G)$ which are adjacent with x, results in a disconnected graph.

Case 2. Let $y \in S_j$ for some j having minimum degree among all vertices of $S'_j s$. If degree of y is less than any other vertex in $E_D(G)$. Then by deleting those vertices which are adjacent with y, results in a disconnected graph. Thus $\kappa(E_D(G)) = \min\{\min_{1\le i\le q}(\deg(e_i)), \min_{1\le j\le n}|S_j|\}$. □

Theorem 3.10. *For any graph G*

$$\lambda(E_D(G)) = \min\{min_{1\le i\le q}(\deg(e_i)), \min_{1\le j\le n}|S_j|\},$$

where $S'_j s$ are the minimal edge dominating sets of G.

Proof. Follows from the same lines in the proof Theorem 3.9. □

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