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# APPLICATIONS OF FRACTIONAL CALCULUS TO $k$-UNIFORMLY STARLIKE AND $k$-UNIFORMLY CONVEX FUNCTIONS OF ORDER $\alpha$

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**Abstract**. A new subclass of analytic functions $k - SP_\lambda(\alpha)$ is introduced by applying certain operators of fractional calculus to $k$-uniformly starlike and $k$-uniformly convex functions of order $\alpha$. Some theorems on coefficient bounds and growth and distortion theorems for this subclass are found. The radii of close to convexity, starlikeness and convexity for this subclass is also derived.

## 1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic and normalized in the open unit disk $U = \{z : |z| < 1\}$ and $S$ denote the subclass of $A$ that are univalent in $U$. Let $T$ denote the subclass of $S$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0 \tag{1.2}$$

We denote by $ST$ and $CV$ the subclasses of $S$ that are respectively starlike and convex. Goodman [2][3]introduced and defined the following subclasses of $CV$ and $ST$.

A function $f(z)$ is uniformly convex (uniformly starlike) in $U$ if $f(z)$ is in $CV(ST)$ and has the property that for every circular arc $\gamma$ contained in $U$, with center $\xi$ also in $U$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions is denoted by $UCV$ and the class of uniformly starlike functions by $S_p$. From Ronning [9]and Ma and Minda [6]it is well known that

$$f \in S_p \Leftrightarrow Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

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and

$$f \in UCV \Leftrightarrow Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|.$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$.

Kanas and Wisniowska [4][5]defined the functions $f \in S$ to be $k$-uniformly convex ($k$-uniformly starlike) if for $0 \leq k < \infty$ the image of every circular arc $\gamma$ contained in $U$ with center $\xi$ where $\xi \leq k$ is convex (starlike). In the same period, Kanas and Srivastava [11]studied extensively on linear operators associated with $k$-uniformly convex functions.

Bharati, Parvatham and Swaminathan [1]defined $k-S_p(\alpha)$ to be the class of functions $f(z)$ of the form (1.1) with $0 \leq k < \infty$ and $0 \leq \alpha < 1$ that satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha. \tag{1.3}$$

In [1], $k - UCV(\alpha)$ is defined to be the class of functions $f(z)$ of the form (1.1) with $0 \leq k < \infty$ and $0 \leq \alpha < 1$ that satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha. \tag{1.4}$$

Owa and Srivastava [7]introduced the operator $\Omega : A \rightarrow A$ defined by

$$\Omega^\lambda f(z) := \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), \qquad (\lambda \neq 2, 3, 4...) \tag{1.5}$$

where $D_z^\lambda f(z)$ is the fractional derivative of $f$ of order $\lambda$ defined by Owa [8]to be

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi, \qquad 0 \leq \lambda < 1. \tag{1.6}$$

Following the work of Srivastava and Mishra [10]we introduce a class of analytic functions related to $k - S_p(\alpha)$ and $k - UCV(\alpha)$ using the operator $\Omega^\lambda$ defined by (1.5).

**Definition 1.1.** Let $f$ be of the form (1.1), $0 \leq k < \infty$, and $0 \leq \alpha < 1$. Then $f \in k - SP_\lambda(\alpha)$ if and only if $f$ satisfies the condition

$$Re\left\{\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}\right\} \geq k\left|\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} - 1\right| + \alpha. \tag{1.7}$$

It follows that $k - SP_0(\alpha) \equiv k - S_p(\alpha)$ and $k - SP_1(\alpha) \equiv k - UCV(\alpha)$.

Define $k - TSP_\lambda(\alpha)$ to be $k - TSP_\lambda(\alpha) = k - SP_\lambda(\alpha) \cap T$.
The aim of this paper is to investigate several basic properties of the class $k - TSP_\lambda(\alpha)$.

## 2. Coefficient estimates

**Theorem 2.1** *Let $f$ be of the form* (1.1). *Then for $0 \le k < \infty$ and $0 \le \alpha < 1$, $f \in k - SP_\lambda(\alpha)$ if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\alpha+k)]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}|a_n| \le 1-\alpha. \tag{2.1}$$

**Proof.** It suffices to show that

$$k\left|\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}-1\right| - Re\left\{\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}-1\right\} \le 1-\alpha.$$

First of all, it is easily seen from (1.6) that

$$D_z^\lambda\left\{z^n\right\} = \frac{\Gamma(n+1)}{\Gamma(n-\lambda+1)}z^{n-\lambda}.$$

We have

$$\begin{aligned} k\left|\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}-1\right| - Re\left\{\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}-1\right\} &\le (1+k)\left|\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}-1\right| \\ &\le \frac{(1+k)\sum_{n=2}^{\infty}(n-1)\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}|a_n|}{1-\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}|a_n|} \end{aligned}$$

The last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}[n(1+k)-(\alpha+k)]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}|a_n| \le 1-\alpha$$

and hence the proof is complete.

**Theorem 2.2.** *A necessary and sufficient condition for $f$ of the form* (1.2) *to be in the class $k - TSP_\lambda(\alpha)$ for $0 \le k < \infty$ and $0 \le \alpha < 1$ is that*

$$\sum_{n=2}^{\infty}[n(1+k)-(\alpha+k)]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}|a_n| \le 1-\alpha. \tag{2.2}$$

**Proof.** In view of Theorem 2.1, we need only to prove that if (2.1) is true then $f(z) \in k - TSP_\lambda(\alpha)$. Now suppose $f(z) \in k - TSP_\lambda(\alpha)$ and $z$ is real, then

$$\frac{1-\sum_{n=2}^{\infty}n\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_n z^{n-1}} - \alpha \ge k\left|\frac{\sum_{n=2}^{\infty}(1-n)\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_n z^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis we obtain

$$\frac{(1-\alpha)-\sum_{n=2}^{\infty}[n(1+k)-(\alpha+k)]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}|a_n|}{1-\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}|a_n|} \geq 0.$$

This is only possible if (2.1) holds. Therefore we obtain the desired results.

## 3. Crowth and distortion theorems

In this section we prove some growth and distortion theorems for the subclass $k-TSP_\lambda(\alpha)$.

**Theorem 3.1.** *If $f(z) \in k-TSP_\lambda(\alpha)$ then*

$$r-\frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)}r^2 \leq |f(z)| \leq r+\frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)}r^2 \quad (|z|=r)$$

*with equality for*

$$f(z)=z-\frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)}z^2 \qquad (z=\mp r).$$

**Proof.** Since $f(z) \in k-TSP_\lambda(\alpha)$ by applying assertion (2.1) of Theorem 2.1, we obtain

$$\frac{[2(1+k)-(\alpha+k)]\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)}\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}[n(1+k)-(\alpha+k)]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}|a_n| \leq 1-\alpha$$

which immediately yields

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)}. \tag{3.1}$$

From (1.2) and (3.1) we have

$$|f(z)| \leq |z|+\sum_{n=2}^{\infty}|a_n|\,|z|^n \leq |z|+\frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)}|z|^2.$$

Also from (1.2) and (3.1)

$$|f(z)| \geq |z|-\sum_{n=2}^{\infty}|a_n|\,|z|^n \geq |z|-\frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)}|z|^2.$$

Thus, the proof of Theorem 3.1 is complete.

**Theorem 3.2.** *If $f(z) \in k - TSP_\lambda(\alpha)$ then*

$$1 - \frac{(1-\alpha)(2-\lambda)}{2-\alpha+k} r \leq |f'(z)| \leq 1 + \frac{(1-\alpha)(2-\lambda)}{2-\alpha+k} r \qquad (|z| = r)$$

*with equality for*

$$f(z) = z - \frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)} z^2 \qquad (z = \mp r).$$

**Proof.** From the proof of Theorem 3.1

$$|f(z)| \leq |z| + \frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)} |z|^2.$$

Therefore

$$|f'(z)| \leq 1 + \frac{2(1-\alpha)(2-\lambda)}{2(2-\alpha+k)} |z|.$$

Also, from the proof of Theorem 3.1,

$$|f(z)| \geq |z| - \frac{(1-\alpha)(2-\lambda)}{2(2-\alpha+k)} |z|^2.$$

Therefore

$$|f'(z)| \geq 1 - \frac{2(1-\alpha)(2-\lambda)}{2(2-\alpha+k)} |z|.$$

## 4. Close-to-convexity, starlikeness and convexity

For some $\delta (0 \leq \delta < 1)$ and all $z \in U$:
A function $f \in T$ is said to be close-to-convex of order $\delta$ if it satisfies

$$Re\{f'(z)\} > \delta. \tag{4.1}$$

A function $f \in T$ is said to be starlike of order $\delta$ if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta. \tag{4.2}$$

A function $f \in T$ is said to be convex of order $\delta$ if and only if $zf'(z)$ is starlike of order $\delta$ that is, if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta. \tag{4.3}$$

**Theorem 4.1.** *If $f(z) \in k - TSP_\lambda(\alpha)$ then $f$ is close-to-convex of order $\delta$ in $|z| < r_1(k, \alpha, \lambda, \delta)$ where*

$$r_1(k, \alpha, \lambda, \delta) = \inf_n \left[\frac{(1-\delta)[n(1+k) - (\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{n(1-\alpha)\Gamma(n-\lambda+1)}\right]^{\frac{1}{n-1}}.$$

**Proof.** It is sufficient to show that

$$|f'(z) - 1| < \sum_{n=2}^{\infty} na_n|z|^{n-1} \leq 1 - \delta. \tag{4.4}$$

But in view of (2.1) the inequality (4.4) holds true if

$$\frac{n|z|^{n-1}}{1-\delta} \leq \frac{[n(1+k) - (\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{(1-\alpha)\Gamma(n-\lambda+1)}. \tag{4.5}$$

Solving (4.5) for $|z|$ we obtain

$$|z| \leq \left[\frac{(1-\delta)[n(1+k) - (\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{n(1-\alpha)\Gamma(n-\lambda+1)}\right]^{\frac{1}{n-1}} \qquad n = 2, 3, 4, \ldots .$$

**Theorem 4.2.** *If $f(z) \in k - TSP_\lambda(\alpha)$ then $f$ is starlike of order $\delta$ in $|z| < r_2(k, \alpha, \lambda, \delta)$ where*

$$r_2(k, \alpha, \lambda, \delta) = \inf_n \left[\frac{(1-\delta)[n(1+k) - (\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{(n-\delta)(1-\alpha)\Gamma(n-\lambda+1)}\right]^{\frac{1}{n-1}}.$$

**Proof.** We must show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}} < 1 - \delta. \tag{4.6}$$

But in view of (2.1) the inequality (4.6) holds true if

$$\frac{(n-\delta)|z|^{n-1}}{1-\delta} \leq \frac{[n(1+k) - (\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{(1-\alpha)\Gamma(n-\lambda+1)}. \tag{4.7}$$

Solving (4.7) for $|z|$ we obtain

$$|z| \leq \left[\frac{(1-\delta)[n(1+k) - (\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{(n-\delta)(1-\alpha)\Gamma(n-\lambda+1)}\right]^{\frac{1}{n-1}} \qquad n = 2, 3, 4, \ldots$$

**Theorem 4.3.** *If $f(z) \in k{-}TSP_\lambda(\alpha)$ then $f$ is convex of order $\delta$ in $|z| < r_3(k, \alpha, \lambda, \delta)$ where*

$$r_3(k, \alpha, \lambda, \delta) = \inf_n \left[\frac{(1-\delta)[n(1+k) - (\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{n(n-\delta)(1-\alpha)\Gamma(n-\lambda+1)}\right]^{\frac{1}{n-1}}.$$

**Proof.** We must show that

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}} < 1 - \delta. \tag{4.8}$$

But in view of (2.1) the inequality (4.8) holds true if

$$\frac{n(n-\delta)|z|^{n-1}}{1-\delta} \leq \frac{[n(1+k)-(\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{(1-\alpha)\Gamma(n-\lambda+1)}. \tag{4.9}$$

Solving (4.9) for $|z|$ we obtain

$$|z| \leq \left[\frac{(1-\delta)[n(1+k)-(\alpha+k)]\Gamma(2-\lambda)\Gamma(n+1)}{n(n-\delta)(1-\alpha)\Gamma(n-\lambda+1)}\right]^{\frac{1}{n-1}} \qquad n=2,3,4,\ldots$$

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