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SOME REMARKS ON RECONSTRUCTION FROM LOCAL WEIGHTED AVERAGES

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Abstract. We solve the convolution equation of the type $f \star \mu = g$, where $f \star \mu$ is the convolution of f and μ defined by $(f \star \mu)(x) = \int_{\mathbb{R}} f(x-y)\,d\mu(y)$, g is a given function and μ is a finite linear combination of translates of an indicator function on an interval.

1. Introduction

We consider the convolution equation of the following type:

$$f \star \mu = g, \tag{1}$$

where g is a known continuous function, μ is a compactly supported measure and f is an unknown continuous function. Delsarte [3] was interested in solving the particular case of equation (1) which is of the type $\frac{1}{\tau}\int_{x-\frac{\tau}{2}}^{x+\frac{\tau}{2}} f(t)\,dt = g(x)$. In the case when f is an integrable function with compact support van der Pol [15, 16] has obtained reconstruction formula using two sided Laplace Transform. But such transform methods can not be used for the case of continuous functions on $\mathbb{R}$. The special case of equation (1), namely $g = 0$, was analyzed by many authors citebag,ber1,deva,Ehr1,Ehr2,kah,schwartz,thangavelu,wei,sze on various groups. The solutions (1) for the particular case when $g = 0$ are called mean periodic functions. Laurent Schwartz [18] gave an intrinsic characterization of such solutions. The corresponding nonhomogeneous type equation is analysed in [14]for the special case of when μ is the indicator function on the interval $[-a, a]$. An explicit construction of a solution is given in [17] for the same equation on the three dimensional Euclidean space when μ is the indicator function of a ball in $\mathbb{R}^3$ using plane wave decomposition. When μ is finitely supported, the equation (1) gets reduced to a non-homogeneous constant coefficient difference equation. Edgar and Rosenblatt [6] have studied the homogeneous equation (ie, when g=0). They have shown that a complex valued function f has linearly independent translates precisely when f does not

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satisfy a nontrivial homogeneous difference equation. An explicit construction of a solution is given in [4] on $\mathbb{R}$ when μ is an arbitrary finitely supported measure and g is a continuous function.

Malgrange [13], Ehrenpreis [9], John [10], and Hörmander [11] have studied the convolution equation of the type analogous to equation (1)

$$P(D)u = f, \tag{2}$$

where $P(D)$ is a constant coefficient partial differential operator and f is a given function. A criterion was given by Hörmander for the existence of solution $u \in D'_F(\Omega)$ for an arbitrary $f \in D'_F(\Omega)$ on an open set $\Omega \subseteq \mathbb{R}^n$.

In general, no necessary and sufficient conditions for the existence of solutions of equation (1) are known. One can easily see the following:

(i) Equation (1) has no solution in $C(\mathbb{R})$ when g is a non smooth function and μ is a compactly supported continuous function.

(ii) If f_0 is a particular solution of equation (1), then every other solution f can be written as $f = f_0 + h$, where $h \star \mu = 0$.

(iii) If the Fourier-Laplace transform $\hat{\mu}(\lambda) = 0$ for some $\lambda \in \mathbb{C}$ and if there exists a solution to equation (1), then there are infinitely many solutions to equation (1).

The methods of [14] can not be extended to the case when μ is a sum of more than one indicator function. In this paper we analyze the case when μ is a finite linear combination of the translates of an indicator function on an interval. A solution $f \in C^r(\mathbb{R})$ is constructed for every $g \in C^{r+1}(\mathbb{R})$.

2. Reconstruction Results

Definition 1 ([4])**.** We say a compactly supported Borel measure on $\mathbb{R}$ is a discrete Borel measure, if there exists a finite set of distinct real numbers $x_1, x_2, \ldots, x_n$ and nonzero complex constants $c_1, c_2, \ldots, c_n$ such that $\mu(E) = \sum_{i=1}^{n} c_i \delta_{x_i}(E)$ for every Borel set E. The set of all compactly supported discrete Borel measures on $\mathbb{R}$ is denoted by $M_{cd}(\mathbb{R})$.

For $a, b \in \mathbb{R}$, the indicator function on the interval $[a, b]$ is denoted by $\chi_{[a,b]}$ and $LST(\chi_{[a,b]})$ denotes the linear span of the translates of $\chi_{[a,b]}$. The set of all compactly supported regular Borel measures on $\mathbb{R}$ is denoted by $M_c(\mathbb{R})$. We note that $LST(\chi_{[a,b]}) \subset M_c(\mathbb{R})$.

Definition 2. For $f \in C(\mathbb{R})$ and $\mu \in M_c(\mathbb{R})$, the convolution of f with μ is defined as

$$(f \star \mu)(x) = \int_{\mathbb{R}} f(x-y)\, d\mu(y).$$

When $\mu = \sum_{i=1}^{n} c_i \chi_{[a_i,b_i]}$, the convolution becomes

$$(f \star \mu)(x) = \sum_{i=1}^{n} c_i \int_{a_i}^{b_i} f(x-y)\,dy.$$

Definition 3. [4] For every real or complex valued function f and discrete measure $\mu = \sum_{i=1}^{n} c_i \delta_{x_i} \in M_{cd}(\mathbb{R})$, the convolution of f and μ is defined by

$$(f \star \mu)(x) = \sum_{i=1}^{n} c_i f(x - x_i).$$

In [4] the special case $r = 0$ of the following lemma is proved. We extend the same for $r > 0$ along the lines of [4].

Lemma 1. *For $\mu, \nu \in M_{cd}(\mathbb{R})$ and $g \in C^r(\mathbb{R})$, the following hold:*

(i) *If $supp(\mu) \subset (-\infty, -\alpha)$ for some $\alpha > 0$ and $supp(g) \subset (-\infty, \beta)$ for some $\beta \in \mathbb{R}$, then there exists $f \in C^r(\mathbb{R})$ such that $f \star (\delta_0 + \mu) = g$.*

(ii) *If $supp(\nu) \subset (\alpha, \infty)$ for some $\alpha > 0$ and $supp(g) \subset (\beta, \infty)$ for some $\beta \in \mathbb{R}$, then there exists $f \in C^r(\mathbb{R})$ such that $f \star (\delta_0 + \nu) = g$.*

Proof. (i) We denote by μ^m the convolution of μ with itself m-times. As $supp(\mu) \subset (-\infty, -\alpha)$, we have $supp(\mu^n) \subset (-\infty, -n\alpha)$. Let $\mu^n = \sum_{i=1}^{l} c_i \delta_{y_i}$. Then

$$(g^{(j)} \star \mu^n)(x) = \sum_{i=1}^{l} c_i g^{(j)}(x - y_i).$$

Since $supp(\mu^n) \subset (-\infty, -n\alpha)$, $y_i < -n\alpha$ and hence $x - y_i > x + n\alpha > \beta$ for sufficiently large n. Therefore for every x, $(g^{(j)} \star \mu^n)(x) = 0$ for n sufficiently large and for $0 \le j \le r$.

We define

$$f(x) := g(x) + \sum_{n=1}^{\infty} (-1)^n (g \star \mu^n)(x).$$

Let us consider the following partial sums:

$$s_n(x) = g(x) + \sum_{k=1}^{n} (-1)^k (g \star \mu^k)(x).$$

Then

$$s_n^{(j)}(x) = g^{(j)}(x) + \sum_{k=1}^{n} (-1)^k (g^{(j)} \star \mu^k)(x).$$

We show that the above sequence converges uniformly on every compact set for $0 \le j \le r$. For, let K be a compact subset of $\mathbb{R}$. Then $K \subset [a,b]$ for some real numbers a and b. Choose N such that $a+n\alpha > \beta$ and $b+n\alpha > \beta$, for $n \ge N$. For $x \ge a$, $x - y_i \ge a+n\alpha > \beta$. Now

$$s_n^{(j)}(x) - s_m^{(j)}(x) = \sum_{k=m+1}^{n} (-1)^k (g^{(j)} \star \mu^k)(x) = 0,$$

for $n \ge m \ge N$.

This implies that the sequence of functions $\{s_k^{(j)}(x)\}$ is uniformly cauchy on every compact set and hence converges uniformly on every compact set for $0 \le j \le r$.

Therefore we get $s_k^{(j)}(x)$ converges uniformly to $f^{(j)}(x)$ on every compact set and

$$f^{(j)}(x) := g^{(j)}(x) + \sum_{n=1}^{\infty} (-1)^n (g^{(j)} \star \mu^n)(x)$$

for $0 \le j \le r$. Hence $f^{(r)}$ is continuous and hence $f^{(r)} \in C^r(\mathbb{R})$. It is very easy to check that $f \star (\delta_0 + \mu) = g$.

(ii) Since $supp(\nu) \subset (\alpha,\infty)$, we have $supp(\nu^n) \subset (n\alpha,\infty)$. Suppose the representation of ν^n is of the form: $\nu^n = \sum_{i=1}^{l} d_i \delta_{z_i}$. Then $(g^{(j)} \star \nu^n)(x) = \sum_{i=1}^{l} d_i g^{(j)}(x - z_i)$. Since $supp(\nu^n) \subset (n\alpha,\infty)$, $z_i > n\alpha$ and hence $x - z_i < x - n\alpha < \beta$ for sufficiently large n. Therefore for every x, $(g^{(j)} \star \nu^n)(x) = 0$ for n sufficiently large. Hence $g^{(j)}(x) + \sum_{m=1}^{\infty} (-1)^m (g^{(j)} \star \nu^m)(x)$ is a finite sum for every x.

We define

$$f(x) := g(x) + \sum_{m=1}^{\infty} (-1)^m (g \star \nu^m)(x). \tag{3}$$

To show

$$f^{(r)}(x) := g^{(r)}(x) + \sum_{m=1}^{\infty} (-1)^m (g^{(r)} \star \nu^m)(x),$$

it is sufficient if we show that the partial sums of the series (3) and their derivatives converge uniformly on compact sets. For, let K be a compact subset of $\mathbb{R}$. Then $K \subset [a,b]$ for some real numbers a and b. Choose N such that $b - n\alpha < \beta$, for $n \ge N$. Let us take the partial sums of the series as

$$t_k(x) = g(x) + \sum_{m=1}^{k} (-1)^m (g \star \nu^m)(x).$$

For $x \le b$, $x - z_i < x - n\alpha \le b - n\alpha$. Choose N such that $b - n\alpha < \beta$ for $n \ge N$. Then

$$t_n^{(j)}(x) - t_m^{(j)}(x) = \sum_{k=m+1}^{n} (-1)^k (g^{(j)} \star \nu^k)(x) = 0,$$

for $n \ge m \ge N$ and $0 \le j \le r$. This implies that the sequence $\{t_n^{(j)}(x)\}$ is uniformly cauchy and hence converges uniformly on every compact set. Hence we get $t_n^{(j)}(x) \to f^{(j)}(x)$. Therefore $f \in C^r(\mathbb{R})$. One easily verifies $f \star (\delta_0 + \nu) = g$. □

Lemma 2. *For $\mu = \chi_{[a,b]}$ and $g \in C^{r+1}(\mathbb{R})$, the following hold: If $supp(g) \subset (-\infty, \beta)$ or $supp(g) \subset (\beta, \infty)$ for some $\beta \in \mathbb{R}$, then there exists $f \in C^r(\mathbb{R})$ such that $f \star \mu = g$.*

Proof. Case(i): Suppose that $supp(g) \subset (-\infty, \beta)$.

We can write

$$f \star \chi_{[a,b]} = f \star \chi_{[\frac{a-b}{2}, \frac{b-a}{2}]} \star \delta_{\frac{a+b}{2}}.$$

Define

$$f_1(x) = -\sum_{n=0}^{\infty} \left(g' \star \delta_{\frac{a-b}{2}}^{2n+1} \right)(x).$$

We show that the above series converges uniformly on compact sets. For, let K be a compact subset of $\mathbb{R}$. Then $K \subset [c,d]$ for some $c,d \in \mathbb{R}$. Let us take

$$s_n(x) = -\sum_{k=0}^{n} \left(g' \star \delta_{\frac{a-b}{2}}^{2k+1} \right)(x).$$

Then

$$s_n^{(j)}(x) = -\sum_{k=0}^{n} \left(g^{(j+1)} \star \delta_{\frac{a-b}{2}}^{2k+1} \right)(x).$$

Now

$$g' \star \delta_{\frac{a-b}{2}}^{2k+1}(x) = g'\left(x + (2k+1)(\frac{b-a}{2}) \right).$$

Choose N such that $c + (2k+1)(\frac{b-a}{2}) > \beta$ for $k \geq N$. Then $g^{(j+1)} \delta_{\frac{a-b}{2}}^{2k+1}(x) = 0$ for $k \geq N$ for all $x \in K$. Therefore $s_n^{(j)}(x) - s_m^{(j)}(x) = 0$ for all $n, m \geq N$, for all $x \in K$ and for $0 \leq j \leq r$. Hence $s_n^{(j)}(x) \to f_1^{(j)}(x)$ uniformly on K.
Therefore $f_1^j(x) = -\sum_{n=0}^{\infty} g^{(j+1)} \star \delta_{\frac{a-b}{2}}^{2n+1}(x)$ and $f_1 \in C^r(\mathbb{R})$. Now

$$\begin{aligned} f_1 \star \chi_{[\frac{a-b}{2}, \frac{b-a}{2}]} &= -\sum_{n=0}^{\infty} \left(\left(g' \star \chi_{[\frac{a-b}{2}, \frac{b-a}{2}]} \right) \star \delta_{\frac{a-b}{2}}^{2n+1} \right)(x) \\ &= -\sum_{n=0}^{\infty} \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} g(x + (2n+1)(\frac{b-a}{2}) - y)\,dy \\ &= \sum_{n=0}^{\infty} \left[g\left(x + (2n+1)(\frac{b-a}{2}) - \frac{b-a}{2} \right) - g\left(x + (2n+1)(\frac{b-a}{2}) + \frac{b-a}{2} \right) \right] \\ &= g(x). \end{aligned}$$

Define $f(x) = f_1 \star \delta_{-(\frac{a+b}{2})}$. One easily verifies $f \star \chi_{[a,b]} = g$.

Case(ii): Suppose that $supp(g) \subset (\beta, \infty)$. Now $f \star \chi_{[a,b]} = f \star \chi_{[\frac{a-b}{2}, \frac{b-a}{2}]} \star \delta_{\frac{a+b}{2}}$. Define $f_1(x) = \sum_{n=0}^{\infty} g' \star \delta_{\frac{b-a}{2}}^{2n+1}(x)$. We show that the above series converges uniformly on compact sets. For, let K be a compact subset of $\mathbb{R}$. Then $K \subset [c,d]$ for some $c,d \in \mathbb{R}$. Let $s_n(x) = \sum_{k=0}^{n} g' \star \delta_{\frac{b-a}{2}}^{2k+1}(x)$. Then $s_n^{(j)}(x) = \sum_{k=0}^{n} g^{(j+1)} \star \delta_{\frac{b-a}{2}}^{2k+1}(x)$.

We can write

$$g \star \delta_{\frac{b-a}{2}}^{2k+1}(x) = g(x-(2k+1)(\frac{b-a}{2})).$$

Choose N such that $c-(2k+1)(\frac{b-a}{2}) < \beta$ for $k \geq N$. Then $g^{(j+1)}\delta_{\frac{b-a}{2}}^{2k+1}(x) = 0$ for $k \geq N$ for all $x \in K$. Therefore $s_n^{(j)}(x) - s_m^{(j)}(x) = 0$ for all $n, m \geq N$, for all $x \in K$ and for $0 \leq j \leq r$. Hence $s_n^{(j)}(x) \to f_1^{(j)}(x)$ uniformly on K.

Therefore $f_1^j(x) = \sum_{n=0}^{\infty} g^{(j+1)} \star \delta_{\frac{b-a}{2}}^{2n+1}(x)$. and $f_1 \in C^r(\mathbb{R})$.

Now

$$\begin{aligned} f_1 \star \chi_{[\frac{a-b}{2},\frac{b-a}{2}]} &= -\sum_{n=0}^{\infty} g' \star \chi_{[\frac{a-b}{2},\frac{b-a}{2}]} \star \delta_{\frac{b-a}{2}}^{2n+1}(x) \\ &= \sum_{n=0}^{\infty} \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} g(x-(2n+1)(\frac{b-a}{2})-y)dy \\ &= -\sum_{n=0}^{\infty} \left[g\left(x-(2n+1)(\frac{b-a}{2})-\frac{b-a}{2}\right) - g\left(x-(2n+1)(\frac{b-a}{2})+\frac{b-a}{2}\right)\right] \\ &= g(x). \end{aligned}$$

Define $f(x) = f_1 \star \delta_{-(\frac{a+b}{2})}$. It is easy to check that $f \star \chi_{[a,b]} = g$. □

Lemma 3. *Let $\mu = \sum_{i=1}^{n} c_i \chi_{[a_i,b_i]}$ be a finite linear combination of indicator functions on intervals. If there exists $r \in \mathbb{R}$ such that $\frac{b_i-a_i}{r} \in \mathbb{Z}$, then the following hold:*

(i) *There exists $g \in LST(\chi_{[a,b]})$ such that $\mu = g$ almost everywhere for some $a, b \in \mathbb{R}$.*

(ii) *There exits $\mu \in M_{cd}(\mathbb{R})$, such that $\mu = \chi_{[a,b]} \star \nu$ a.e and $f \star \mu = f \star \chi_{[a,b]} \star \nu$ for all $f \in C(\mathbb{R})$.*

Proof. (i) Let $\frac{b_i-a_i}{r} = m_i$. Then $\chi_{[a_i,b_i]} = \sum_{j=1}^{m_i} \chi_{[a_i+(j-1)r, a_i+jr]}$ a.e. As the indicator functions $\chi_{[a_i+(j-1)r,a_i+jr]}$ are translates of the indicator function on $[0,r]$, we have $\chi_{[a_i+(j-1)r,a_i+jr]} \in LST(\chi_{[0,r]})$. Hence $g_i = \sum_{j=1}^{m_i} \chi_{[a_i+(j-1)r,a_i+jr]} \in LST(\chi_{[0,r]})$. Therefore $g = \sum_{i=1}^{n} c_i g_i \in LST(\chi_{[0,r]})$ and hence $\mu = \sum_{i=1}^{n} c_i \chi_{[a_i,b_i]} = g$ a.e.

(ii) In the above proof,

$$\begin{aligned} g_i &= \sum_{j=1}^{m_i} \chi_{[a_i+(j-1)r,a_i+jr]} \\ &= \sum_{j=1}^{m_i} \chi_{[0,r]} \star \delta_{a_i+(j-1)r} \\ &= \chi_{[0,r]} \star (\sum_{j=1}^{m_i} \delta_{a_i+(j-1)r}) \\ &= \chi_{[0,r]} \star \nu_i, \end{aligned}$$

where $\nu_i = \sum_{j=1}^{m_i} \delta_{a_i+(j-1)r} \in M_{cd}(\mathbb{R})$. But $g = \sum_{i=1}^{n} c_i g_i$ a.e. Therefore

$$\begin{aligned} g &= \sum_{i=1}^{n} c_i g_i \\ &= \sum_{i=1}^{n} \chi_{[0,r]} \star c_i \nu_i \\ &= \chi_{[0,r]} \star (\sum_{i=1}^{n} c_i \nu_i) \\ &= \chi_{[0,r]} \star \nu, \end{aligned}$$

where $\nu = \sum_{i=1}^{n} c_i \nu_i \in M_{cd}(\mathbb{R})$. Since $\mu = g$ a.e, we have $\mu = \chi_{[0,r]} \star \nu$ a.e.

Also we have

$$f \star \chi_{[a_i,b_i]}(x) = \int_{a_i}^{b_i} f(x-y)dy = \sum_{j=1}^{m_i} \int_{a_i+(j-1)r}^{a_i+jr} f(x-y)dy = (f \star g_i)(x).$$

Therefore $f \star \mu = f \star g = f \star \chi_{[0,r]} \star \nu$. □

The first part of the following theorem is an extension of [4] and the second part is a simple extension of [14].

Theorem 2.1. (i) *For every $g \in C^r(\mathbb{R})$ and every $\nu \in M_{cd}(\mathbb{R})$ with $\nu \neq 0$ there exists $f \in C^r(\mathbb{R})$ such that $f \star \nu = g$.*

(ii) *For every $g \in C^{r+1}(\mathbb{R})$, there exists $f \in C^r(\mathbb{R})$ such that $f \star \chi_{[a,b]} = g$.*

Proof. For $\epsilon > 0$, choose $\phi_\epsilon \in C^{r+1}(\mathbb{R})$ such that $supp(\phi_\epsilon) \subset (-\epsilon,\epsilon)$ and $\int_{-\epsilon}^{\epsilon} \phi_\epsilon(x)dx = 1$.

Define $\eta_1, \eta_2 \in C(\mathbb{R})$ by

$$\eta_1(x) := \begin{cases} 0 & \text{if } x \le -k \\ \frac{x+k}{2k} & \text{if } -k \le x \le k \\ 1 & \text{if } x \ge k \end{cases}$$

$$\eta_2(x) := \begin{cases} 1 & \text{if } x \le -k \\ \frac{k-x}{2k} & \text{if } -k \le x \le k \\ 0 & \text{if } x \ge k \end{cases}.$$

It is simple to check that $\eta_1(x) + \eta_2(x) = 1$ for all $x \in \mathbb{R}$. Convolving both sides with ϕ_ϵ, we get $(\eta_1 \star \phi_\epsilon)(x) + (\eta_2 \star \phi_\epsilon)(x) = 1$ for all $x \in \mathbb{R}$. Define

$$g_1(x) = g(x)(\eta_1 \star \phi_\epsilon)(x) \text{ and } g_2(x) = g(x)(\eta_2 \star \phi_\epsilon)(x).$$

Then $g_1, g_2 \in C^{r+1}(\mathbb{R})$ and $supp(g_1) \subset [-k-2\epsilon, \infty)$ and $supp(g_2) \subset (-\infty, k+2\epsilon]$. Also we have $g_1 + g_2 = g$.

(i) We show that $f_1 \star \mu = g_1$ and $f_2 \star \mu = g_2$ have solutions f_1 and f_2 respectively in $C^r(\mathbb{R})$. These f_1 and f_2 are then used to construct a solution of the equation $f \star \mu = g$.

Let $\mu = \sum_{i=1}^{n} c_i \delta_{x_i}$, $x_{i_0} = Min\{x_1, x_2, \ldots, x_n\}$ and $x_{j_0} = Max\{x_1, x_2, \ldots, x_n\}$.

We can write

$$\begin{aligned}
\mu &= \sum_{i=1}^{n} c_i \delta_{x_i} \\
&= \sum_{i=1}^{n} c_i \delta_{x_i} \star \delta_{-x_{i_0}} \star \delta_{x_{i_0}} \\
&= \delta_{x_{i_0}} \star \sum_{i=1}^{n} c_i \delta_{x_i - x_{i_0}} \\
&= c_{i_0} \delta_{x_{i_0}} \star (\delta_0 + \sum_{i=1, i \neq i_0}^{n} \frac{c_i}{c_{i_0}} \delta_{x_i - x_{i_0}}) \\
&= c_{i_0} \delta_{x_{i_0}} \star (\delta_0 + \nu), \quad \text{where } \nu = \sum_{i=1, i \neq i_0}^{n} \frac{c_i}{c_{i_0}} \delta_{x_i - x_{i_0}}.
\end{aligned}$$

Also we can write μ as

$$\begin{aligned}
\mu &= \sum_{j=1}^{n} c_j \delta_{x_j} \\
&= \sum_{j=1}^{n} c_j \delta_{x_j} \star \delta_{-x_{j_0}} \star \delta_{x_{j_0}} \\
&= \delta_{x_{j_0}} \star \sum_{j=1}^{n} c_j \delta_{x_j - x_{j_0}} \\
&= c_{j_0} \delta_{x_{j_0}} \star (\delta_0 + \sum_{j=1, j \neq j_0}^{n} \frac{c_j}{c_{j_0}} \delta_{x_j - x_{j_0}}) \\
&= c_{j_0} \delta_{x_{j_0}} \star (\delta_0 + \psi), \quad \text{where } \psi = \sum_{j=1, j \neq j_0}^{n} \frac{c_j}{c_{j_0}} \delta_{x_j - x_{j_0}}.
\end{aligned}$$

Define $\alpha = \frac{1}{2} \min\{x_i - x_{i_0} / 1 \le i \le n, i \neq i_0\}$ and $\beta = \frac{1}{2} \min\{x_{j_0} - x_j / 1 \le j \le n, j \neq j_0\}$.

Then $x_i - x_{i_0} > \alpha$, $x_j - x_{j_0} < -\beta$. Hence $supp(\nu) \subset (\alpha, \infty)$ and $supp(\psi) \subset (-\infty, -\beta)$.

Using lemma 1, we get $h_1, h_2 \in C^r(\mathbb{R})$ such that

$$h_1 * (\delta_o + \nu) = g_1 \tag{4}$$

and

$$h_2 * (\delta_o + \psi) = g_2. \tag{5}$$

Convolving both sides of the equation (4) with $c_{i_0} \delta_{x_{i_0}}$ and the equation (5) with $c_{j_0} \delta_{x_{j_0}}$, we get $h_1 * (\delta_o + \nu) \star c_{i_0} \delta_{x_{i_0}} = c_{i_0} g_1 \star \delta_{x_{i_0}}$ and $h_2 * (\delta_o + \psi) \star c_{j_0} \delta_{x_{j_0}} = c_{j_0} g_2 \star \delta_{x_{j_0}}$. That is

$$h_1 * \mu = c_{i_0} g_1 \star \delta_{x_{i_0}} \tag{6}$$

and

$$h_2 * \mu = c_{j_0} g_2 \star \delta_{x_{j_0}}. \tag{7}$$

Equations (6) and (7) imply $(\frac{1}{c_{i_0}} h_1 \star \delta_{-x_{i_0}}) \star \mu = g_1$ and $(\frac{1}{c_{j_0}} h_2 \star \delta_{-x_{j_0}}) \star \mu = g_2$. Define $f = \frac{1}{c_{i_0}} h_1 \star \delta_{-x_{i_0}} + \frac{1}{c_{j_0}} h_2 \star \delta_{-x_{j_0}}$. Then $f \in C^r(\mathbb{R})$. Now

$$\begin{aligned} f \star \mu &= (\frac{1}{c_{i_0}} h_1 \star \delta_{-x_{i_0}}) \star \mu + (\frac{1}{c_n} h_2 \star \delta_{-x_{j_0}}) \star \mu \\ &= g_1 + g_2 \\ &= g. \end{aligned}$$

(ii) Using Lemma 2, we get $f_1, f_2 \in C^r(\mathbb{R})$ such that $f_1 \star \chi_{[a,b]} = g_1$ and $f_2 \star \chi_{[a,b]} = g_2$. Then $f = f_1 + f_2 \in C^r(\mathbb{R})$ will satisfy $f \star \chi_{[a,b]} = g$. □

Theorem 2.2. *For $g \in C^{r+1}(\mathbb{R})$, the following hold:*

(i) *For every $\mu \in LST(\chi_{[a,b]})$ with $\mu \neq 0$ a.e., there exists $f \in C^r(\mathbb{R})$ such that $f \star \mu = g$.*

(ii) *If there exists $r \in \mathbb{R}$ such that $\frac{b_i - a_i}{r} \in \mathbb{Z}$ and $\mu = \sum_{i=1}^{n} c_i \chi_{[a_i, b_i]} \neq 0$ a.e., then there exists $f \in C^r(\mathbb{R})$ such that $f \star \mu = g$.*

Proof. (i) By lemma 3, there exists $\nu \in M_{cd}(\mathbb{R})$ such that $\mu = \chi_{[a,b]} \star \nu$. Applying Theorem 2.1, we get a $h \in C^{r+1}(\mathbb{R})$ and $f \in C^r(\mathbb{R})$ such that $h \star \nu = g$ and $f \star \chi_{[a,b]} = h$. It is simple to verify that $f \star \mu = g$.

(ii) Using Lemma 3, we can write $\mu = \chi_{[a,b]} \star \nu$ a.e for some $\nu \in M_{cd}(\mathbb{R})$. As in previous part we obtain $f \in C^r(\mathbb{R})$ such that $f \star \mu = g$. □

Remark 1. The operator T_μ defined by $T_\mu(f) = f \star \mu$ is 1-1 if we restrict the domain of T_μ to the space of integrable functions $L_1(\mathbb{R})$. This can be seen as follows:Suppose $f \star \mu = 0$ and $f \in L_1(\mathbb{R})$. Since f is integrable and μ is compactly supported, the Fourier transforms of both f and μ namely $\hat{f}$ and $\hat{\mu}$ are holomorphic on $\mathbb{C}$. Hence the corresponding zero sets $z(\hat{f})$ and $z(\hat{\mu})$ are of measure zero. Therefore we get $f = 0$ a.e.

Remark 2. When $\mu \in LST(\chi_{[a,b]})$ or $\mu = g$ a.e for some $g \in LST(\chi_{[a,b]})$, the kernel of the operator T_μ is a nontrivial subspace of $C(\mathbb{R})$. For, since μ can be written as $\mu = \chi_{[0,r]} \star \nu$ for some $\nu \in M_{cd}(\mathbb{R})$. This implies that $\lambda = \frac{2n\pi}{r} \in z(\hat{\mu})$ for $n \in \mathbb{Z}$. Therefore $e^{i\lambda x} \in Ker(T_\mu)$. Hence there are infinitely many solutions to the convolution equation $f \star \mu = g$.

Remark 3. Theorem 2.2 is possible even if $g \in L_1(\mathbb{R})$ with $\hat{g}(\lambda) \neq 0$ and the Fourier-Laplace transform $\hat{\mu}(\lambda) = 0$ for some $\lambda \in \mathbb{C}$.

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